Problem 4: A Baby Chess

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Abstract

In the following text we solve the whole game without a trace for an $m \times n$-rectangle with an arbitrary starting sqaure.

In addition we will solve the game with a trace for an $n \times n$-rectangle with the starting square at the boundary for the bishop and the rook. Furthermore, we present some further ideas for the bishop in an $n \times n$-rectangle with an arbitrary starting field.

In the first section, the main idea is the principle of the free neighbor, i.e. one player can pair the squares such that he is able to move in every situation and because of the finiteness of the game, the other player has to lose. Moreover, we reduce the cases when m and n are large to little and hence easy ones. Considering the knight, we will see a nice idea with a "pseudo-starting" square.

In the second section, the main idea is to move along the longest possible way to generate a rectangle in which one person moves always along the short side and the other along the longer side and thus, the latter has to win. An interesting method is also the robbing of a strategy (if one player had a strategy, the other would have a strategy too). We will use that when considering the bishop.

Illustrations are to be found at the end.

Without loss of generality we may assume that the starting square is colored black (if not we can recolor the whole rectangle).

Problem 4.1: Without a trace

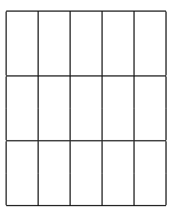
The main idea in all of the following solutions is the principle of the free neighbor. That is that one player can form pairs of the squares such that he can always execute a step. Because of the finiteness of the game the other player has to loose.

The King

1. case: m or n is even

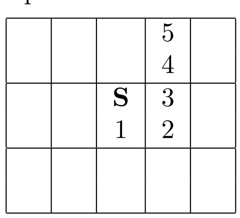
   Let m be even without loss of generality.

   Then there exist such a pattern with 1×2-dominos:

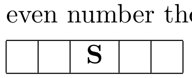
   The dominos are placed vertically. Because of the fact that m is even, the pattern covers the whole rectangle.

   Wherever the starting square is, the player who moves at first (we will call him player A, the other student will be called player B) has a winning strategy. He always move to the free square of the same domino. The second player has to change the domino and on the new one, there is still a free square. So player A is always able to move. Because of the finiteness of the game player A is the winner. An example (S denotes the starting square and the numbers are in the same order as the moves):

2. case: m and n are odd

   If m or n is 1 then the situation is trivial because each player has only one possibility (exception is the first move). If the starting square has an even number then player A can win, otherwise it is player B.

   From now consider m and n bigger than 1.

I claim that player B has now a winning strategy because he can cover the rectangle with 1×2-dominos and with "diagonal dominos" (that are two squares which are connected only by a corner).

First of all, player B can cover $2 \times k$-rectangles which haven't got the starting square in it until only a 3×3-rectangle is left over. Then there are only three possibilities because of symmetry:

a	b	c
a	b	c
S	d	d

d	d	c
a	c	b
a	**S**	b

a	a	c
b	**S**	c
b	d	d

The same letters denotes the pairs of the squares.

So player B has a winning strategy.

The Queen

1. case: m or n is even

   We have the same situation as when we considered the king. So player A wins.

2. case: m and n are odd

   If m or n is 1 then the game can't stop before all squares are visited. So player B wins (number of squares is odd).

   If m and n are both bigger than 1, we have the same situation as when we considered the king. So player B wins.

The Rook

1. case: m or n is even

   Analogously to the king player A has a winning strategy.

2. case: m and n are odd

   If m or n is 1 we have the same situation as for the queen.

   If m and n are both bigger than 1 then player B can reduce the pattern to a 3×3-square in an analogous way as in the king case. But the rook can't move diagonal. So one of the three possibilities has changed:

a	b	c
a	b	c
d	**S**	d

   So player B is winning.

The Bishop

If m or n is 1 then nobody can move, so B wins.

1. case: number of black squares is even

   Note that if the bishop starts on a black square, he can only ever move to black squares. Consider the rectangle diagonally. You see diagonal black rows. If we talk about "rows" in the following (until the end of the bishop), these diagonal rows are ment. A row is called even if the number of black squares in it is even. Analogously, we define odd rows. The parity of the numbers of black squares in a row can change only twice.

   Player A can now build pairs as for the king, but now we consider the diagonal rows. If there is an even row, the pairing is obviously possible. If there are two odd rows, the pairing is also possible with one "connecting pair". Because of the fact that the number of black squares is even, the number of odd rows is even too. Since the parity changes only twice the pairing works like it is described above except the following case: at first the rows are odd, then even and then odd again. Then player A has to create connecting pairs to even rows too. Then one pair inside the even row is destroyed. Use the free partner for the next connecting pair.

   So, if the number of black squares is even there is a possibility of pairing the black squares such that player A has a winning strategy (principle of the free neighbor again).

   Example:

a		b		d	
	a		d		e
b		c		f	
	c		e		f

2. case: number of black squares is odd

   Player B executes the above method. At the end one square in an odd row stays free. Player B can now create connecting pairs such that this free square is in the same row as the starting square. Within there is a pairing without the starting square.

   That is why player B now has a winning strategy.

The Knight

Obviously m and n should be greater than 1 (otherwise, B wins).

1. case: m and n are even

   If m or n is equal to 2, the situation is trivial because there is only one possibility for a motion (except the first move). So if the number of available squares to the left and the right of the starting square is even, player B wins. Otherwise player A.

If m or n is divisible by 4: the whole rectangle can be covered with 2×4-rectangles which can be paired. So A wins.

1	3	2	4
2	4	1	3

If m and n both have residue 2 mod 4: A can split the whole rectangle in a 6×6-square and three other rectangles with at least one sidelength divisible by 4 (see Figure 1, attached at the end of the solution). In a 6×6-square, a pairing is possible (see Figure 2). As a consequence A wins too.

In the following we often want to cover rectangles so we introduce some shapes which can easily be covered with a pairing. See Figure 3.

2. case: m is even and n is odd

   If m is divisible by 4, A can cover the rectangle with one row of 3×4-rectangles and the rest with 2×4-rectangles.

   If m has residue 2 mod 4 and $n \equiv 1$ mod 4 then A can cover the rectangle like it is shown in the upper part of Figure 4 .

   If m isn't divisible by 4 and n has residue 3 mod 4 then A can cover the rectangle like it is presented in the lower part of Figure 4.

   The case m is odd and n is even is analogous.

   So we see that player A has a winning strategy if m or n is even.

3. case: m and n are odd

   If $m = n = 3$ and the starting square is the middle, B wins (no motion possible). Otherwise A wins because every player has only one possible move (except the first) and there are seven moves altogether.

   I will show below: if the number of black squares (remember: the starting square is black) is greater than the number of the white ones then B has a winning strategy. With that, it is easy to prove that A has a winning strategy if the number of white squares is greater than the number of black ones.

   The strategy for B is pairing the squares like we have often done it above. If there are more white squares, A choose one of them and state it to be the pseudo-starting square. Then A can pair the rest, because he is in the same situation as B if there are more black squares. Then A plays like at all times (going to the free neighbor). Because of the move of the knight player A moves always to a white square and B to a black square. So B can't reach the white pseudo-starting square. Thus, A wins because he can always execute a move.

   Now, we prove that player B wins if the number of black squares is greater than the white ones.

   If $m = 3$ and $n \equiv 3$ mod 4 B can pair 3×4-rectangles until only a 3×7 -rectangle is left which has got the starting square inside. There are now only 4 possibilities because of symmetry and because of the fact that the corners are colored black (number of black squares is bigger than number of white ones). See Figure 5.

If $m = 3$ and n has residue 1 mod 4 then there is one exception: if $n = 5$ and the starting square is in the middle of the n-direction, then player A wins (see Figure 6a). Otherwise B can pair 3×4-rectangles until only 3×5 or in the one case a 3×9 -rectangle is left. See Figure 6b.

If m and n are both greater than 3, B has a winning strategy by pairing 3×4 -rectangles until a 5×5 or a 5×7 or a 7×7-rectangle is left. Because of symmetry and because of the fact that a corner is black, there are only a few possibilities. See Figure 7.

So the knight is also analysed.

Problem 4.2: With a trace

The King

This is the same as above, as the king always only moves by one square and thus doesn't leave a trace.

The Rook

We consider an $m \times n$-rectangle where $m \geq n$. The starting square is the upper right corner.

S					

The idea is now, that A has the following strategy: he goes always until "the end" (until he can't move longer) in the direction of m. Then B has to go in the direction of n. A follows his strategy. So B has always to go in the direction of n and A always in the direction of m. Because of $m \geq n$, A has to win (because the fact that the number of lines of the rectangle is smaller than the number of columns stays invariant, so we can proceed by induction). an example is shown in Figure 8a.

So we know: if we consider an $m \times n$-rectangle with $m \geq n$ and the starting square is along the n direction, player A wins. In particular, if we consider an $n \times n$-square and the starting square is on the boundary, A wins (see Figure 8b).

The Bishop

We consider an $n \times n$-square and we assume that the starting square is at the boundary. We will prove that A wins.

We watch in the diagonal direction so we see again the diagonal rows. The argument with the rectangle we applied for the rook is still correct for diagonal rectangles.

As you can see from the first picture of Figure 9 there is a rectangle R (because of symmetry).

The strategy for A: at the beginning, he chooses the longest possible way and he moves unil the end (until the edge of the $n \times n$-square is reached). A goes always parallel to this first motion, always until the end of R (so not outside of R). That will be in the normal case, we will see what is meant by this.

The long side of R is in the same direction as player A moves (see Figure 9a). Either all motions stay in R, but that would create a winning situation for A because of the argument from the rook, or player B has to go in the area 3 or 4 because he can't go into area 1 or 2 (see Figure 9b). That could only happen if player B moves one time in the same direction as player A. If A is then in a loosing situation, A could have done that move just before (that would be the unnormal case). Thus, player B is in that loosing situation.

So player A can win.

Now, we consider an $n \times n$-square and the starting square isn't at the boundary.

When A executes the longest possible move at the beginning, a rectangle R exists again. The direction of the move is along the longer side of the rectangle (see Figure 10a). So we have got nearly the same situation like above. Thus, the only chance for player B to win is to move one time like it is shown in Figure 10b (B has to move to the lower right corner of R). As you can see from Figure 10b a rectangle R_1 arises. If the longer side of that rectangle shows in the same direction as the longer side of R, we have again the same situation like above, so A wins. Only if the starting square is in the near of the middle of the $n \times n$-rectangle R_1 hasn't got this property. That doesn't imply automatically that B wins.

If the proportion of R_1 is $x - 1 \geq \max - x + 2$ (see Figure 10b), it follows that $\max \leq 2x - 3$. With $x = l - \max$ we get $3 \max \leq 2l - 3$. Because of the maximality of max, we have $\max \geq \frac{l}{2} + 1$ (max can't be exactly $\frac{l}{2}$ because if the starting square lies on a main diagonal of the $n \times n$-square player A moves into a corner and the game is over), so we obtain

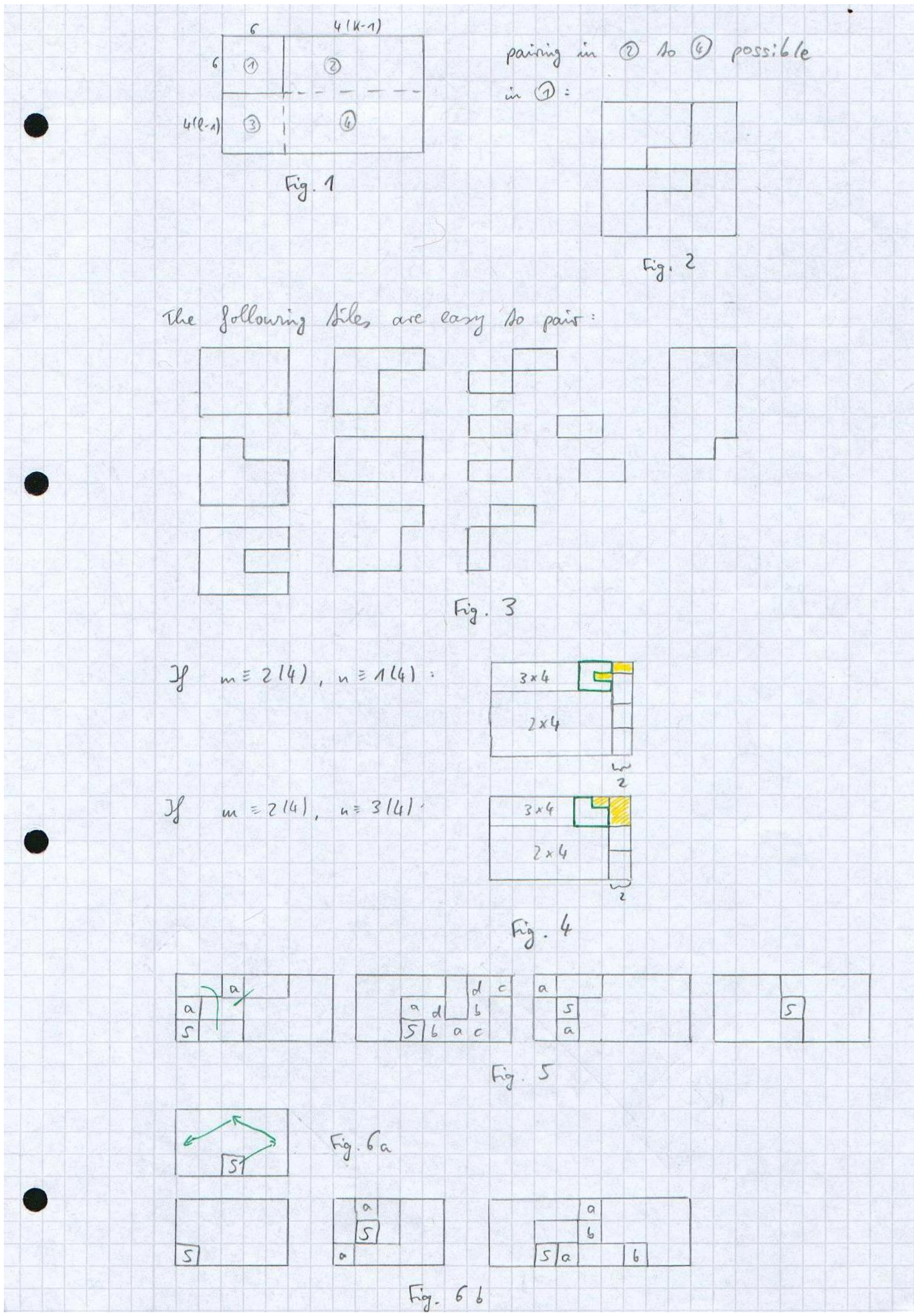
$$2l - 3 \geq 3 \max \geq \frac{3l}{2} + 3$$

and from this finally

$$l \geq 12$$

Therefore, in an normal chess field (8×8-rectangle) player A wins whereever the starting square is.

Figures


6
4(k-1)
4(l-1)
Fig. 1
pairing in ② to ④ possible
in ①:
Fig. 2
The following tiles are easy to pair:
Fig. 3
If m ≡ 2(4), n ≡ 1(4):
3x4
2x4
If m ≡ 2(4), n ≡ 3(4):
Fig. 4
Fig. 5
Fig. 6a
Fig. 6b

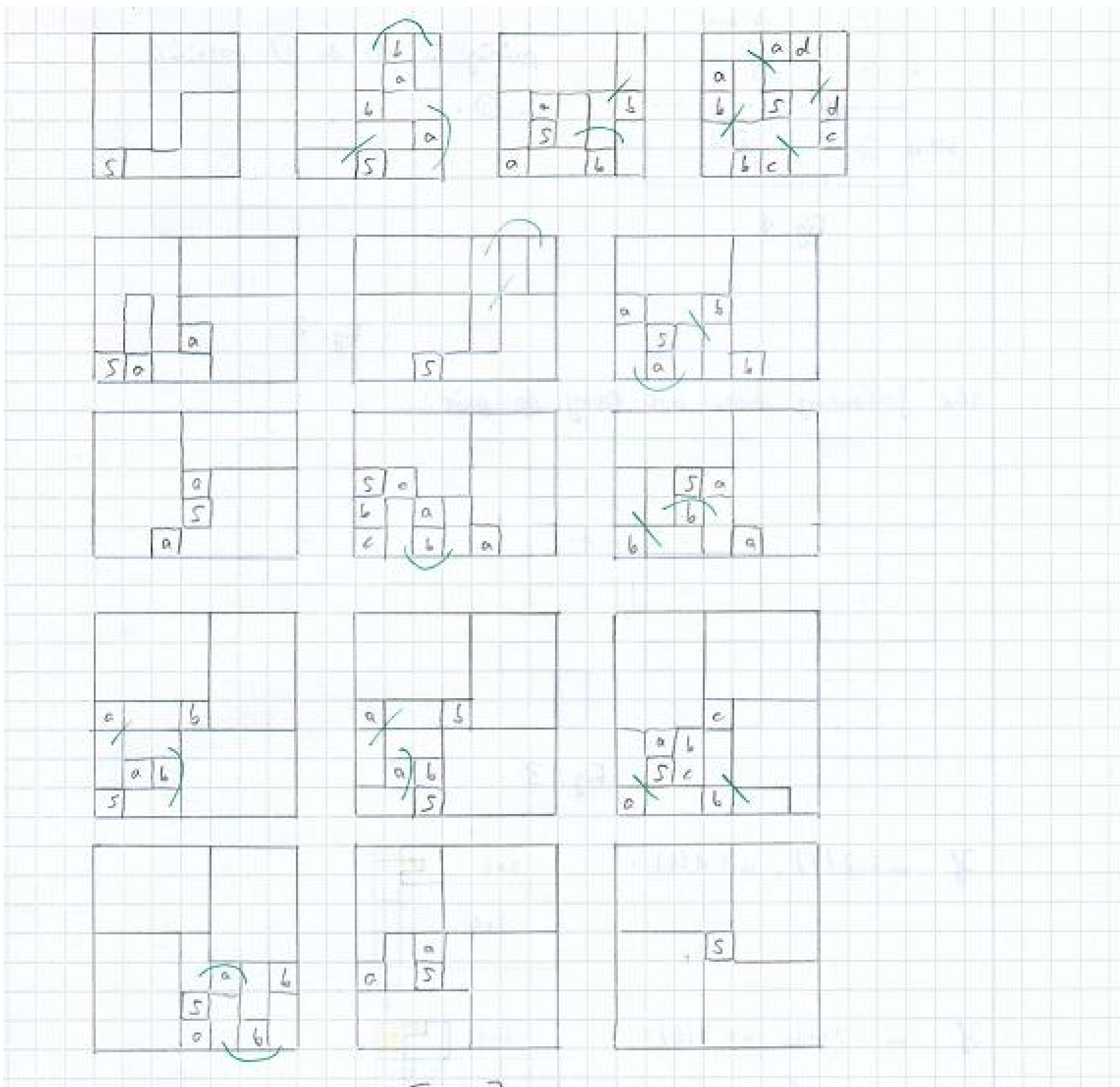
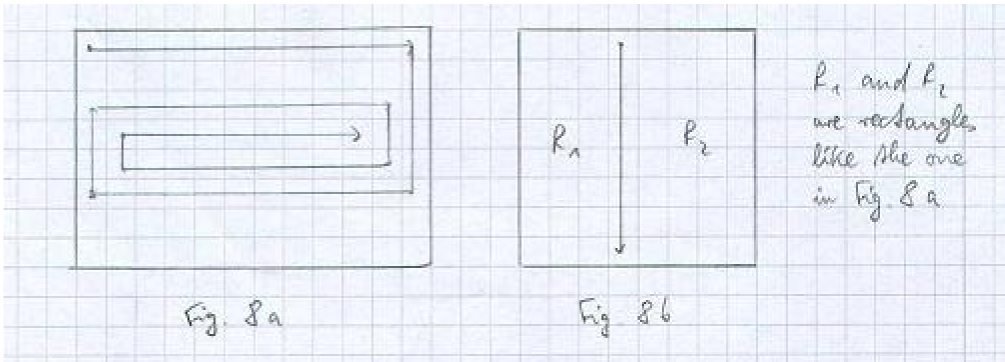
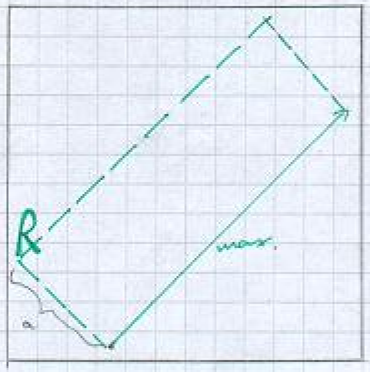


Fig. 7


R1
R2
R1 and R2 are rectangles like the one in Fig. 8a
Fig. 8a
Fig. 8b



$a \leq max$

Fig. 9a

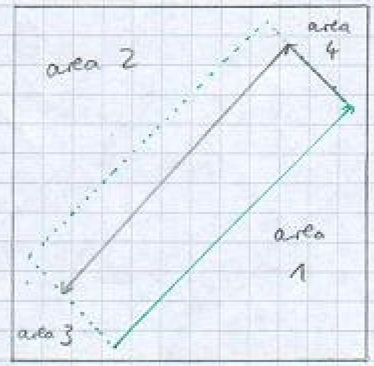
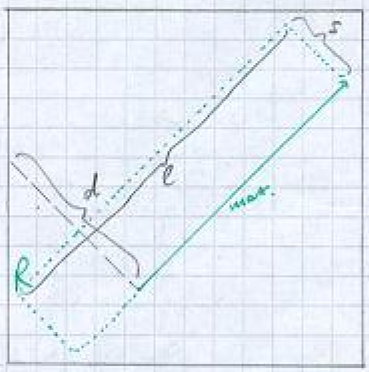


Fig. 9b



$l \geq max \geq d \geq s$

Fig. 10a

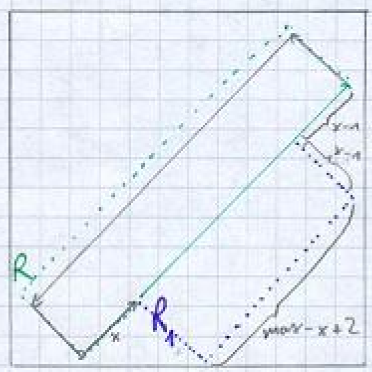


Fig. 10b