# Problem 3: A Cyclic Inequality

Team Germany

**Abstract**

We have worked on the problems 3.1 and 3.2.

To prove the inequality in Problem 3.1, we have noticed that by an easy substitution, it looks like a weighted Jensen inequality for the function $f : t \mapsto \frac{1}{1+at^k}$. Unfortunately, this function is only convex if $0 \leq k \leq 1$, so in all other cases, one has to be more careful. In this case, we have proved the inequality by imitating one of the proofs of Jensen's inequality and showing that the graph of $f$ is above the tangent to the graph at $t = 1$. This suffices because the weighted sum of the values we want to plug into our "Jensen" inequality is 1.

As for Problem 3.2, we have given two simple examples where the inequality signs $<$ and $>$ hold. One inequality sign can trivially be achieved by setting $x_1 = 0$ and $x_2 = \ldots = x_n = 1$; for the other sign, one sets again $x_2 = \ldots = x_n$ and shows that if $x_1$ is chosen close enough to 1 (but not equal to 1, where there will be equality), then we will have $C_{k,a}(x) < A(x)$. We have done this by calculating first and second derivatives.

For the whole problem, we make the additional assumption that $k \geq 0$. This was not given explicitely in the statement of the problem, but it can easily be seen that it's necessary to have this condition as otherwise (for example) the statement of Problem 3.1 would not be true: Setting $a = 1$ and $n = 2$, we obtain the inequality

$$\frac{x_1^{k+1} + x_2^{k+1}}{x_1^k + x_2^k} \geq \frac{x_1 + x_2}{2} \Leftrightarrow x_1^{k+1} + x_2^{k+1} \geq x_1^k x_2 + x_1 x_2^k.$$

Now, if $k \leq 0$, then the sequences $(x_1, x_2)$ and $(x_2^k, x_1^k)$ are ordered in the same direction, so by the rearrangement inequality, the same inequality holds with $\leq$ instead of $\geq$, and we have strict inequality if $x_1 \neq x_2$. Thus, for the problem statement to be true, we have to have $k \geq 0$.

## Problem 3.1

Let $\mathbb{R}_0^+$ denote the nonnegative real numbers.
Set $f : \mathbb{R}_0^+ \to \mathbb{R}, t \mapsto \frac{1}{1+at^k}$. Then the problem statement can be formulated as

$$x_1 f\left(\frac{x_2}{x_1}\right) + x_2 f\left(\frac{x_3}{x_2}\right) + \cdots + x_n f\left(\frac{x_1}{x_n}\right) \geq (x_1 + x_2 + \cdots + x_n) f(1),$$

the right hand side of which can be written as

$$(x_1 + x_2 + \cdots + x_n) f(1) = (x_1 + x_2 + \cdots + x_n) f\left(\frac{x_1 \frac{x_2}{x_1} + x_2 \frac{x_3}{x_2} + \cdots + x_n \frac{x_1}{x_n}}{x_1 + x_2 + \cdots + x_n}\right)$$

which reminds one of the weighted Jensen inequality. However, we cannot directly apply Jensen here as $f$ is in general not convex. In fact, deriving twice (it is obvious that $f$ is twice continuously differentiable) we obtain

$$\begin{aligned}
f'(x) &= -\frac{akt^{k-1}}{(1+at^k)^2} < 0 \\
f''(x) &= -ak\frac{(k-1)t^{k-2}(1+at^k)^2 - t^{k-1} \cdot 2akt^{k-1}(1+at^k)}{(1+at^k)^4} \\
&= ak\frac{2akt^{2k-2} - (k-1)t^{k-2}(1+at^k)}{(1+at^k)^3} \\
&= \frac{akt^{k-2}}{(1+at^k)^3} \cdot (a(k+1)t^k - (k-1)).
\end{aligned}$$

As we only consider positive $t$ and we have assumed that $k \geq 0$, the sign of this expression is determined by the second factor, which is an increasing and unbounded function of $t$. If $k \leq 1$, then both factors are surely nonnegative on $\mathbb{R}_0^+$ and $f$ is convex on $\mathbb{R}_0^+$ so that we can apply Jensen's inequality and are finished. So let us assume that $k > 1$ in the sequel. Let $x_0$ be the unique positive zero of $z(t) = a(k+1)t^k - (k-1)$. Then $f$ is convex for $x \geq x_0$ and concave for $x \leq x_0$.
Consider the function $g : \mathbb{R}_0^+ + \to \mathbb{R}, t \mapsto f(1) + (t-1)f'(1)$ which is the tangent to $f$ in $t = 1$. As $g$ is linear, it satisfies

$$\omega_1 g(t_1) + \omega_2 g(t_2) + \cdots + \omega_n g(t_n) = (\omega_1 + \omega_2 + \cdots + \omega_n) g\left(\frac{\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right).$$

Setting $\omega_i = x_i, t_i = \frac{x_{i+1}}{x_i}$ with $x_{n+1} = x_1$ and noting that $\frac{x_1 \frac{x_2}{x_1} + x_2 \frac{x_3}{x_2} + \cdots + x_n \frac{x_1}{x_n}}{x_1 + x_2 + \cdots + x_n} = 1$, we obtain

$$\begin{aligned} x_1 g\left(\frac{x_2}{x_1}\right) + x_2 g\left(\frac{x_3}{x_2}\right) + \cdots + x_n g\left(\frac{x_1}{x_n}\right) &\geq (x_1 + x_2 + \cdots + x_n) g(1) \\ &= (x_1 + x_2 + \cdots + x_n) f(1). \end{aligned}$$

If we can prove that $f(t) \geq g(t) \forall t \in \mathbb{R}_0^+$ under the assumption $a \geq k-1$, part 1 of the problem is solved. To prove this, note that since $a \geq k-1$, one must have $z(1) = a(k+1) - (k-1) \geq k(k-1) > 0$, where we used that $k > 1$, and therefore $x_0 < 1$. Now $f$ is convex on $[x_0, \infty)$, and as $g$ is the tangent to the point $t = 1$ of that interval we immediately obtain $t \geq x_0 \implies f(t) \geq g(t)$. On the other hand the function $[0, x_0] \to \mathbb{R}, t \to f(t) - g(t)$ is a concave function on a compact interval and thus obtains its minimum on the boundary of this interval. But we already showed that $f(x_0) \geq g(x_0)$, and in addition,

$$\begin{aligned} f(0) - g(0) &= 1 - f(1) + f'(1) \\ &= 1 - \frac{1}{1+a} - \frac{ak}{(1+a)^2} \\ &= \frac{(1+a)^2 - (1+a) - ak}{(1+a)^2} \\ &\geq \frac{a(1+a) - a(a+1)}{(1+a)^2} = 0 \\ &\Rightarrow f(t) \geq g(t) \forall t \in [0, x_0] \end{aligned}$$

This proves that $f(t) \geq g(t) \forall t \in \mathbb{R}_0^+$. As was said before, this implies part 1 of the problem.

## Problem 3.2

Suppose that $0 < a < \frac{k-1}{k+1}$ and $n > 1$.
Consider an $n$-tuple $x = (x_1, \dots, x_n)$ with $x_2 = x_3 = \dots = x_n = 1$. Then

$$C_{k,a}(x) = \frac{1+a}{n}\left(\frac{x_1^{k+1}}{x_1^k + a} + \frac{1}{1+ax_1^k} + \frac{n-2}{1+a}\right) =: L(x_1)$$

and $A(x) = \frac{x_1}{n} + \frac{n-1}{n} =: R(x_1)$.
We will now first show that there exists a nonnegative real number $x_1$ such that $L(x_1) > R(x_1)$, and then we will show the existence of a nonnegative real number $x_1$ with $L(x_1) < R(x_1)$. From now on, we write $x$ instead of $x_1$ for simplicity, as we won't need the $n$-tuple $x$ any more. Thus, we have to show that the function $D : \mathbb{R}_0^+ \to \mathbb{R}, x \mapsto L(x) - R(x)$, takes positive and negative values.
One of the two claims is trivial: We have

$$L(0) = \frac{1+a}{n}\left(0 + 1 + \frac{n-2}{1+a}\right) = \frac{1+a+n-2}{n} = \frac{n-1+a}{n} > \frac{n-1}{n} = R(0).$$

For the other claim, we note that $L$ and $R$ are twice continuously differentiable and calculate their first and second derivatives: We have $R'(x) = \frac{1}{n}$ and $R''(x) = 0$, and

$$\begin{aligned}
L'(x) &= \frac{1+a}{n}\left(\frac{(k+1)x^k(x^k+a) - x^{k+1}\cdot kx^{k-1}}{(x^k+a)^2} - \frac{akx^{k-1}}{(1+ax^k)^2}\right) \\
&= \frac{1+a}{n}\left(\frac{x^{2k} + a(k+1)x^k}{(x^k+a)^2} - \frac{akx^{k-1}}{(1+ax^k)^2}\right), \\
L''(x) &= \frac{1+a}{n}\left(\frac{(2kx^{2k-1} + ak(k+1)x^k)(x^k+a)^2 - 2(x^{2k} + a(k+1)x^k)kx^{k-1}(x^k+a)}{(x^k+a)^4}\right. \\
&\qquad \left. - \frac{ak(k-1)x^{k-2}(1+ax^k)^2 - 2a^2k^2x^{2k-2}(1+ax^k)}{(1+ax^k)^4}\right).
\end{aligned}$$

Now, we have $L(1) = 1 = R(1)$,

$$L'(1) = \frac{1+a}{n}\frac{1+a(k+1)-ak}{(1+a)^2} = \frac{1}{n} = R'(1)$$

and

$$\begin{aligned}
L''(1) &= \frac{1}{n(1+a)^2}\big((2k+ak(k+1))(1+a)-2k(1+a(k+1))-ak(k-1)(1+a)+2a^2k^2\big)\\
&= \frac{1}{n(1+a)^2}\big((2k+ak(k+1)-ak(k-1))(1+a)-2k(1+a(k+1)-a^2k)\big)\\
&= \frac{2k}{n(1+a)^2}\big((1+a)^2-1-ak-a+a^2k\big) = \frac{2ak}{n(1+a)^2}\,(1+a-k+ak)\\
&= \frac{2ak}{n(1+a)^2}\,(a(k+1)-(k-1))\,.
\end{aligned}$$

Since $0 < a < \frac{k-1}{k+1}$, we have $L''(1) < 0$. Thus, we have $D(1) = D'(1) = 0$ and $D''(1) < 0$, so $D$ has a local maximum at $x = 1$. Thus, if we choose $x$ close enough to 1, then we will have $D(x) < 0$.
Thus, we have proved that $D$ takes positive and negative values, so we can have strict inequalities in both directions between $C_{k,a}(x)$ and $A(x)$.