Problem 1: Blocking Sets

Team Germany

Abstract

We worked on the problems 1.1, 1.4 and 1.5. In Problem 1.1, we show that a blocking set has to have at least 8 elements by a simple elementary argument: We consider a blocking set with the minimal number of elements and show that from the point $(0,0)$, we can reach at least three pairwise different points, and there are three pairwise different points from which we can reach $(1,1)$. As there can't be any point which can be reached from $(0,0)$ and from which we can reach $(1,1)$ at the same time, we get that our minimal blocking set has at least 8 elements. By giving an example for an 8 element blocking set, we showed that this is indeed the minimal number of elements a blocking set can have.

In Problem 1.4, we showed that it's trivial to see that every set has to be blocking if condition *ii)* is changed as proposed.

Generalizing Problem 1.1 to higher dimensions, one could perhaps expect that the cardinality of a minimal blocking set gets bigger in higher dimensions. But thinking only a little bit, one sees that you can just put a copy of any two-dimensional blocking set into a three-dimensional cube in a suitable way (formally, you can do this for example by copying the second coordinate), yielding a three-dimensional set with the same number of elements. The same works for any higher dimension. Thus, the minimal number of elements of a blocking set always decreases if the dimension gets higher. We have shown that it is 8 for $N \in \{2,3\}$ and 6 for $N \geq 4$, by giving an example of a four-dimensional blocking set with 6 elements. (The proof that a blocking set has to have at least 8 elements easily generalizes from $N = 2$ to $N = 3$, and the proof that a blocking set in any dimension has at least 6 elements is a modification of the very easy beginning of this proof.)

Problem 1.1

Answer: The minimal number of elements of a blocking set is 8.

Proof. Why is there no blocking set with less than 8 *elements?* Consider a blocking set S with the minimal number of elements. We show that from both $(0,0)$ and $(1,1)$ we can reach at least three points (going forward from $(0,0)$ and backward from $(1,1)$), thus there are at least 8 points in total because the points we can reach from $(0,0)$ are different from those we can reach from $(1,1)$.
We will only show our claim for $(0,0)$; it follows for $(1,1)$ if we reflect all points at $(1/2,1/2)$.
Let (a,b) be the point in S with the smallest (positive) y coordinate. Then there is an arrow pointing from $(0,0)$ to (a,b) (one easily verifies conditions *i)* and *ii)*). Similarly, there is an arrow from $(0,0)$ to the point (c,d) having the smallest positive x coordinate. Can we have $(a,b)=(c,d)$? No! In this case, we could just throw away the point $(0,0)$ and cut off the stripes $[0,1]\times[0,b)$ and $[0,a)\times[0,1]$ where no other points of the blocking set could lie, and (after rescaling) we would get a blocking set with one element less, contradicting our assumption that S has the minimal number of elements.
Thus, $(a,b)\neq(c,d)$. Moreover, there can be no arrow pointing from (a,b) to (c,d) because $a\geq c$ by definition of (c,d). Now, let (e,f) be the point in S with minimal x coordinate which is greater than a. Then there is an arrow from (a,b) to (e,f): Condition *i)* is fulfilled by definition of (a,b) and (e,f), and there is no point having an x coordinate between a and e, by the choice of (e,f).
Thus, there are at least the three points (a,b), (c,d) and (e,f) which can be reached from $(0,0)$.

Why is there a blocking set with 8 *elements?* An example for such a set is $S=\{(0,0),(1/9,5/9),(1/3,1/3),(5/9,1/9),(4/9,8/9),(2/3,2/3),(8/9,4/9),(1,1)\}$. Figure 1 shows the set S together with the arrows that have to be drawn. It is easy to check for every single point that all arrows are already drawn, and it is also easy to see that there is no path from $(0,0)$ to $(1,1)$.

□

Problem 1.4

Answer: If the problem statement is changed as proposed, then no accepted set is blocking.

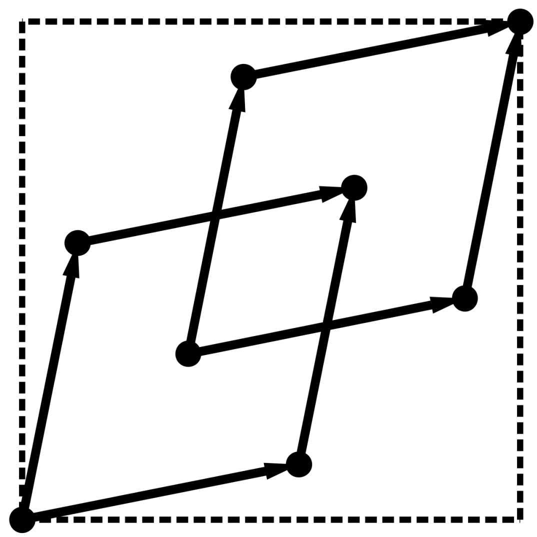

Figure 1: A blocking set with 8 elements.

Proof. We show that there is at least one arrow starting from each vertex (except $(1,1)$). Let $(a,b) \neq (1,1)$ be any vertex. Let M be the set of all points $(x,y) \in S$ satisfying $a < x$ and $b < y$. Note that $M \neq \emptyset$ as $(1,1) \in M$. Thus, M contains a point (c,d) with minimal x coordinate. Now, there must be an arrow pointing from (a,b) to (c,d): Condition *i)* is surely satisfied because $(c,d) \in M$; as for *ii)*, suppose that there is a point $(x,y) \in S$ such that $a < x < c$ and $b < y$. Then $(a,b) \in M$ as $a < x$ and $b < y$ and thus, by the minimality of (c,d), we have $c \leq x$, contradiction!
Thus, every point is connected to at least one other point with an arrow and because the x coordinate of the point at the tip of the arrow is greater than the x coordinate of the original point and S is finite, any path of arrows has to reach $(1,1)$ after a finite number of steps. □

This argumentation can easily be modified such that it holds for any higher dimension.

Problem 1.5

For any integer $N \geq 2$, we consider the following problem: Let S be a finite set of points from $[0,1]^N$ with $(0,0,\dots 0) \in S$ and $(1,1,\dots,1) \in S$ such that any two points in S differ in all of their coordinates. We draw an arrow from a point $(x_1,\dots,x_N) \in S$ to a point $(y_1,\dots,y_N) \in S$ if and only if the following two conditions are satisfied:

i) For all $i \in \{1,\dots,N\}$ we have $x_i < y_i$.

ii) There exists an $i \in \{1,\dots,N\}$ such that there is no point $(z_1,\dots,z_N) \in S$ with $x_i < z_i < y_i$.

Moreover, as in the original problem, a diagonal is a path following the arrows and connecting $(0,0,\dots,0)$ to $(1,1,\dots,1)$, and the set S is called blocking if it has no diagonal.
We now prove the following theorem generalizing Problem 1.1:

Theorem 1. The minimal cardinality of a blocking set is 8 if $N \in \{2,3\}$ and 6 if $N \geq 4$.

Proof. The proof is a slight variation of the original proof:

Why is there no blocking set with less than 6 *elements (less than* 8 *elements if $N \leq 3$)?* Consider a minimal blocking set. We will show (similar to the above proof for $N = 2$) that at least two points are connected to $(0,0,\dots,0)$. Let $x = (x_1,\dots,x_N)$ be the point with the minimal first coordinate. If x has minimal i-th coordinate for all $i \in \{1,\dots,N\}$, then we can throw away $(0,0,\dots,0)$ and the parts of the hypercube which are smaller than x in at least one coordinate, and we will get a smaller blocking set (again, after rescaling). Thus, without loss of generality, there is a point $y \neq x$ with minimal second coordinate. It is obvious that $(0,0,\dots,0)$ has to be linked to both x and y, and we are done.
Now, suppose that $N = 3$ (we have already proved the theorem for $N = 2$ in the solution of Problem 1.1). We show that $(0,0,0)$ is connected to at least three points by paths of arrows. Let z be the point with minimal third coordinate. If $z \neq x, y$, then we are done, as there has to be an arrow from $(0,0,0)$ to z. Thus, we can assume that $z = y = (y_1,y_2,y_3)$ has minimal second and third coordinate. Let w be the point with minimal first coordinate which is greater than y_1 (note that there is at least one point with the first coordinate greater than y_1: $(1,1,1)$). As the first coordinate of x is not greater than y_1, we must have $w \neq x$. Furthermore, y has to be connected to w: Condition *i)* is trivially satisfied (for the first coordinate by

the choice of w; for the second and third coordinate, by the minimality of $y = z$) and condition *ii)* is satisfied if we consider the first coordinate, again by choice of w. Thus, $(0, 0, 0)$ has to be connected to the three pairwise different points x, y and w.

Why is there a blocking set with 6 *elements (*8 *elements for* $N \leq 3$*)?* For $N = 2$, we have already shown the existence of such a set with 8 elements. For $N = 3$, just take the two-dimensional set and 'copy' the third coordinate (i.e., consider the set of all points (x, y, y), where (x, y) is an element of the two-dimensional blocking set with 8 elements). It's easy to see that no arrow will be added (or deleted) by this operation, so we get a three-dimensional blocking set with 8 elements. (One can imagine this as follows: We take the two-dimensional blocking set which is a subset of the square $[0, 1]^2$, and put the square into the plane given by the equation $y = z$, so that the two corners $(0, 0)$ and $(1, 1)$ come to lie on the corners $(0, 0, 0)$ and $(1, 1, 1)$ of the cube.)
For $N = 4$, choose a small positive real number ε (so small that $2\varepsilon < 1 - 2\varepsilon$; thus, for example, $\varepsilon = \frac{1}{5}$ will do), and consider the set

$$S = \{(0, 0, 0, 0), (\varepsilon, \varepsilon, 1 - 2\varepsilon, 1 - 2\varepsilon), (1 - 2\varepsilon, 1 - 2\varepsilon, \varepsilon, \varepsilon), \\ (1 - \varepsilon, 2\varepsilon, 1 - \varepsilon, 2\varepsilon), (2\varepsilon, 1 - \varepsilon, 2\varepsilon, 1 - \varepsilon), (1, 1, 1, 1)\}.$$

This set is blocking: There are only arrows from $(0, 0, 0, 0)$ to $(\varepsilon, \varepsilon, 1 - 2\varepsilon, 1 - 2\varepsilon)$ and $(1 - 2\varepsilon, 1 - 2\varepsilon, \varepsilon, \varepsilon)$ as well as from $(1 - \varepsilon, 2\varepsilon, 1 - \varepsilon, 2\varepsilon)$ and $(2\varepsilon, 1 - \varepsilon, 2\varepsilon, 1 - \varepsilon)$ to $(1, 1, 1, 1)$. As by these arrows, one cannot get from $(0, 0, 0, 0)$ to $(1, 1, 1, 1)$, it suffices to show that there are no more arrows (we don't have to show that these arrows are indeed drawn in, although this is quite obvious). It is easy to check that there are no more arrows starting in $(0, 0, 0, 0)$ or going to $(1, 1, 1, 1)$ because these are linked exactly to the points with some minimal (respectively, maximal) coordinate. We will prove that there is no other arrow starting or ending in $X := (\varepsilon, \varepsilon, 1 - 2\varepsilon, 1 - 2\varepsilon)$. Then the same statement will follow for the other elements of S (except $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$, where we already gave a different argument) by swapping coordinates and applying the symmetry of the 4-dimensional hypercube. As X has minimal first coordinate, there can be no arrow ending in X except the one coming from $(0, 0, 0, 0)$. There can be no arrow from X to $(0, 0, 0, 0), (1-2\varepsilon, 1-2\varepsilon, \varepsilon, \varepsilon), (1-\varepsilon, 2\varepsilon, 1-\varepsilon, 2\varepsilon)$ or $(2\varepsilon, 1 - \varepsilon, 2\varepsilon, 1 - \varepsilon)$ because the first condition is never fulfilled (for the first, third, fourth and third coordinate, respectively), and there can be no arrow from X to $(1, 1, 1, 1)$ because condition *ii)* is not satisfied: The points between those two are $(1 - \varepsilon, 2\varepsilon, 1 - \varepsilon, 2\varepsilon)$ for the first, second and third

coordinate, and $(2\varepsilon, 1-\varepsilon, 2\varepsilon, 1-\varepsilon)$ for the fourth coordinate. Thus, the set S is blocking.
For $N \geq 5$, just take the four-dimensional blocking set we just described and 'copy' the fourth coordinate $N-4$ times. As above, no arrows can be added (or deleted), so we get an N-dimensional blocking set with 6 elements.

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