Problem 9 : A Topological Problem

Team : France 3

Abstract

We have solved the first question by finding general properties of the Cl() and int() functions. These properties allowed us to prove that the number of distinct sets that one can obtain from these properties, starting from one set, is smaller than 6 (without counting the starting set). We then give a example of a set from which we can build 6 different sets. We finally conclude that the answer to the second question is 6.

In question 2a, we demonstrate the following inclusions:
Int (conv) C Int (conv (cl)) C Int (Cl (conv))
and : Int (Cl (conv)) C Int (conv)
We get Int (conv) = Int (Cl (conv)) = Int (conv (cl)) which answers to question 2a.

In question 2b, we only addressed the case where int (conv (A)) is not empty. We use the same argument as in question 1, only using additional properties of function conv and the property demonstrated in question 2a. We thus find an upper boundary of 16 for the number of distinct sets that one can build by using Cl, Int, and Conv. We also find an example to confirm that the answer to questions 2b is 16.

Question 1. Case $n = 1$. What is the maximal number of distinct sets that can be obtained from a set $A \subset \mathbb{R}$ using the operations **int** and **cl**

Answer. In this section, all sets are subsets in $\mathbb{R}$. We consider a subset A of $\mathbb{R}$. We know that

$$\begin{aligned} \mathbf{int}(A) &= \mathbf{int}(\mathbf{int}(A)) \\ \mathbf{cl}(A) &= \mathbf{cl}(\mathbf{cl}(A)) \end{aligned}$$

So, the really significant operations are represented by the sequences of operations such as

$$\begin{aligned} &\mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \cdots \\ &\mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \cdots \end{aligned}$$

where a empty sequence means identity operation. We have the following theorem:

Theorem. We have the two identities:

$$\mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} = \mathbf{int} \circ \mathbf{cl} \qquad (1)$$
$$\mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} = \mathbf{cl} \circ \mathbf{int} \qquad (2)$$

To give a proof, we need the lemma:

Lemma Let B and C two sets. If $B \subset C$, then $\mathbf{int}(B) \subset \mathbf{int}(C)$, and $\mathbf{cl}(B) \subset \mathbf{cl}(C)$.

Proof It is a direct consequence of the definitions. ∎

Proof of the theorem Now turn back to the proof of the theorem. By definition, we have

$$\mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A)$$

which implies

$$\mathbf{int} \circ \mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A)$$

i.e.

$$\mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A)$$

On the other hand, in a similar way, we have successively

$$\begin{aligned} & \mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{cl}(A) \\ \Rightarrow \quad & \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{cl} \circ \mathbf{cl}(A) = \mathbf{cl}(A) \\ \Rightarrow \quad & \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A) \subset \mathbf{int} \circ \mathbf{cl}(A) \end{aligned}$$

This proves the identity(1). As for the identity(2), the idea is the same. We write

$$\begin{aligned} & \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int}(A) \subset \mathbf{cl} \circ \mathbf{int}(A) \\ \Rightarrow \quad & \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int}(A) \subset \mathbf{cl} \circ \mathbf{int}(A) \end{aligned}$$

and

$$\begin{array}{rl} & \mathbf{int}(A) \subset \mathbf{cl} \circ \mathbf{int}(A) \\ \Rightarrow & \mathbf{int}(A) \subset \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int}(A) \\ \Rightarrow & \mathbf{cl} \circ \mathbf{int}(A) \subset \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int}(A) \end{array}$$

This proves the identity(2). The theorem is proved. ∎

As a consequence of the above analysis, the family of significant is reduced to the seven operations :

$$\begin{array}{l} \mathbf{id} \\ \mathbf{cl} \\ \mathbf{int} \\ \mathbf{cl} \circ \mathbf{int} \\ \mathbf{int} \circ \mathbf{cl} \\ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl} \\ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int} \end{array}$$

Let us prove that this family can no longer be reduced by a concret example. Let

$$A = [0, 1) \cup (1, 2] \cup ([2, 3] \cap \mathbb{Q}) \cup \{4\}.$$

Then we obtain

$$\begin{array}{lcl} \mathbf{id}(A) & = & [0, 1) \cup (1, 2] \cup ([2, 3] \cap \mathbb{Q}) \cup \{4\} \\ \mathbf{cl}(A) & = & [0, 3] \cup \{4\} \\ \mathbf{int}(A) & = & (0, 1) \cup (1, 2) \\ \mathbf{cl} \circ \mathbf{int}(A) & = & [0, 2] \\ \mathbf{int} \circ \mathbf{cl}(A) & = & (0, 3) \\ \mathbf{cl} \circ \mathbf{int} \circ \mathbf{cl}(A) & = & [0, 3] \\ \mathbf{int} \circ \mathbf{cl} \circ \mathbf{int}(A) & = & (0, 2) \end{array}$$

Question 2. Case $n = 2$.

a. Show that for any set $A \subset \mathbb{R}^2$ such that $\mathbf{int}(\mathbf{conv}(A)) \neq \emptyset$ we have

$$\mathbf{int}(\mathbf{conv}(A)) = \mathbf{int}(\mathbf{conv}(\mathbf{cl}(A)))$$

b.

In this section the sets are all subsets of $\mathbb{R}^2$. Let us first prove some lemmae.

Lemma For any set A, $\mathbf{conv}(A)$ is equal to the set of all barycenters (with positive coefficients) of finite families of points in A. (Note that a single point in A is considered as a barycenter of itself).

Proof. Let $\hat{A}$ the set of all barycenters (with positive coefficients) of finite families of points in A. It is clear that $\hat{A}$ contains A and is convex (by the associativity of barycenters), so that $\hat{A} \supset \mathbf{conv}(A)$. And inversely, if p is a barycenter of $(p_1, p_2, \ldots, p_k) \subset A$, we will have $(p_1, p_2, \ldots, p_k) \subset \mathbf{conv}(A)$. And since the latter is convex, $p \in \mathbf{conv}(A)$ (a fact that we can prove by an induction argument together with the associativity of the barycenters). The lemma is proved. ∎

Lemma For any A, $\mathbf{conv} \circ \mathbf{cl}(A) \subset \mathbf{cl} \circ \mathbf{conv}(A)$.

Proof. Consider a $p \in \mathbf{conv} \circ \mathbf{cl}(A)$. By the preceding lemma, there exist a finite family $(z_1, z_2, \ldots, z_k) \subset \mathbf{cl}(A)$ such that

$$p = \text{bar}\{(z_i, \alpha_i) : i = 1, 2, \ldots, k\}.$$

For each i, the point z_i being in $\mathbf{cl}(A)$, there exists a sequence $(z_{i,j})_{j \geq 1}$ which converges to z_i. It is straightforward to see that

$$p_j := \text{bar}\{(z_i, \alpha_i) : i = 1, 2, \ldots, k\},\ j \geq 1,$$

converges to p. We note that the points p_j belong to $\mathbf{conv}(A)$ which implies $p \in \mathbf{cl}(\mathbf{conv}(A))$. The lemma is proved. ∎

Lemma For any A

$$\mathbf{int} \circ \mathbf{conv}(A) \subset \mathbf{int} \circ \mathbf{conv} \circ \mathbf{cl}(A) \subset \mathbf{int} \circ \mathbf{cl} \circ \mathbf{conv}(A)$$

Proof. Notice that the operation $\mathbf{conv}$ is an increasing operation, i.e., if $B \subset C$, we have $\mathbf{conv}(B) \subset \mathbf{conv}(C)$. For the first inclusion, it is obtained as follows:

$$\begin{array}{ll} & A \subset \mathbf{cl}(A) \\ \Rightarrow & \mathbf{conv}(A) \subset \mathbf{conv} \circ \mathbf{cl}(A) \\ \Rightarrow & \mathbf{int} \circ \mathbf{conv}(A) \subset \mathbf{int} \circ \mathbf{conv} \circ \mathbf{cl}(A) \end{array}$$

As for the second inclusion, it is simply because $\mathbf{conv} \circ \mathbf{cl}(A) \subset \mathbf{cl} \circ \mathbf{conv}(A)$. The lemma is proved. ∎

Lemma For any A,

$$\mathbf{int} \circ \mathbf{cl} \circ \mathbf{conv}(A) \subset \mathbf{int} \circ \mathbf{conv}(A)$$

Proof. Let $p \in \mathbf{int} \circ \mathbf{cl} \circ \mathbf{conv}(A)$. There exists an open ball $B_r(p)$ centered at p contained in $\mathbf{cl} \circ \mathbf{conv}(A)$. We find then three no aligned points x, y, z in $B_r(p)$

such that p is an interior point of the open triangle formed by (x, y, z). Let $(x_j)_{j\geq 1}, (y_j)_{j\geq 1}, (z_j)_{j\geq 1}$ be three sequences converging respectively to x, y, z. It is clear that for j big enough, p becomes an interior point of the open triangle formed by (x_j, y_j, z_j). But the latter is an open subset of $\mathbf{conv}(A)$. We can conclude that $p \in \mathbf{int} \circ \mathbf{conv}(A)$. This being true for all $p \in \mathbf{int} \circ \mathbf{cl} \circ \mathbf{conv}(A)$, the lemma is proved. ∎

Proof of the theorem Now, combining the two last lemmae, we obtain immediately the equalities

$$\mathbf{int} \circ \mathbf{cl} \circ \mathbf{conv}(A) = \mathbf{int} \circ \mathbf{conv} \circ \mathbf{cl}(A) = \mathbf{int} \circ \mathbf{conv}(A)$$

The theorem is proved. ∎

Question 2.b.

Theorem 1. The closure of a convex is a convex.

Proof. Let A be a convex set and P and Q 2 points of its closure. Let R be a point of the line segment connecting P and Q.So R is a barycenter of P and Q (with nonnegative coefficients x and y). The closure of A is a closed so we have 2 sequences u and v that converge to P and Q. Let w be the sequence defined by w□ = bar {(u□,x),(v□,y)} .
w converges to R so R □ Cl(A). So the line segment connecting 2 points of Cl(a) is in Cl(A) so Cl(A) is convex.

Theorem 2. The interior of a convex is convex.

Proof. Let P and Q be 2 points of the interior of a convex A. P and Q are also in A. Let's show that the line segment connecting P and Q is in an open included in A. We know that there exists two open balls B□(p) and B□(q) included in A and centered in P and Q. We can choose one point R in B□(p) and S in B□(q) such as RPSQ is a « quadrolatère non croisé ». The line segment [P;Q] is one of its diagonals so [P;Q] is in the interior of RPSQ which is in A so the line segment connecting P and Q is in the interior of A. So the interior of A is convex.

Theorem 3. The convex hull of an open is an open.

Proof. Let A be an open set. A C conv(A). So A C Int(conv(A)). We know that int(conv(A)) is a convex (Theorem 2). So Int (conv (A)) is a convex containing A and included in conv (A). So Int (conv(A)) = conv (A). Int (conv(A)) is an open so conv (A) is an open.

Theorem 4. Let A be a set. Cl (int (conv (A))) = Cl (conv (A))

Proof. Cl (Int (Conv)) C Cl (Conv) is obvious.

For the reverse, we take a point P belonging to Cl (conv).
There is a sequence of points Zi belonging to conv which goes towards P.
If this sequence belongs to int conv then it is finished.
Otherwise, we take a point Q belonging to int conv, the full quadrilaters with summits Q, Zn-1, Zn, Zn+1 belong to conv.
There is thus one open set belonging to conv containing segment [Q; Zn [for any n.
Thus [Q, Zn [C int (conv) for any n.
Now the interval [Q, Zn [contains a sequence of points which goes towards Zn, thus Zn belongs to Cl (int (conv)).
Thus there is a sequence of points belonging to Cl(int (conv)) going towards P thus

P belongs to Cl (int (conv)) because Cl (int (conv)) is a closed set.

From theorems 1 and 2 we derive that conv (F (conv (A)) where F is a compound of Cl and Int can read as: f (conv (A)) because F (conv (A)) is convex thus equal to its convex envelope. We can thus simplify any composition involving several times the convex function.

We also know according to question 1 that any composition involving more than 3 times the functions cl and int following are reducible.

Finally, from the theorem 4 and the question 2b we deduce that if conv is not in last or second last position in the composition, the composition is reducible.

The maximal number of functions in a irreducible compound would thus be 5. We have 4 compositions of this type:

Cl ○ conv ○ cl ○ int ○ cl
Cl ○ conv ○ int ○ cl ○ int
Int○ conv ○ int ○ cl ○ int
Int ○ conv ○ cl ○ int ○ cl

All are reductible :

Cl ○ conv ○ int ○ cl ○ int = cl ○ int ○ conv ○ cl ○ int ○ cl ○ int
= cl ○ conv ○ int

Cl ○ conv ○ cl ○ int ○ cl = cl ○ int ○ conv ○ cl ○ int ○ cl
= cl ○ conv ○ cl

Int○ conv ○ int ○ cl ○ int = conv ○ int ○ cl ○ int

Int ○ conv ○ cl ○ int ○ cl = Int ○ conv ○ int ○ cl
= conv ○ int ○ cl

Thus let us pass in the compositions in 4 functions:

Cl ○ conv ○ int ○ cl
Cl ○ conv ○ cl ○ int
Int○ conv ○ int ○ cl
Int ○ conv ○ cl ○ int
conv ○ int ○ cl ○ int
conv ○ cl ○ int ○ cl

Some are reductible :

Cl ○ conv ○ cl ○ int = Cl ○ Int ○ conv ○ cl ○ int
= Cl ○ conv ○ int

Int○ conv ○ int ○ cl = conv ○ int ○ cl

Int ○ conv ○ cl ○ int = Int ○ conv ○ int

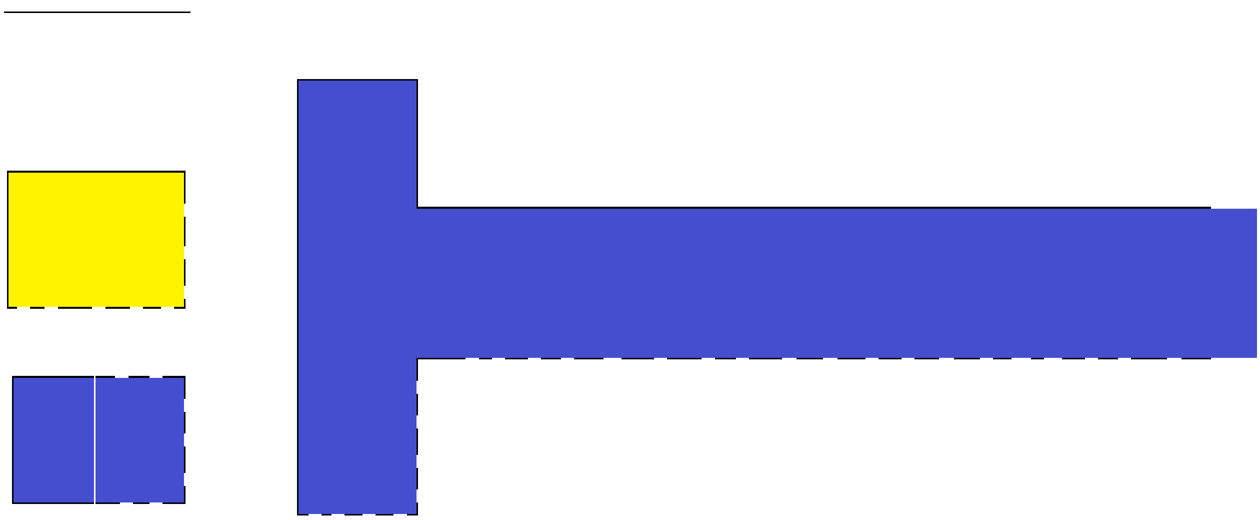
conv ○ int ○ cl ○ int = Int ○ conv ○ int ○ cl ○ int
= Int ○ conv ○ cl ○ int ○ cl ○ int
= Int ○ conv ○ cl ○ int
= Int ○ conv ○ int

Il reste donc : - Cl ○ conv ○ int ○ cl
- conv ○ cl ○ int ○ cl

Thus let us pass in the compositions in 3 functions:
–Cl ○ conv ○ int
–Cl ○ conv ○ cl = Cl ○ Int ○ conv ○ cl = Cl ○ conv donc éliminée
–Cl ○ int ○ cl
–Int ○ conv ○ int = conv ○ int donc éliminée
–Int ○ conv ○ cl = int ○ conv donc éliminée
–Int ○ cl ○ int
–conv ○ int ○ cl
–conv ○ cl ○ int
Thus let us pass in the compositions in 2 functions:
Cl ○ conv
conv ○ int
Cl ○ int
int ○ cl
Int ○ conv
conv ○ cl
Finally, basic fonctions :
int, cl et conv
16 compositions stay :
int, cl et conv
Cl ○ conv
conv ○ int
Cl ○ int
int ○ cl
Int ○ conv
conv ○ cl
Cl ○ conv ○ int
Cl ○ int ○ cl

Int ○ cl ○ int
conv ○ int ○ cl
conv ○ cl ○ int
Cl ○ conv ○ int ○ cl
conv ○ cl ○ int ○ cl

We raised the number of compositions possible for 16. Let us show that there is a set which can give 16 different sets with an example:
We choose the following set:



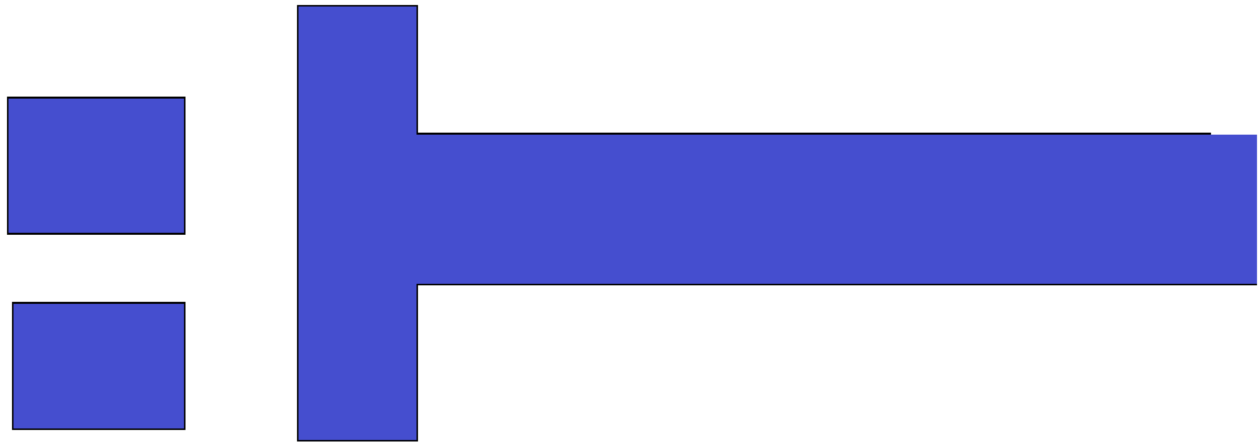
The blue indicates all the real numbers included in the coloured surface.

The yellow rectangle indicates rational understood(included) in this rectangle.

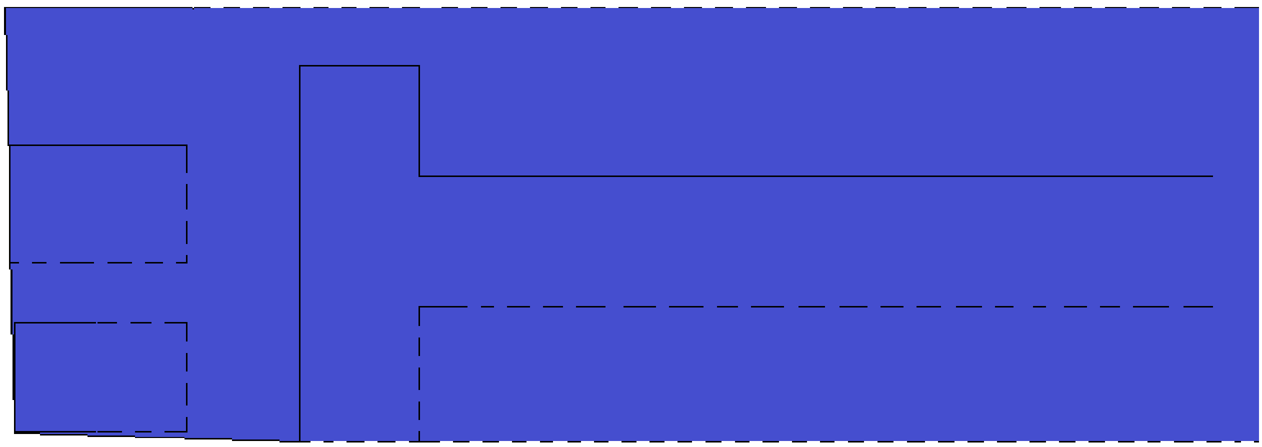
The branch which leaves towards the right-hand side goes to the infinity.

Int

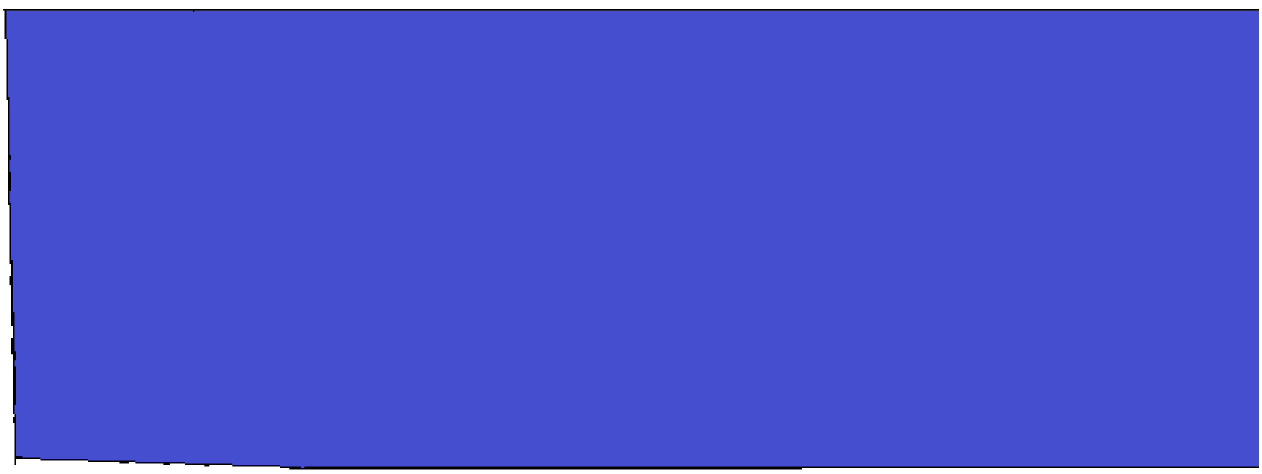
cl



Conv



Cl ○ conv

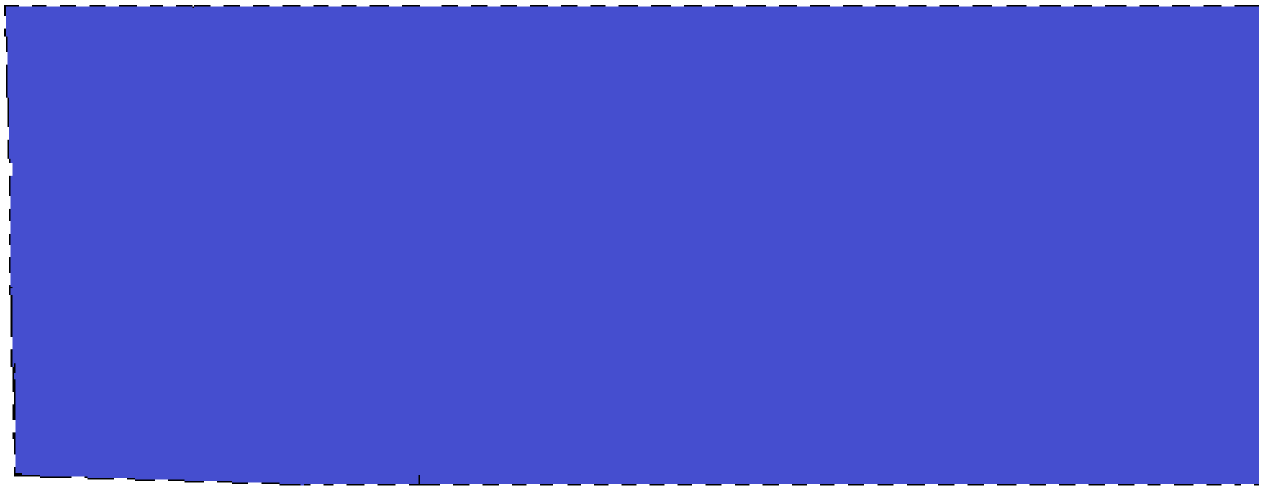


conv ○ int

Cl ○ int

int ○ cl

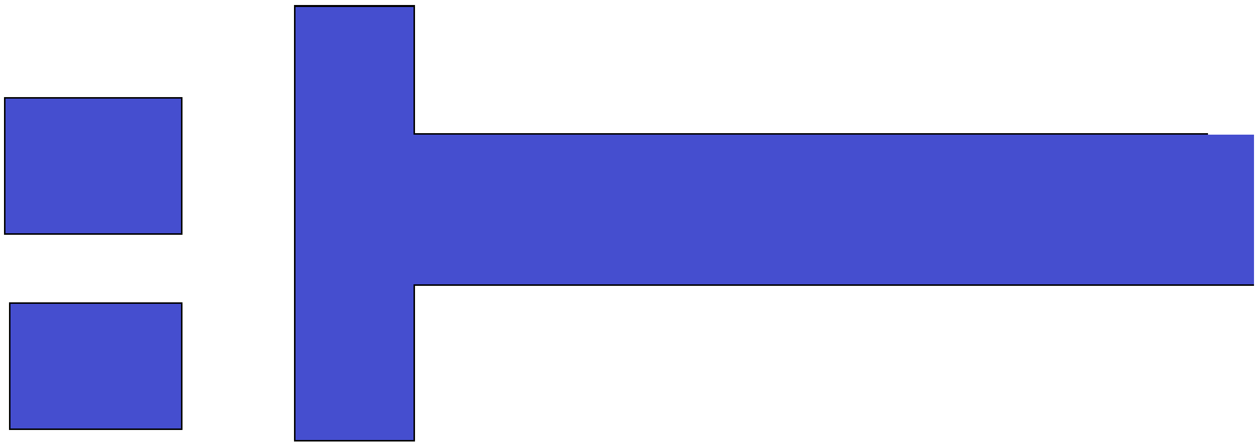
Int ○ conv



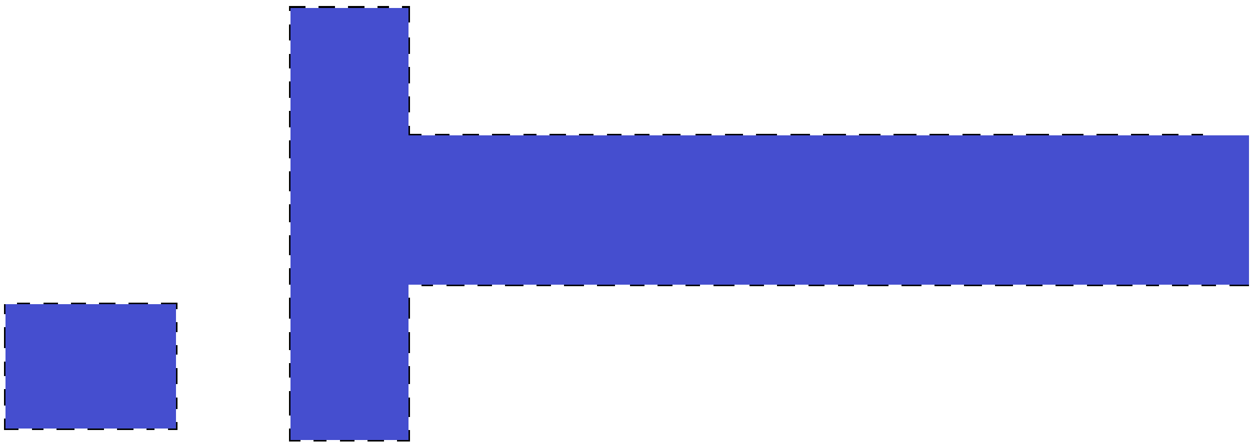
conv ○ cl

Cl ○ conv ○ int

Cl ○ int ○ cl



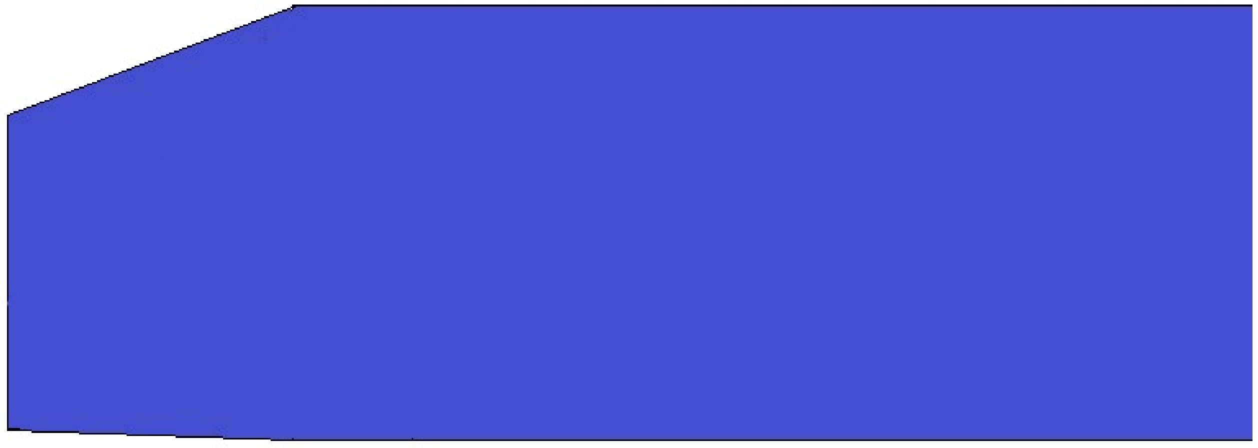
Int ○ cl ○ int



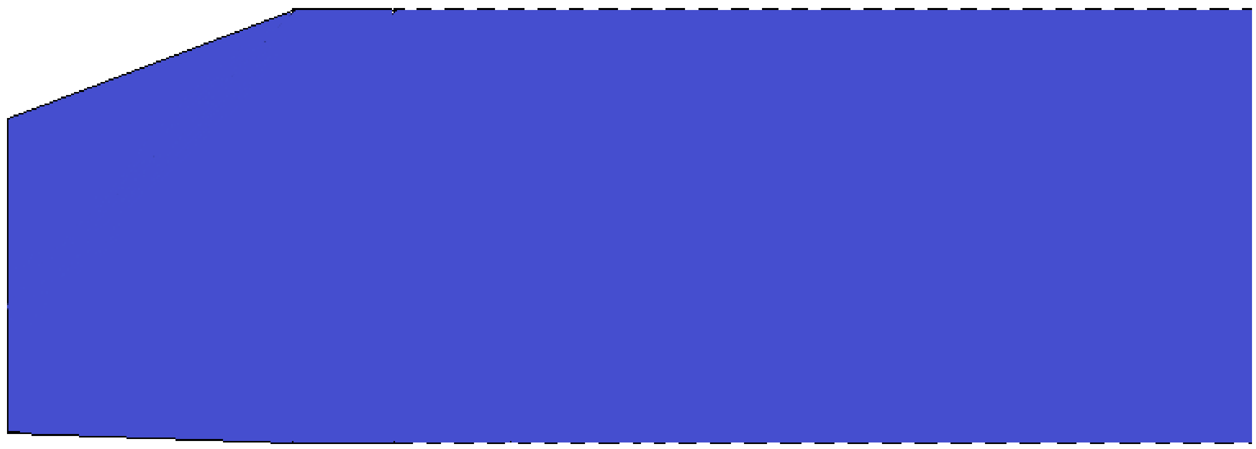
conv ○ int ○ cl

conv ○ cl ○ int

Cl ○ conv ○ int ○ cl



conv ○ cl ○ int ○ cl



So the answer is 16.