Problem 4 : A Baby Chess

Team : France 3

Abstract

We have developed a method to solve the problem for all chessmen with no trace. In this method, it is necessary to find a division of the chessboard for each chessman. We have described in detail the divisions specific to each chessman, and corresponding winning strategies.

For the second question, we could not find a winning strategy. We therefore created a computer program which explores all possible situations and find winning strategies, for a chessboard size up to 10x10 squares.

Baby Chess

Solution question 1 + the other chessmen without tracks

1 How to be sure to win

It is enough to have an answer to each of the moves of the opponent, during the whole duration of the game, whatever happens. One is then certain to play last. There is a simple way to make sure of it: associate all the squares by pairs in such a way that:

- Every square appears only in a single pair
- All squares belong to a pair
- The pairs correspond to 2 squares that the movement of the chessman considered can connect (we can here wonder about the odd chessboards where the association by pair is obviously impossible; this question will be solved later).

If we succeed in cutting the chessboard in this way, we obtain a winning strategy for the player who begins the game: **when the knight is situated on a square belonging to a certain pair, he must move to the other square which establishes this pair**.

Let us argue by induction to show it (we show that, whatever the square from where the winning player must start, its associated square is free, and that for all other squares, if a square is "blocked", its associated square is blocked as well). We of course suppose that the chessboard was cut in the previously described way.

Initialization: at the beginning of the game, the knight is put on a square at random. No other square is "surrounded", so the square associated to the starting square is free and the assertion « for all other squares, if a square is “blocked” its associated square is also blocked» is true.

Heredity: we suppose that all the conditions are respected for the N-th move of the player who follows the strategy. Let us show that they will be respected at the N+1-th move if we follow the winning strategy.

We are in the situation where:

- It is the "winning" player’s turn to play
- He is on a square for which the associated square is free
- For all other squares, if a square is "blocked" its associated square is also “blocked”

We follow the strategy: we play on the associated square, we are then in the situation where for any square, if a square is "blocked", its associated square is also blocked (it was true for all the squares except for the pair where the knight is, now it is true for the pair where the knight is, so it is true for all the squares).

It is now the opponent’s turn. He can go on any free square, for which the associated square is necessarily free, because if it was blocked the assertion « if a square « is blocked » its associated square is also blocked» would be false.

It is then "winning" player’s turn. We have just shown that the condition « the associated square is free » is respected. Since the last move, only the pair on which the knight is has been modified and we had: « if a square « is blocked » its associated square is also blocked». We can thus assert that:

- The knight is on a square for which the associated square is free

• For all other squares, if a square is "blocked" its associated square is also blocked.

The property is therefore true for the N+1-th move.

Conclusion: for all the times when it will belong to the "winning" player to move the knight, it will have a free square, thus he cannot lose. Now, the chessboard is not infinitely large, so there will necessarily be a loser: it will be the opponent.

Therefore we have modified the problem: **it is now necessary to find divisions for every chessman.**

The chessboards with an even number of squares.

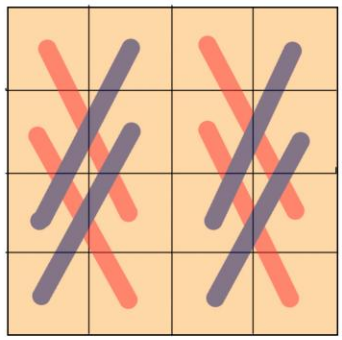
For the rook, the queen and the king, one of the chessboard dimensions is inevitably even, we can thus make the following division:

Without loss of generality, for a chessboard of size m x n, let us suppose that m is even (m being the horizontal dimension of the chessboard). All the lines then have an even number of squares. We can thus associate, according to the movement of three pieces considered, the first square of every line to the second, the third to the fourth, the M-1th to the M-th square. We obtain a division which uses all the squares.

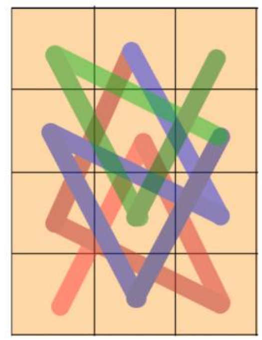
For the bishop, we play only on the squares of a single colour. Diagonals can have two different directions: from bottom to top and from left to right or from bottom to top and from right to left. We notice that all the diagonals following the same direction have the same parity. If the number of squares of a colour is even, then at least all the diagonals along one direction are even. We can thus apply a division similar to that of the rook, the king and the queen.
NB: for the bishop, we study here the chessboards for which the number of squares of a colour is even and not the total number of squares as for the other chessmen.

For the knight, we look for divisions on small rectangular subparts of the chessboard to then build divisions on bigger chessboards by induction. On the chessboards of size 4x4 to 7x8, a program allows to draw routes passing through all the squares, by associating the first square of the road to the second, then the third to the fourth… and finally the next to last to the last. We find the other divisions by induction, by separating a chessboard of size MxN in 4 smaller chessboards: 4x4, (M-4) x4, 4x (N-4) and (M-4) x (N-4). We have the initialization with the divisions given by the computer and the heredity with a strong induction (we have divisions on 4 small chessboards and because they cover the entire big chessboard, every square of the big chessboard is associated to the other one: we have a division).

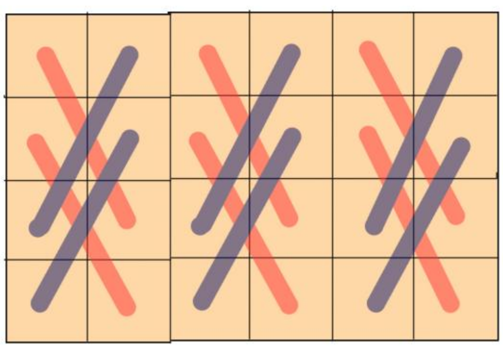
4*4



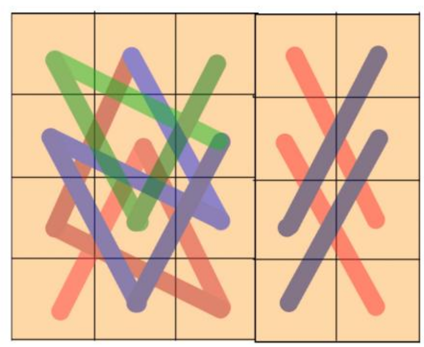
3*4



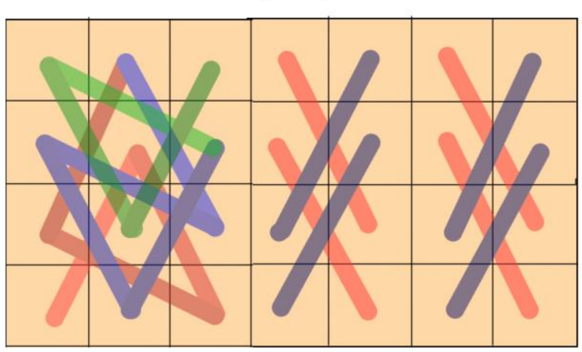
6*4



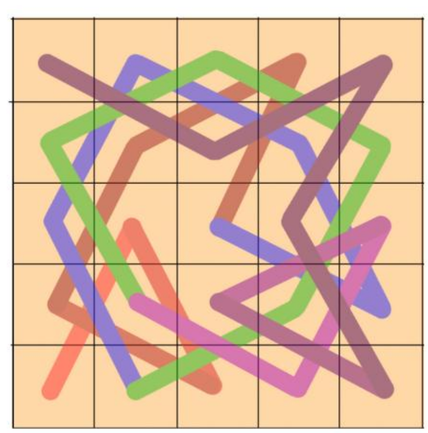
5*4

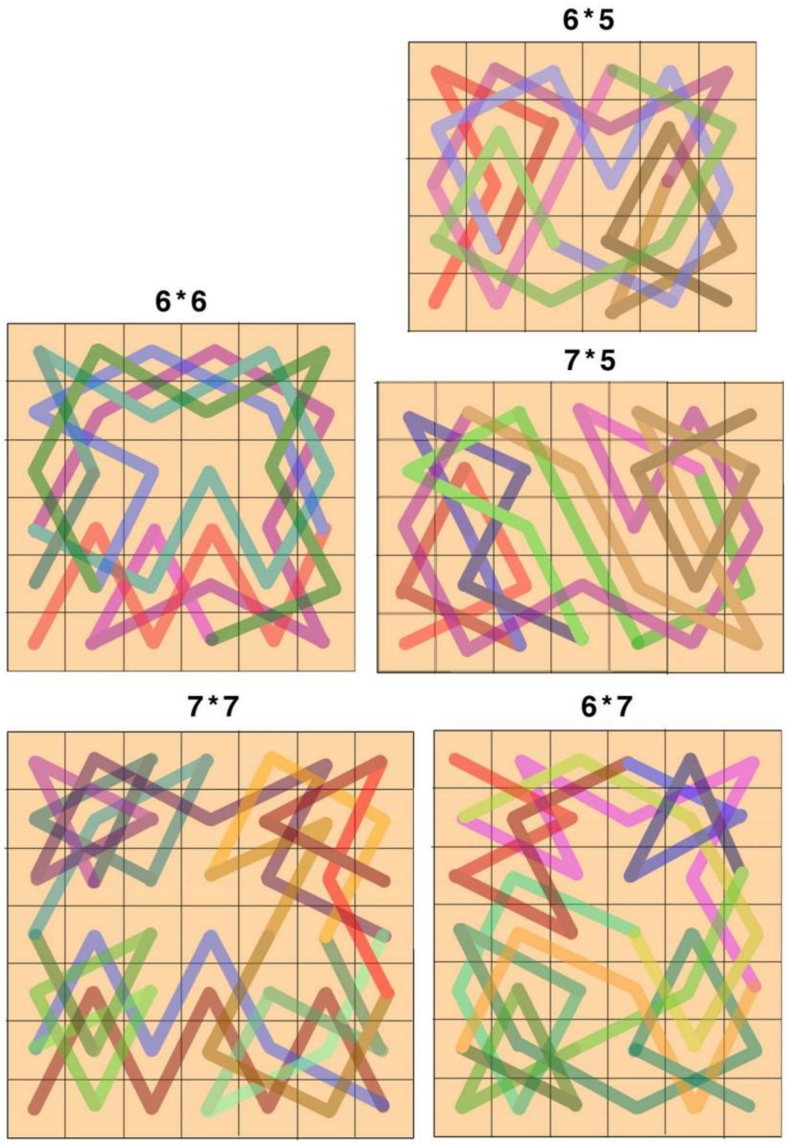


7*4



5*5





Last problem: the chessboards with an odd number of squares.

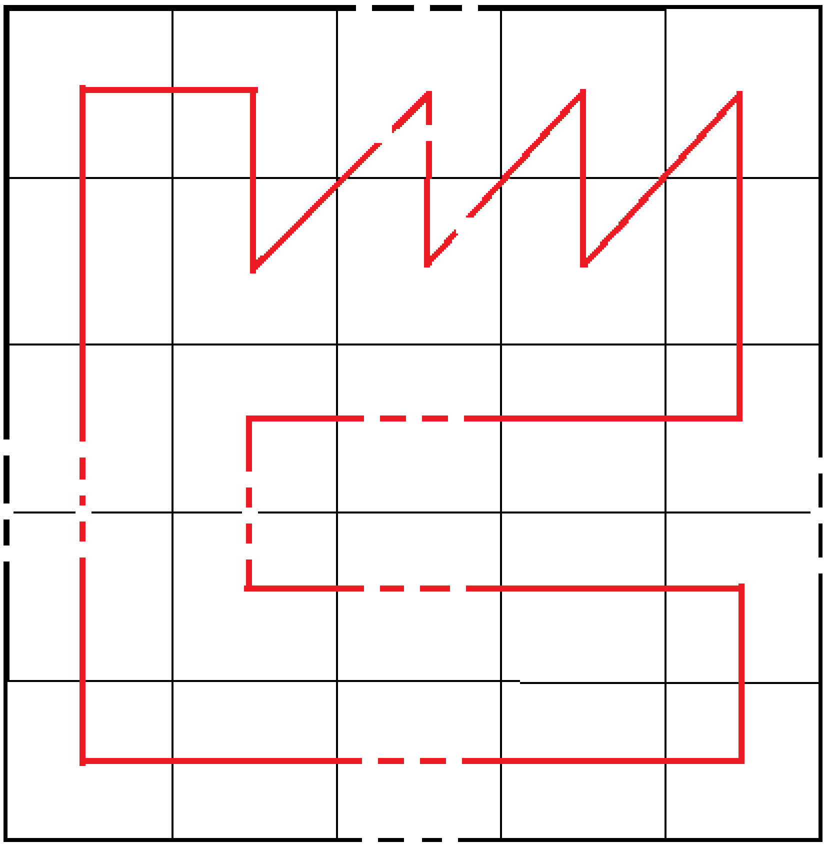
If we succeed in finding a division which excludes the square on which the knight starts, we obtain a winning strategy for the second player. Indeed, it is just as if a square of the chessboard was missing and that the second player had begun. The square on which ends the first move of the first player is considered as the starting square. We can also say that, because the first square is not part of a pair, the first player plays on the first square of a pair, the second completes this pair. The game then goes on as on an even chessboard, except from the fact that the second player follows the winning strategy.

Such a division must be found independent of the starting square.

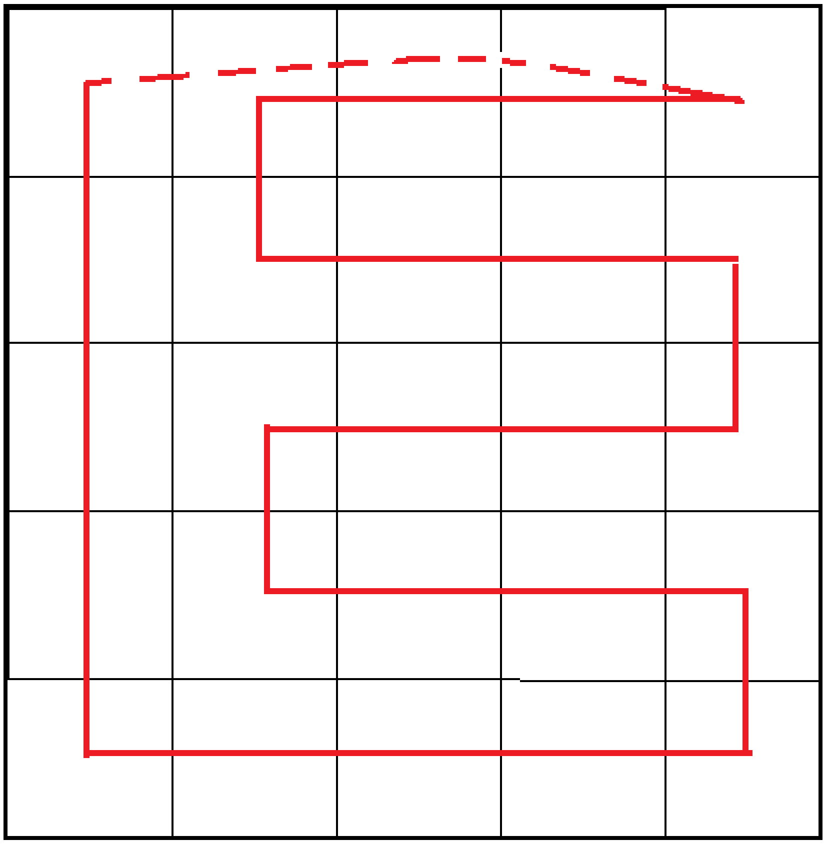
Rook, king and queen

We saw that a route which crosses all the squares allows making a division. We thus need routes passing through all the squares except one. Let us define as a cycle a route which passes through all the squares, and for which the last square is connected to the first one. Such a cycle allows finding roads which exclude any square. We start from the square following the excluded square and can reach the square before the excluded square. We thus passed through the rest of the chessboard.

Cycle for king and queen:



Cycle for the rook:



The Knight

We cannot find cycles for the knight because its movement makes it necessary to change square colour and thus after an even number of movements, it will end up on the colour of departure. A route which contains an odd number of squares will begin and finish with the same colour. Therefore the square of the beginning and the square of the end cannot connect.

Nevertheless we can find a division by removing some squares, but not all, by means of the route. We know that the road begins and ends by squares of same colour. Let us suppose that they are black. Let us remove a black square from the route. We obtain then two roads with a white end (near the removed square) and a black end (pre-existent ends) thus two even routes, thus a division that is a winning strategy for the second player.

If the knight is put on a white square this reasoning does not work. On the other hand, we notice that the player who begins will always play on the black squares and the other one on the white squares (demonstrable by induction but obvious during a game). The first player can thus decide to never play on one of the black squares, so none of 2 players can reach it; it is a "dead" square. He can see to it that this square is an extremity of the route; we obtain then an even route and thus a winning strategy for the player who begins.

We just needed one route to find a winning strategy:

- for the first player, when the knight is put on a square of colour opposite to that of the ends of

the route.
• for the 2nd player, when the knight is put on a square of the same colour as that of the ends of the route.

What about the induction?
We cannot find a route in MxN but we can guarantee a division in the same way as for the even chessboards. It is only necessary to make sure that we find a winning strategy for any the square of departure. By applying the same induction we obtain a demarcation of our chessboard in 3 even chessboards and an odd one. We can by symmetry, make sure that the odd chessboard covers any given square of the big chessboard, and thus find divisions for any the given square of departure, by including it in the odd chessboard, a division being always guaranteed in the even chessboards.

The Bishop

We do not try to find cycles for the bishop either; let us only assume that the bishop is on a square (no matter which one) and let us find a division.

The number of squares of the colour on which moves the bishop is odd, so the diagonals have an odd number of squares in both directions and the number of diagonals in both directions is also odd.

We consider diagonals in a single direction. One of these diagonals has an even number of squares because a square was removed from it. There is thus an even number of diagonals possessing an odd number of squares. We associate a square of each of these diagonal to the neighboring diagonal: the first one with the second, the 3rd with the 4th … We thus obtain only diagonals with an even number of squares which can be associated as in the previous paragraph.

Example of division for the bishop: