ITYM2010
Team : France3

Problem 3: A Cyclic Inequality

The problem is divided in four parts. We start with proving that $a \geq k-1$ is a sufficient condition to have the inequality $C_{k,a}(x) \geq A(x)$ for any positive n-tuple x. It will be proved by an induction reasoning. Then, we study what happens when $a \in (0, \frac{k-1}{k+1})$. We show that there exist n-tuples x and y such that $C_{k,a}(x) > A(x)$ and $C_{k,a}(y) < A(y)$. Next we turn back to the condition $a \geq k-1$ and show it is also a necessary condition for the cyclic inequality. For this we study the asymptotic behaviour of the difference $C_{k,a}(x) - A(x)$ when n tends to infinity. As for the last question, we note that $C_{k,l,a}(x) = C_{k,1,a}(y)$ if $y_i = x_i^l$. This enables us to prove that, if $a \geq \frac{k}{l} - 1$, the inequality $C_{k,l,a}(x) \geq A^l(x)$ holds, thanks to the first question and the Jensen's inequality.

Question 1. Show that, if $a \geq k-1$, then $C_{k,a}(x) \geq A(x)$ for any n-tuple $x = (x_i)_{1 \leq i \leq n}$ of positive numbers.

Proof. Let us name the proposition:

> The inequality $C_{k,a}(x) \geq A(x)$ holds for any $a \geq k-1$ and any positive n-tuple x.

by proposition $\mathcal{C}_n$. We use an inductive reasoning on n (where n is the size of the n-tuple) to prove the proposition $\mathcal{C}_n$.

- For $n = 1$ we have $C_{k,a}(x) = A(x)$ for any positive n-tuple x and $a > 0$. The proposition $\mathcal{C}_1$ is true.
- Suppose that, for a $n \geq 1$, the proposition $\mathcal{C}_n$ is true.
- Let us prove now that the proposition $\mathcal{C}_{n+1}$ is also true. Consider any positive $(n+1)$-tuples $x = \{x_1, \cdots, x_n, x_{n+1}\}$ and its reduction to n-tuple $z = \{x_1, \cdots, x_n\}$. We introduce

$$
\begin{aligned}
\mu_i = & \tfrac{(1+a)x_i^k}{x_i^k + a x_{i+1}^k} = \tfrac{(1+a)}{1+a(\frac{x_{i+1}}{x_i})^k}, \quad 1 \leq i \leq n-1 \\
\mu_n = & \tfrac{(1+a)x_n^k}{x_n^k + a x_1^k} = \tfrac{(1+a)}{1+a(\frac{x_1}{x_n})^k}, \\
\lambda_i = & \tfrac{(1+a)x_i^k}{x_i^k + a x_{i+1}^k} = \tfrac{1+a}{1+a(\frac{x_{i+1}}{x_i})^k}, \quad 1 \leq i \leq n+1, x_{n+2} = x_1
\end{aligned}
$$

By the induction hypothesis, $C_{k,a}(z) \geq A(z)$ if $a \geq k-1$. We write the equality

$$(n+1)C_{k,a}(x) - (n+1)A(x) = nC_{k,a}(z) - nA(z) + (\lambda_n - \mu_n)x_n + (\lambda_{n+1} - 1)x_{n+1}$$

So, the proposition $\mathcal{C}_{n+1}$ will be proved, if we prove that

$$(*) : \quad (\lambda_n - \mu_n)x_n + (\lambda_{n+1} - 1)x_{n+1} \geq 0$$

whenever $a \geq k-1$. Introducing

$$\tau_1 = \frac{x_1}{x_{n+1}} \quad \text{and} \quad \tau_2 = \frac{x_n}{x_{n+1}}$$

we can prove that the above inequality(∗) is equivalent to

$$a^2\big(\tau_1^k(1 - \tau_2^{k+1})\big) + a\big(\tau_2^k(1 - \tau_2)(1 + \tau_1^k)\big) - \tau_2^{k+1}(1 - \tau_2^{k-1}) \geq 0$$

Now let us call an $(n+1)$-tuple $y = (y_i)_{1 \leq i \leq n+1}$ a translation of $x = (x_i)_{1 \leq i \leq n+1}$, if $y_i = x_{i+1}$ for $1 \leq i \leq n$, and $y_{n+1} = x_1$. It is clear that the quantities $C_{k,a}(x)$ and $A(x)$ are invariant with respect to the translations.

We can therefore suppose, without loss of generality, that $x_{n+1} = \max_i x_i$. Under this condition, we have

$$M := \tau_1^k(1 - \tau_2^{k+1}) \geq 0$$
$$N := \tau_2^k(1 - \tau_2)(1 + \tau_1^k) \geq 0$$
$$P := \tau_2^{k+1}(1 - \tau_2^{k-1}) \geq 0$$

We consider the trinom $a^2M + aN - P$. The discriminant of this trinom is

$$\Delta = N^2 + 4MP \geq 0$$

We compute its roots

$$a_- = \frac{-N-\sqrt{\Delta}}{2M}$$
$$a_+ = \frac{-N+\sqrt{\Delta}}{2M}$$

We check that a_+ has another expression

$$a_+ = \frac{2P}{N + \sqrt{\Delta}}$$

We note that $a^2M+aN-P \geq 0$ whenever $a_+ \leq a$. Therefore, the proposition $\mathcal{C}_{n+1}$ will be proved, if $a_+ \leq k-1$. But then,

$$\begin{aligned} a_+ &= \frac{2P}{N+\sqrt{\Delta}} \\ &\leq \frac{2P}{2N} \\ &= \frac{\tau_2^{k+1}(1-\tau_2^{k-1})}{\tau_2^k(1-\tau_2)(1+\tau_1^k)} \\ &\leq \frac{\tau_2(1-\tau_2^{k-1})}{(1-\tau_2)} \end{aligned}$$

Consider the function $g(t) := \frac{t(1-t^{k-1})}{(1-t)}$ for $0 < t < 1$. The derivative of g is:

$$\begin{aligned} g'(t) &= \frac{(1-kt^{k-1})(1-t)+(t-t^k)}{(1-t)^2} \\ &= \frac{(1-t^k)-kt^{k-1}(1-t)}{(1-t)^2} \\ &= \frac{k\alpha^{k-1}(1-t)-kt^{k-1}(1-t)}{(1-t)^2} \end{aligned}$$

for some $\alpha \in (t, 1)$. This shows that $g'(t) \geq 0$ and therefore

$$g(t) \leq \lim_{s\uparrow 1} g(s) = k-1.$$

We prove thus $a_+ \leq k-1$, and consequently the proposition $\mathcal{C}_{n+1}$ is proved.

The Question 1. is proved.

Question 2. Prove that, if $n > 1$ and $0 < a < \frac{k-1}{k+1}$, there eixsts two n-tuples $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ such that

$$C_{k,a}(x) > A(x)$$
$$C_{k,a}(y) < A(y)$$

Proof. The idea here is to consider a particular kind of n-tuple $x^{[b,c]} = (x_i)_{1 \le i \le n}$ with $x_1 = x_2 = \ldots = x_{n-1} = b > 0$ and $x_n = c > 0$, and to make vary the value b with respect to a. For such an n-tuple,

$$\begin{aligned}
C_{k,a}(x^{[b,c]}) &= \tfrac{1+a}{n}\left[\sum_{i=1}^{n-2} \tfrac{x_i^{k+1}}{x_i^k + a x_{i+1}^k} + \tfrac{x_{n-1}^{k+1}}{x_{n-1}^k + a x_n^k} + \tfrac{x_n^{k+1}}{x_n^k + a x_1^k}\right] \\
&= \tfrac{1+a}{n}\left[(n-2)\tfrac{b}{1+a} + \tfrac{b^{k+1}}{b^k + a c^k} + \tfrac{c^{k+1}}{c^k + a b^k}\right] \\
&= \tfrac{1}{n}\left[(n-2)b + \tfrac{1+a}{1+a(\frac{c}{b})^k} b + \tfrac{1+a}{1+a(\frac{b}{c})^k} c\right] \\
A(x^{[b,c]}) &= \tfrac{1}{n}\left((n-1)b + c\right)
\end{aligned}$$

Set $r = \frac{b}{c}$ and suppose $r < 1$. We write

$$\begin{aligned}
C_{k,a}(x^{[b,c]}) - A(x^{[b,c]}) &= \tfrac{1}{n}\left[-b + \tfrac{1+a}{1+a(\frac{c}{b})^k} b + \tfrac{1+a}{1+a(\frac{b}{c})^k} c - c\right] \\
&= \tfrac{1}{n}\tfrac{1}{c}\left[-r + \tfrac{1+a}{1+a(\frac{1}{r})^k} r + \tfrac{1+a}{1+ar^k} - 1\right] \\
&= \tfrac{1}{n}\tfrac{1}{c} a\left[r\tfrac{1-\frac{1}{r^k}}{1+a\frac{1}{r^k}} + \tfrac{1-r^k}{1+ar^k}\right] \\
&= \tfrac{1}{n}\tfrac{1}{c} a\left[r\tfrac{r^k-1}{r^k+a} + \tfrac{1-r^k}{1+ar^k}\right] \\
&= \tfrac{1}{n}\tfrac{1}{c} a(1-r^k)\left[-\tfrac{r}{r^k+a} + \tfrac{1}{1+ar^k}\right] \\
&= \tfrac{1}{n}\tfrac{1}{c} a(1-r^k)\tfrac{(r^k+a) - r(1+ar^k)}{(r^k+a)(1+ar^k)} \\
&= \tfrac{1}{n}\tfrac{1}{c} a(1-r^k)\tfrac{a(1-r^{k+1}) - r(1-r^{k-1})}{(r^k+a)(1+ar^k)} \\
&= \tfrac{1}{n}\tfrac{1}{c} a\tfrac{(1-r^k)(1-r^{k+1})}{(r^k+a)(1+ar^k)}\left(a - \tfrac{r(1-r^{k-1})}{(1-r^{k+1})}\right)
\end{aligned}$$

Consider the function

$$f(r) := \frac{r(1-r^{k-1})}{(1-r^{k+1})},\ r \in (0,1)$$

We see that, for $0 < r < 1$, $C_{k,a}(x^{[r,1]}) - A(x^{[r,1]}) > 0$, if and only if $a > f(r)$. As a consequence, the Question 2. is solved whenever we show that neither of the two subsets $\{0 < r < 1; f(r) < a\}$ and $\{0 < r < 1; f(r) > a\}$ are empty.

The function f has two limits: (using Hospital rule)

$$f(0+) = 0, \quad f(1-) = \frac{k-1}{k+1}.$$

It is continuously differentiable on $(0,1)$. We have

$$f'(r) = \frac{1 - kr^{k-1} + kr^{k+1} - r^{2k}}{(1 - r^{k+1})^2}$$

Set $u(r) := 1 - kr^{k-1} + kr^{k+1} - r^{2k}$ for all $r \in \mathbb{R}$. We have

$$\begin{aligned} u'(r) &= -k(k-1)r^{k-2} + k(k+1)r^k - 2kr^{2k-1} \\ &= kr^{k-2}(-(k-1) + (k+1)r^2 - 2r^{k+1}) \end{aligned}$$

Set $v(r) := -(k-1) + (k+1)r^2 - 2r^{k+1}$. We have

$$v'(r) = 2(k+1)r - 2(k+1)r^k = 2(k+1)r(1 - r^{k-1}) > 0.$$

Therefore, for $0 < r < 1$, $v(r) < v(1) = 0$. It follows that $u'(r) < 0$ so that $u(r) > u(1) = 0, 0 < r < 1$. This implies that the derivative $f'(r)$ is strictly positve and the function f is strictly increasing. In other words, f is a one-to-one map from $(0,1)$ onto $(0, \frac{k-1}{k+1})$. Now, a being a point in $(0, \frac{k-1}{k+1})$, neither of the two subsets $\{0 < r < 1; f(r) < a\}$ and $\{0 < r < 1; f(r) > a\}$ can be empty. The Question 2. is solved. ∎

Question 3. In the Question 1., we have proved that the condition $a \geq k-1$ is a sufficient condition to have the cyclic inequality

$$C_{k,a}(x) \geq A(x)$$

where x runs over the set of all n-tuple of positive numbers. We will prove in this section that $a \geq k-1$ is also a necessary condition for the cyclic inequaltiy to hold.

Proof. The idea is to consider the sequence of differences on n-tuples $x = (x_i)_{1 \leq i \leq n}$:

$$D_n = \sum_{i=1}^{n} \frac{(1+a)x_i^{k+1}}{x_i^k + ax_{i+1}^k} - \sum_{i=1}^{n} x_i = a \sum_{i=1}^{n} \frac{x_i^k - x_{i+1}^k}{x_i^k + ax_{i+1}^k} x_i,$$

where $n \geq 2$ and $x_{n+1} = x_1$. This is because the assertion that $C_{k,a}(x) \geq A(x)$ for an n-tuple x is equivalent to say that $D_n \geq 0$. Concretely, we will consider the sequence (D_n) with the particular n-tuples: $x_i = \frac{i}{n}, 1 \leq i \leq n$, and we will show that the $\lim_{n\to\infty} D_n$ can be strictly negative, if $a < k-1$. This gives a proof of our assertion.

Let us compute first the $\lim_{n\to\infty} D_n$. Applying the mean value theorem, we find $\alpha_i \in [x_i, x_{i+1}]$ for $1 \le i \le n-1$ such that

$$x_i^k - x_{i+1}^k = k\alpha_i^{k-1}(x_i - x_{i+1}) = -\frac{k}{n}\alpha_i^{k-1}$$

We can write

$$D_n = -a\sum_{i=1}^{n-1} \frac{\frac{k}{n}\alpha_i^{k-1}}{x_i^k + ax_{i+1}^k} x_i + a\frac{(1-\frac{1}{n^k})}{1^k + a\frac{1}{n^k}}$$

Let $1 < m < n-1$. Introduce

$$A_n := \frac{ak}{n}\sum_{i=1}^{m} \frac{\alpha_i^{k-1}}{x_i^k + ax_{i+1}^k} x_i,\ \ B_n := \frac{ak}{n}\sum_{i=m+1}^{n-1} \frac{\alpha_i^{k-1}}{x_i^k + ax_{i+1}^k} x_i$$

We compare B_n with $\frac{n-1-m}{n}\frac{ak}{1+a}$:

$$\begin{aligned}
& \left|B_n - \tfrac{n-1-m}{n}\tfrac{ak}{1+a}\right| \\
= \ & \tfrac{ak}{n}\left|\textstyle\sum_{i=m+1}^{n-1}\left(\tfrac{\alpha_i^{k-1}}{x_i^k+ax_{i+1}^k}x_i - \tfrac{1}{1+a}\right)\right| \\
= \ & \tfrac{ak}{n(1+a)}\left|\textstyle\sum_{i=m+1}^{n-1}\tfrac{\alpha_i^{k-1}x_i(1+a)-(x_i^k+ax_{i+1}^k)}{x_i^k+ax_{i+1}^k}\right| \\
\le \ & \tfrac{ak}{n(1+a)}\textstyle\sum_{i=m+1}^{n-1}\left|\tfrac{(\alpha_i^{k-1}-x_i^{k-1})x_i+a(\alpha_i^{k-1}x_i-x_{i+1}^k)}{x_i^k+ax_{i+1}^k}\right| \\
\le \ & \tfrac{ak}{n(1+a)}\textstyle\sum_{i=m+1}^{n-1}\tfrac{(x_{i+1}^{k-1}-x_i^{k-1})x_i+a(x_{i+1}^k-x_i^k)}{x_i^k+ax_{i+1}^k} \\
\le \ & \tfrac{ak}{n(1+a)}\textstyle\sum_{i=m+1}^{n-1}\tfrac{k-1+ak}{n}\ \tfrac{x_{i+1}^{k-1}}{x_i^k+ax_{i+1}^k} \\
\le \ & \tfrac{ak}{n(1+a)}(n-1-m)\tfrac{k-1+ak}{n}\ \tfrac{1}{1+a}\tfrac{1}{(\frac{m}{n})^k} \\
\le \ & \tfrac{ak}{(1+a)^2}\ \tfrac{k-1+ak}{(\frac{m}{n})^k}\ \tfrac{1}{n}
\end{aligned}$$

Next we estimate A_n:

$$\begin{aligned}
|A_n| \ &= \ \left|\tfrac{ak}{n}\textstyle\sum_{i=1}^{m}\tfrac{\alpha_i^{k-1}}{x_i^k+ax_{i+1}^k}x_i\right| \\
&\le \ \tfrac{ak}{n}\textstyle\sum_{i=1}^{m}\tfrac{x_{i+1}^k}{x_i^k+ax_{i+1}^k} \\
&\le \ \tfrac{km}{n}
\end{aligned}$$

Putting together these estimations, we obtain

$$\begin{aligned}
& \left|A_n + B_n - \tfrac{ak}{1+a}\right| \\
\le \ & |A_n| + \left|B_n - \tfrac{n-1-m}{n}\tfrac{ak}{1+a}\right| + \tfrac{m+1}{n}\tfrac{ak}{1+a} \\
\le \ & \tfrac{km}{n} + \tfrac{ak}{(1+a)^2}\ \tfrac{k-1+ak}{(\frac{m}{n})^k}\ \tfrac{1}{n} + \tfrac{m+1}{n}\tfrac{ak}{1+a}
\end{aligned}$$

Let $0 < c < 1$ and $m = [c \cdot n] + 1$, so that $c \leq \frac{m}{n} \leq c + \frac{1}{n}$. Incorporating c into the above estimation, we obtain

$$\begin{array}{rcl} \left|A_n + B_n - \frac{ak}{1+a}\right| & \leq & k(c + \frac{1}{n}) + \frac{ak}{(1+a)^2} \ \frac{k-1+ak}{c^k} \ \frac{1}{n} + (c + \frac{2}{n})\frac{ak}{1+a} \\ & = & c(k + \frac{ak}{1+a}) + \frac{1}{n}(k + \frac{ak}{(1+a)^2} \ \frac{k-1+ak}{c^k} \ + 2\frac{ak}{1+a}) \end{array}$$

Let $\epsilon > 0$ be any positive number. Choose c such that $c(k + \frac{ak}{1+a}) < \frac{\epsilon}{2}$. Choose next n_0 such that, for any $n \geq n_0$,

$$\frac{1}{n}(k + \frac{ak}{(1+a)^2} \ \frac{k-1+ak}{c^k} \ + 2\frac{ak}{1+a}) < \frac{\epsilon}{2}.$$

We see then, for $n \geq n_0$,

$$\left|a\sum_{i=1}^{n-1} \frac{\frac{k}{n}\alpha_i^{k-1}}{x_i^k + ax_{i+1}^k} x_i - \frac{ak}{1+a}\right| = \left|A_n + B_n - \frac{ak}{1+a}\right| < \epsilon.$$

This means

$$\lim_{n\to\infty} D_n = -\frac{ak}{1+a} + a.$$

Now, if $a < k-1$, $-\frac{ak}{1+a} + a = \frac{a}{1+a}((1+a) - k) < 0$. There will exist a n such that $D_n < 0$. The Question 3. is proved. ∎

Question 4. Find a_0 the smallest $a > 0$ such that the inequality

$$C_{k,l,a}(x) \geq A^l(x)$$

holds for any n-tuple of positive real numbers.

Proof. Let $x = (x_i)_{1\leq i\leq n}$ be any n-tuple of positive real numbers. We introduce $y_i = x_i^l$, $1 \leq i \leq n$. We check immediately the equality

$$C_{k,l,a}(x) = C_{\frac{k}{l},1,a}(y)$$

So, if $a \geq \frac{k}{l} - 1$, according to Question 1., using Jensen's inequality, we can write

$$\begin{array}{rcl} C_{k,l,a}(x) & = & C_{\frac{k}{l},1,a}(y) \\ & \geq & A(y) \\ & = & \frac{1}{n}\sum_{i=1}^n x_i^l \\ & \geq & \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^l \\ & = & A^l(x) \end{array}$$

This means that $a_0 \leq \frac{k}{l} - 1$. ∎