Problème 9

ITYM- FRANCE 2

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Abstract

We try in this exercice to compute the number of distinct set reachable with diffreent operators in some euclidian's spaces. The method is always the same: We define a monoïd, where the generators are the combinaisons of operators, then we determine the cardinal of the set of this generators, and fianlly we find an exemple set to finish our enumeration.

1 Notations

Let's define the terms we are going to use:

- i refers to the interior operator
- e refers to the convex hull operator
- c refers to the closure operator
- Every combinaison of these previous operators will be denoted by a concatenation of i, c or e, for instance conv(int(A)) will be written: $ei(A)$

2 Question 2

2.1 SubQuestion b

A simple enumeration gives us a total number of 3^l with l the maximal lenth of the operators.
Like peviously we should find some rules between the generators of this monoïd, in order to reduce the number of potential operators. Rules which are applied on only c and i operators are the same one as these explained on question 1.

2.1.1 Conservation of the connexity by i and c

This rule allow us to reduce operators by letting the last occurence of the primary operator conv(). Try first to prove that closure and interior conserv the convexity.

For the closure: Let $a, b \in ce(A)^2$. Let (u_n), (v_n) be two sequences, such that $\lim_{\infty} u_n = a$ and $\lim_{\infty} v_n = b$.

We now consider the segment $[a;b]$ which equation is: $\forall z \in [a;b], \exists \epsilon \in [0;1] \quad z = \epsilon a + (1-\epsilon)b$ therefore:

$$z = \lim_{\infty} \epsilon u_n + (1-\epsilon) \lim_{\infty} v_n = \lim_{\infty} [\epsilon u_n + (1-\epsilon) v_n]$$

So z is the limit of a sequence which is in $ce(A)$, so, $z \in cce(A)$ finally $cce(A) \subseteq ce(A)$ and the converse inclusion is trivial so $cce(A) = ce(A)$. We deduce that: $[a;b] \subseteq ce(A)$ so:

$ce(A)$ is convex. Therefore, closure conserves the convexity.

For the interior: let $(a, b) \in ie(A)^2$, by definition $ie(A)$ is an open set so we can find an open ball $\mathbf{B}(a, \epsilon) \in ie(A)$ Let $z \in [a; b]$, so $\forall z \in [a; b], \exists \phi \in [0; 1] \quad z = \phi a + (1 - \phi)b$. Therefore, $\mathbf{B}(a, \epsilon\phi) \in ie(A)$. So, $[a; b] \in ie(A)$ finally:

$ie(A)$ is convex. Therefore, interior conserves the convexity.

We have just proved that closure and interior conserve the convexity of a convex set, so, if there is a convex hull in an operator, only the first one will be efficient (e(A) = A if A is convex), therefore we can reduce operators by letting the last occurence of the primary operator conv() (the last occurence is the first one to be used)

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2.1.2 $ce = cie$

The inclusion $cie(A) \subseteq ce(A)$ is trivial (because for all A we have $i(A) \subseteq A$), therefore, let's work on $ce(A) \subseteq cie(A)$.

Let $x \in ce(A)$ we also consider that $i(A)$ is not empty. Therefore we can find two points $y, z \in ce(A)$ such that x, y, z are not collinear (because if such points do not exist then A is include in a straight line and $i(A)$ is empty). Since $ce(A)$ is convex then the triangle x, y, z is include in $ce(A)$.

Let's apply the operator i: we conserve the interior of our triangle (only the edges can disapear). We apply now the closure: we can recover the edges of our previous shape.

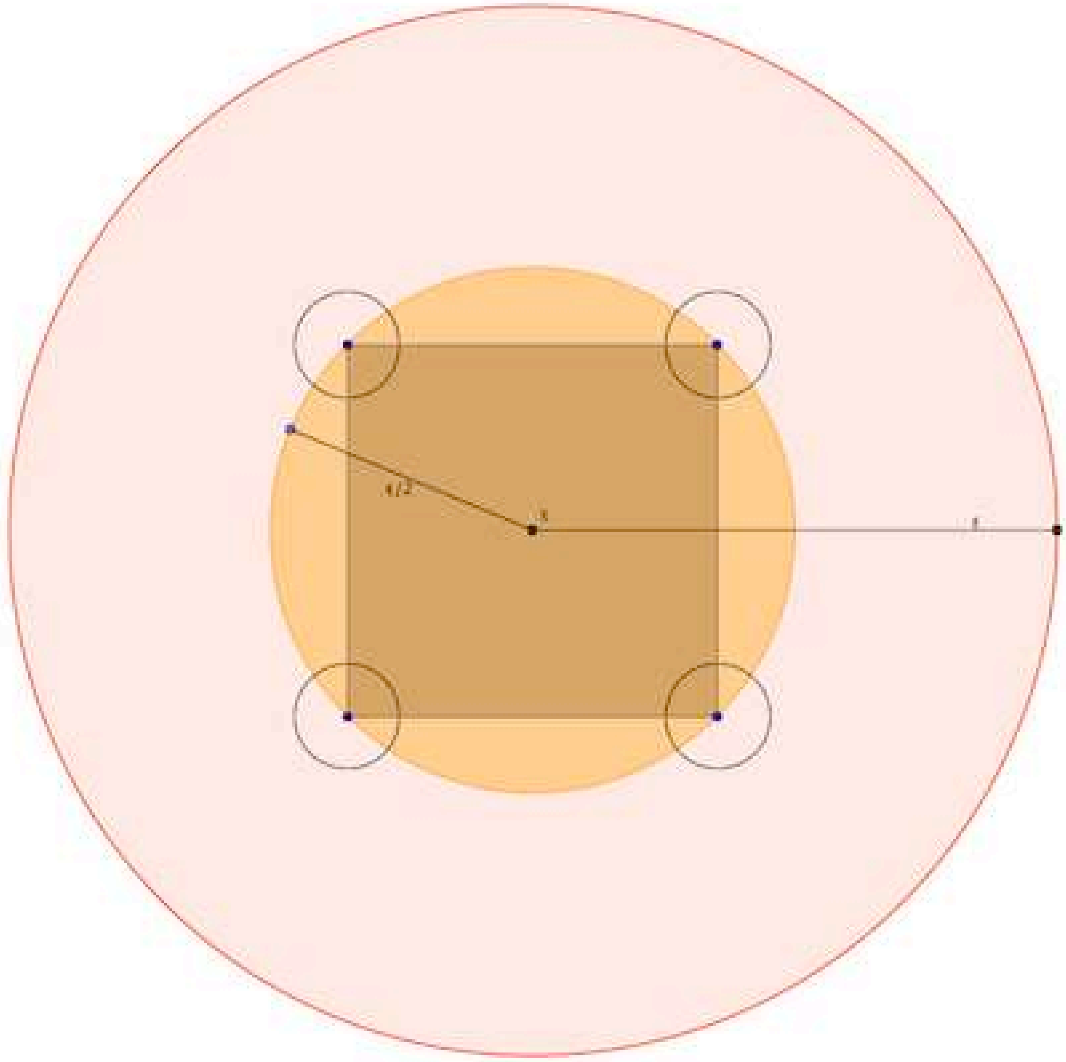
So if $x \in ce(A)$ we have shown that $x \in cie(A) \qquad ce(A) \subseteq cie(A)$

$ce = cie$

2.1.3 $ice = ie$

Like previously we only work on the non-trivial inlusion ($ce \subseteq ice$ is clear because $i(A) \subseteq A$ for all A): $ic(A) \subseteq e(A)$ ($ice \subseteq ie$ is a consequence of this fact because we have shown that $ee = e$ and $ii = i$).

Let $x \in ic(A)$, so,by definition of i , we can find an open ball $\mathbf{B}(x, \epsilon) \subseteq ic(A)$, and this ball contains only sequence's limits of A.



We consider 4 others ball of radius $\epsilon/10$ include in $\mathbf{B}(x,\epsilon)$, which are situated on the edge of $\mathbf{B}(x,\epsilon/2)$ and with centers on the vertex of a square.

This smaller ball are a small radius (i.e $\lambda = \epsilon/10$), and each point in these balls are sequence's limits of A so we can find a least one point of A in each such ball. And finally x is in the quadrilateral formed by these four points, so $x \in e(A)$.

We deduce that: $ic(A) \subseteq e(A)$:

$$ice \subseteq ie(A), \text{ so } ice = ie$$

2.1.4 $iec = ie$ (question 2 a/)

- Let's start with the inclusion $ie(A) \subseteq iec(A)$:

  Like previously we have $ic(A) \subseteq e(A)$
  So: $icc(A) = ic(A) \subseteq ec(A)$
  $ice(A) \subseteq ece(A)$
  $ice(A) \subseteq iece(A)$ Therefore, $ice(A) \subseteq iec(A)$
  But we've just demonstrate that $ice(A) = ie(A)$:
  $ice(A) = ie(A) \subseteq iec(A)$.

- And now let's continue with $iec(A) \subseteq ie(A)$

  We have: $id(A) \subseteq e(A)$ therefore, because c is an increasing function:
  $c(A) \subseteq ce(A)$
  $ec(A) \subseteq ece(A) = ce(A)$
  therefore: $ec \subseteq ce(A)$
  Let's apply i: $iec(A) \subseteq ice(A) = ie(A)$
  $iec(A) \subseteq ie(A)$

We can now deduce that: $iec = ie$

2.1.5 eici = ei

Je ne sais pas si la règle existe

2.1.6 A particular Set

With the new rules, we are likely to define only 15 (or 14 depends on the last rule).

In order to finish this demonstration we need to find a set which react in a particular way for each of the operators. Let's try to construct it with the union of particular subsets.

[schéma de l'ensemble]

We could theoricly made 15 distinct set with i,e and c.

2.2 With $int(conv(A)) = \emptyset$

The vacuity imposes us to only work on line segments(in order to have line segments for convex hull and thus an empty interior), therefore, $int(A) = \emptyset$.

We can make a new rule: $ec = ce$.

2.2.1 $ce = ec$

- We consider a set made with the union of line segments. If we apply the convex hull operator, we'll obtain a new line segment, which could be like $[a; b],]a; b[, [a; b[$ or $]a; b]$ (where a and b define the two points such as $d(a, b)$ is maximum). Then we apply the closure: the only set we could obtain is $[a; b]$.

- If we apply first the clusure, we obtain a set which is the union of closed line segments. Then if we apply the convex hull, we cvan only obtain a closed segment.

therefore $ce = ec$.

We deduce that, the resulting operators are id, i, e, c, ec

Let's try to find a particular subset (like above). We choose: $]a; b[\bigcup[c; d[$,with $x_a < x_b < x_c < x_d$

id	$]a;b[\bigcup[c;d[$
e	$]a;d[$
c	$[a;b]\bigcup[c;d]$
i	$\emptyset$
ec	$[a;d]$

We could theoricly made 5 distinct sets with i,e and c, when $int(conv(A)) = \emptyset$.

3 Globalisation

Our methods alows us to define the generators in any euclidian's space, therefore, we just have to create an exemple set which has the same properties as sets we've made above (denseset, convex dense open set, ...) at N dimention for a N-Space.

Exercise 9: Topological Spaces

France 2

Question 1

Let c, i, id denote respectively the closure, interior, and identity operators on $\mathcal{P}(\mathbb{R}^n)$ and define the set $\mathcal{M}$ generated by these operators with the composition $\circ$ which is defined by

$$\forall X \in E,\ \forall (o_1, o_2) \in \mathcal{M}^2, \qquad o_1 \circ o_2(X) = o_1(o_2(X)).$$

In order to lighten the notation, we shall write $o_1 o_2$ for $o_1 \circ o_2$.

From the definition of the closure, interior and composition, we deduce that these operators are associative, therefore, $(\mathcal{M}, \circ)$ is a **monoid.** Let's investigate some relationships between generators of this monoid.

Lemma

- $ii = i$
- $cc = c$
- $cici = ci$
- $icic = ic$

Proof of the two first idempotence properties:
From the definitions, the interior (resp. closure) of a set A is the biggest open set contained in A (resp. the smallest closed set which contains A).
Consequently, $i(A)$ is open and $c(A)$ is closed and, for any open (resp. closed) set B, $i(B) = B$ (resp. $c(B) = B$).
Therefore, $ii(A) = i(A)$ and $cc(A) = c(A)$. ■

Let's define a partial order relation $\leq$ on $\mathcal{M}$.

Definition 1

Forall $(o_1, o_2) \in \mathcal{M}^2$, $o_1 \leq o_2$ means that

$$\forall A \subseteq E, \qquad o_1(A) \subseteq o_2(A).$$

In order to show that $icic = ic$, we will check that $icic \leq ic$ and $ic \leq icic$.

Proof of $cici \leq ci$:
Let $A \in \mathcal{P}(\mathbb{R}^n)$ and $B = ic(A)$.
Since $B \subseteq c(B)$, we deduce $i(B) \subseteq ic(B)$ (the application i is increasing for the inclusion).
Then, we obtain $iic(A) \subseteq icic(A)$; we previously showed that $ii = i$ so

$$\forall A \in \mathcal{P}(\mathbb{R}^n), \qquad ic(A) \subseteq icic(A).$$

■

Proof of $cici \geq ci$:
Let $A \in \mathcal{P}(\mathbb{R}^n)$ and $B = ci(A)$.
Since $i(B) \subseteq B$, we deduce $ci(B) \subseteq c(B)$ (the application c is increasing for the inclusion).
Then, we obtain $cici(A) \subseteq cci(A)$; we previously showed that $cc = c$ so

$$\forall A \in \mathcal{P}(\mathbb{R}^n), \qquad cici(A) \subseteq ci(A).$$

■

We obtain, by combining the two inequalities,

$$cici = ci$$

The other property $icic = ic$ is proved in the same way.

We deduce from these relationships that the set $\mathcal{M}$ is finite; for example, $ii\dots i$ and $cc\dots c$ are i and c; $cicic\dots cic$ and $icic\dots i$ are cic and ici.

Proposition 2

The cardinal of $\mathcal{M}$, the monoid of the operators, is 7.
More precisely,

$$\mathcal{M} = \{\mathrm{id}, i, c, ic, ci, ici, cic\}.$$

As a consequence, we can potentially obtain *up to* 7 different sets.

In order to finish the proof, we shall find a set which have different images for each of the seven operators of $\mathcal{M}$.
Let's construct such a set as the union of some particular subsets with different topological properties (closed set with non empty interior, dense set with empty interior, closed set with empty interior, dense set).

id	$[p;q]$	$\mathbb{Q}\cap[a;b]$	$\{\alpha\}$	$\mathbb{R}_+\setminus\{2^{-n},\ n\in\mathbb{N}\}$
i	$]p;q[$	$\varnothing$	$\varnothing$	$\mathbb{R}_+\setminus\{2^{-n},\ n\in\mathbb{N}\}$
c	$[p;q]$	$[a;b]$	$\{\alpha\}$	$\mathbb{R}_+$
ic	$]p;q[$	$]a;b[$	$\varnothing$	$\mathbb{R}_+^*$
ci	$[p;q]$	$\varnothing$	$\varnothing$	$\mathbb{R}_+$
ici	$]p;q[$	$\varnothing$	$\varnothing$	$\mathbb{R}_+^*$
cic	$[p;q]$	$[a;b]$	$\varnothing$	$\mathbb{R}_+$

Therefore, we can obtain 7 different sets starting with the set:

$$[p;q]\cup\mathbb{Q}\cap[a;b]\cup\{\alpha\}\cup\mathbb{R}\setminus\{2^{-n}\ |\in\mathbb{N}\},$$

with $p<q<a<b<\alpha<0$.

Proposition 3

The maximal number of distinct sets that can be obtained with the interior and the closure of a set $A\subset\mathbb{R}$ is **7**.

Question 2

Let's note h the operator of convex hull and $\mathcal{M}'$ the monoid generated by i, c and h.
We first prove:

Lemma

Let $A \in \mathcal{P}(\mathbb{R}^2)$ such that $ih(A) \neq \emptyset$.
Then

$$ihc = ih$$