Problem 6: Min/Max Questions
Team: France 2

June 22, 2010

Abstract

We will search the maximal ratio $\frac{l_{min}}{l_{max}}$ for the different cases in different propositions.

The proposition 1.1 studies the case of the line. After having ordered the points as A_1, A_2, ... , A_N, we show that $l_{max} \geq l_{min}$ thanks to two lemmas:

- $\exists 1 \leq i \leq N-1$ such that $l_{min} = A_i A_{i+1}$
- $l_{max} = A_1 A_N$

After this, the inequality becomes obvious.

The proposition 1.2 studies the cases of the plane for $N = 4$. The plane is the "worse" situation so every other set will verify the same inequality. So, studying the case of the plane for $N = 4$ allows us to conclude for the cases of the circle, the convex polygon and the grids if there is an equality situation for each case.

We will show that $\frac{l_{min}}{l_{max}} \leq \frac{1}{\sqrt{2}}$

For this result, we will use the following lemmas:

- pigeonhole theorem : if there are k elements in p sets, then a set contains less than $\lfloor \frac{k}{p} \rfloor$ elements and a set contains more than $1 + \lfloor \frac{k}{p} \rfloor$ where $\lfloor X \rfloor$ is the integer part of X.
- Al-Kashi theorem : in a triangle ABC, with $\alpha = \widehat{BAC}$, we have $BC^2 = AB^2 + AC^2 - 2AB.AC.cos(\alpha)$.
- Every triangle can be covered by two discus whose diameters are two sides of the triangle.

We use this theorems to show that there is always one triangle with an angle greater than 90°, and that in this triangle, and so in the whole plane, we have $\frac{l_{min}}{l_{max}} \leq \frac{1}{\sqrt{2}}$ with equality for the square.

The square can be found in the circle, the grids and in some convex polygons so for the cases b, c, d, e and f, we have $\frac{l_{min}}{l_{max}} \leq \frac{1}{\sqrt{2}}$ for $N = 4$.

The proposition 1.3 studies the case of the circle. We use basic geometry.

opening In the proposition 1.3.1, we study this case for every N. We show that the maximum ratio $\frac{l_{min}}{l_{max}}$ is $\frac{l_{min}}{l_{max}} = \frac{2r\sin(\pi/n)}{2r\sin(\lfloor n/2 \rfloor \pi/n)} = \frac{\sin(\pi/n)}{\sin(\lfloor n/2 \rfloor \pi/n)}$.

In the proposition 1.3.2, we reduce the study to the case $N = 5$, and we prove $\frac{l_{min}}{l_{max}} = \frac{1}{2sin(\frac{3\pi}{10})} = \frac{\sqrt{5}-1}{2}$ for $N = 5$ in the circle.

In the proposition 1.3.3, we study the least upper bound of $\frac{l_{min}}{l_{max}}$ when N become great. We prove that $\lim_{N \to \infty} \frac{l_{min}}{l_{max}} = 0$. We use the pigeonhole theorem.

opening The proposition 1.4 studies generalizations of the case of the plane. The proposition 1.4.1 compares the case of the plane to those of the grids.

The proposition 1.4.2 gives an upper bound for the plane with $N = 5$.

The proposition 1.4.3 generalizes the upper bound of the plane.

Question2:

The proposition 2.1 finds the maximal ratio for the case of the line and the plane.

opening The proposition 2.2 is an adaptation of the opening solution for the circle in question 1.

opening The propositions 2.3 and 2.4 give an inequality for the case of the grids (d and e).

Remark:

You can find the figures of this problem in the Appendix. Some propositions are an opening of the question. We precise it beneath their title in the abstract and in the proof.

1 the ratio $\frac{l_{min}}{l_{max}}$

We denote $C_S(N)$ the supremum (the least upper bound) of $\frac{l_{min}}{l_{max}}$ with N points in S set. We often denote i the set S which will be treated in the subquestion i), for example b is a circumference.

Some Remarks:

- $S_1 \subset S_2 \Rightarrow C_{S_1}(N) \leq C_{S_2}(N)$, in particular $C_b(N) \leq C_f(N)$.
- $C_S(N+1) \leq C_S(N)$ because if there exists $N+1$ points, there are N points
- So in particular, $(C_S(N))_N$ is decreasing and bounding below so convergent

Proposition 1.1. **(Case of a.)** $l_{max} \geq (N-1)l_{min}$
We order the points on the line and we name them $A_1, A_2, ..., A_N$.

Lemma 1 $\exists 1 \leq i \leq N-1$ *such that* $l_{min} = A_iA_{i+1}$

Proof:
We have $l_{min} = A_iA_j$. Suppose $j > i+1$. Then we have (by definition) $A_iA_j = A_iA_{i+1} + A_{i+1}A_j > A_iA_{i+1}$. Contradiction.

□

Lemma 2 $l_{max} = A_1A_N$

Proof:

We suppose that $l_{max} = A_iA_j$ with $1 < i < j < N$. Then we have $A_iA_j < A_1A_N = A_1A_{i-1} + A_iA_j + A_{j+1}A_N$ (by definition). Contradiction.

□

Thus we have

$$A_1A_N = \sum_{k=1}^{N-1} A_kA_{k+1} \geq \sum_{k=1}^{N-1} l_{min} = (N-1)l_{min}$$

So, $l_{max} \geq (N-1)l_{min}$

Proposition 1.2. **(Case of the plane b, c, d, e, f for $N = 4$)** $C_d(4) = C_e(4) = C_f(4) = \frac{1}{\sqrt{2}}$

Proof
We first consider the case in which the convex hull of the four points is a convex quadrilateral. The sum of the angles in a convex quadrilateral is 360° so there is at least one angle greater than $\frac{360}{4} = 90°$ (pigeonhole theorem). We consider

the triangle of this angle. It is clear that the ratio $\frac{l_{min}}{l_{max}}$ in this triangle is less than $\frac{1}{\sqrt{2}}$ (We can use Al-Kashi).

Then we must treat the case in which the convex hull of the for points is a triangle. If this triangle is [*obtusangle*], it is trivial (it is the case before).
We suppose that every angle between three points of these four points is littler than 90°. It means that there exists a triangle for which the third point is in the inner of the discus whose diameter is built by the two other points. We will show it is impossible.

Lemma 3 *Every triangle can be covered by two discus whose diameters are two sides of the triangle.*

Proof:

Let ABC a triangle, I and J the middles of $[AB]$ and $[AC]$. Let M be the point such that $IA = IM$ and $JA = JM$. M is the symmetric of C by $[IJ]$. Thanks to the theorem of the middle's line, we have $M \in (BC)$. Only two points (M and A) in the triangle verify $IM = IA$ and $JM = JA$, so the whole triangle can be covered by the two discus of diameter $[AB]$ and $[AC]$. (The two triangles ABM and ACM are included in these discus, which are their circumscribe circle and $ABC = ABM \cup ACM$).

Figure 1.2.

□

Then, by considering the convex hull of the four points, we see that the fourth point is in a discus of diameter in the side of the triangle : there is an angle greater than 90°.

So, we have : $C_f(4) = \frac{1}{\sqrt{2}}$.

We have $b \subset f$, $c \subset f$, ... , and for all this cases, the square verifies the equality, so we showed $C_b(4) = C_d(4) = C_e(4) = C_f(4) = \frac{1}{\sqrt{2}}$

Propositions 1.3. (Case of b)

Proposition 1.3.1

opening

$$\frac{l_{min}}{l_{max}} = \frac{2r\sin(\pi/n)}{2r\sin(\lfloor n/2 \rfloor \pi/n)} = \frac{\sin(\pi/n)}{\sin(\lfloor n/2 \rfloor \pi/n)} \tag{1}$$

Proof

The distance between two points on a circle is given by the formula:

$$d(A, B) = 2r\sin(\alpha/2) \tag{2}$$

where $\alpha = \widehat{AOB}$.

Therefore, the maximum distance between n points on the circle will be obtain with the maximum angle between two points which is inferior to π. In order to obtain the maximum value of the minimum distance,(and therefore a value of the maximum distance, because in this case, min and max are linked by an additive law), we must try to put our points such as there is a maximum number of minimum distance (like in the first case , point on the line). Therefore we deduce that the pattern that we must obtain is a regular polygon: all the centre angles will be equal, and therefore we will have a maximum value for the minimum. (The ratio will be maximum because of the additive law between the angle: the minimum value of the maximum angle is obtain for the regular polygon because the angle make the most equilibrated partition of the 2π center angle, we therefore will have the smallest maximum angle lower than π).

Let's compute l_{min}/l_{max}:

- l_{min} is really simple to compute, it's the distance between two successive points, which define a center angle of the polygon, so:

$$l_{min} = 2r\sin(2\pi/2n) = 2r\sin(\pi/n) \tag{3}$$

- For l_{max} we need to define the number of center angle we can add to each other until the total angle is equal or lower than π. We obviously notice that this angle is equal to $n/2$ when n is even and $(n-1)/2$ when n is odd. We have now:

$$l_{max} = 2r\sin(\lfloor n/2 \rfloor \pi/n) \tag{4}$$

We obtain therefore:

$$\frac{l_{min}}{l_{max}} = \frac{2r\sin(\pi/n)}{2r\sin(\lfloor n/2 \rfloor \pi/n)} = \frac{\sin(\pi/n)}{\sin(\lfloor n/2 \rfloor \pi/n)} \tag{5}$$

Proposition 1.3.2
$C_b(5) = \frac{1}{2sin(\frac{3\pi}{10})} = \frac{\sqrt{5}-1}{2}$

Proof
Five points on a circumference give a convex pentagon. The sum of angles of this polygon is 3π, so there exists an $\alpha \geq \frac{3\pi}{5}$. Let be the triangle formed by this angle with both neighbouring sides.
Figure 1.3.2.a

By symmetry, we can suppose $AC < AB$
So, $\frac{AB}{BC} \leq \frac{AC}{B'C} = \frac{1}{2\times cos(\frac{\pi-\alpha}{2})} = \frac{1}{2\times sin(\frac{\alpha}{2})} \leq \frac{1}{2\times sin(\frac{3\pi}{10})} = \frac{\sqrt{5}-1}{2}$

A fortiori, $\frac{l_{min}}{l_{max}} \leq \frac{\sqrt{5}-1}{2}$, but the case of equality is realised with this **figure 1.3.2.b**

□

Proposition 1.3.3

$$\lim_{N \to \infty} C_b = 0$$

Proof
Let $S = b$ then S is a circumference. All the points have a modulus 1, and a argument θ. Let be $\alpha > 0$ and $K \in \mathbb{N}, K \geq 3$. As soon as $N > \frac{2K\pi}{\alpha}$, there exists an interval $[\theta, \theta + \alpha]$ which has K points (pigeonhole theorem). These points are almost aligned because α is small,so about these K points, $\frac{l_{min}}{l_{max}} \leq \frac{1}{K-1+\varepsilon}$ with ε so small as we want. So $\lim_{N \to \infty} C_b = 0$

□

Propositions 1.4. (Case of f)

opening

Proposition 1.4.1

$C_f(N) = C_e(N) \approx C_d(N)$
There is not equality between C_f and C_d because the covering d is supposed finished by size $m \times n$. Therefore, a figure with a lot of points cannot always be included in d. There cannot be equality between them insofar as the covering is supposed finished by size $m \times n$. When m and n become great, their difference becomes negligible.

Proposition 1.4.2

$C_f(5) = tan(\frac{\pi}{5})$
The proof in the case the convex hull of 5 points is a pentagon is the same as for the circle with 5 points. We show the other cases in the same way by proving that there exists an angle bigger that $\frac{3\pi}{5}$ when:

- convex hull is a quadrilateral
- convex hull is a triangle

□

Proposition 1.4.3
Figures 1.4.3.a correspond with lower bounding $C_f(N)$ for $N = 5...9$

If N is even :
$l_{max} = 2$ and $l_{min} = 2 \times sin(\frac{\pi}{N})$

So, $C_f(N) \geq sin(\frac{\pi}{N})$

If N is odd :
$l_{max} = 2 \times cos(\frac{\pi}{2N})$ and $l_{min} = 2 \times sin(\frac{\pi}{N})$

So, $C_f(N) \geq \frac{cos(\frac{\pi}{2N})}{sin(\frac{\pi}{N})}$

Figure 1.4.3. b.

$C_f(N)$ is decreasing and bounded below, so convergent
For $N \geq 4$, $C_f(N) \leq \frac{1}{\sqrt{2}}$

Figure 1.4.3. c.

2 the ratio $\frac{c_{min}}{c_{max}}$

We denote $D_S(N)$ the smallest upper bound of $\frac{c_{min}}{c_{max}}$ with N points in S set. We often denote i the set S will be treated in the sub question i), for example b is a circumference.

Proposition 2.1. case of a
$D_a(N) = C_a(N) = \frac{1}{N-1}$

Proof
It is trivial, the distance l and c coincide

□

Proposition 2.2. case of b and c
$D_c(N) = \frac{2}{N}$
Proof
The first equality is trivial. For the second, we come down the case of a circumference (center O and radius 1) We can suppose that for all line Δ passing by O, there exists points in the two half-plane bounded by Δ otherwise we come down to the case of the line

$D_b(N) = \frac{2}{\lfloor \frac{N}{2} \rfloor}$
with : $\lfloor N \rfloor$ integer part
Proof
$l_{min} = \frac{2\pi}{N}$ and $l_{max} = \frac{\lfloor \frac{N}{2} \rfloor \times 2\pi}{N}$

□

Proposition 2.3. case of d
$D_d(N) \geq \frac{1}{\sqrt{2}} \times C_d(N)$

<u>Lemma :</u>
We denote $l(x,y)$ the Euclidean distance in the plane between x and y and $c(x,y)$ the distance between x and y in the square grid, so $l(x,y) \leq c(x,y) \leq l(x,y) \times \sqrt{2}$

Proof :
for all $(a,b) > 0, 1 \leq \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}} \leq \sqrt{2}$

□

Proof of the proposition
We deduce that, $c_{min} \geq l_{min}$ and $c_{max} \leq l_0 \times \sqrt{2}$ with $l_0 \leq l_{max}$, so $c_{max} \leq l_{max} \times \sqrt{2}$ In conclusion, $D_d(N) \geq \frac{1}{\sqrt{2}} \times C_d(N)$

□

Proposition 2.4. case of e
We change the constant $\sqrt{2}$ by $\sqrt{3}$ in the proposition 2.3. for the triangular grid.

Proof :
Figure 2.4.

□

In conclusion, $D_e(N) \geq \frac{1}{\sqrt{2}} \times C_e(N)$

3 the ratio $\frac{c_{min}}{c_{max}}$

We denote $D_S(N)$ the smallest upper bound of $\frac{c_{min}}{c_{max}}$ with N points in S set. We often denote i the set S will be treated in the sub question i), for example b is a circumference.

Proposition 2.1. (Cases of a. and f.)
$D_a(N) = C_a(N) = \frac{1}{N-1}$
$D_f(4) = C_f(4) = \frac{1}{\sqrt{2}}$

Proof
It is trivial, the distance l and c coincide for both cases.

□

Proposition 2.2. case of b

opening

$D_b(N) = \frac{1}{\lfloor \frac{N}{2} \rfloor}$
with : $\lfloor N \rfloor$ integer part of N
Proof
It is an adaptation of the solution of the question 1 for the circle : the greatest circle bow is the one which intercepts the greatest segment, and it is the same thing for the littlest. The best case is the regular polygon so we have $c_{min} = \frac{2\pi r}{N}$ and $c_{max} = \lfloor \frac{N}{2} \rfloor \frac{2\pi r}{N}$ so $D_b(N) = \frac{1}{\lfloor \frac{N}{2} \rfloor}$

□

Proposition 2.3. case of d

opening $D_d(N) \geq \frac{1}{\sqrt{2}} \times C_d(N)$

<u>Lemma :</u>
We denote $l(x,y)$ the Euclidean distance in the plane between x and y and $c(x,y)$ the distance between x and y in the square grid, so $l(x,y) \leq c(x,y) \leq l(x,y) \times \sqrt{2}$

Proof :
for all $(a,b) > 0, 1 \leq \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}} \leq \sqrt{2}$

□

Proof of the proposition
We deduce that, $c_{min} \geq l_{min}$ and $c_{max} \leq l_0 \times \sqrt{2}$ with $l_0 \leq l_{max}$, so $c_{max} \leq l_{max} \times \sqrt{2}$ In conclusion, $D_d(N) \geq \frac{1}{\sqrt{2}} \times C_d(N)$

□

Proposition 2.4. case of e
We change the constant $\sqrt{2}$ by $\sqrt{3}$ in the proposition 2.3. for the triangular grid.

Proof :
Figure 2.4.

□

In conclusion, $D_e(N) \geq \frac{1}{\sqrt{2}} \times C_e(N)$

PROBLEM 6 : MIN/ MAX QUESTIONS

France TEAM 2

APPENDIX

Figure 1.2.

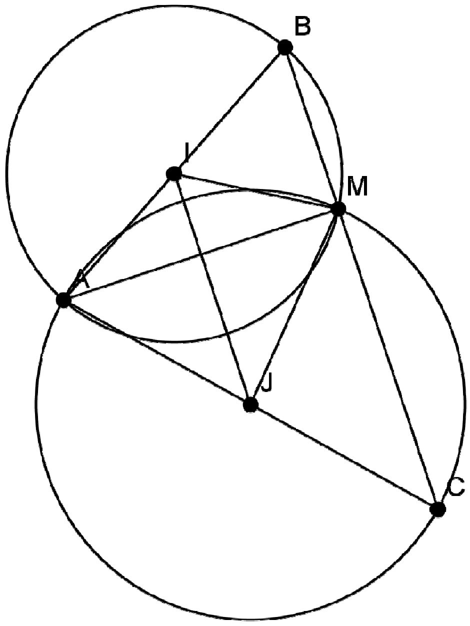


Figure 1.3.2.a

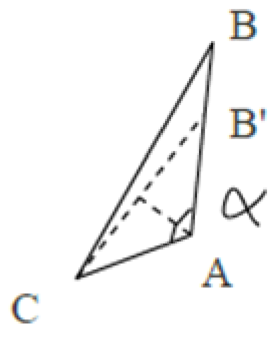
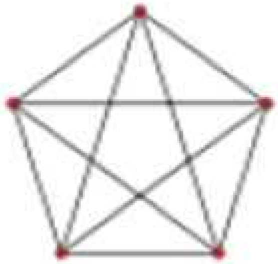


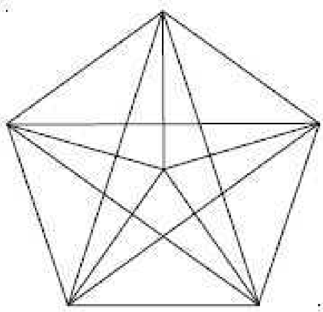
Figure 1.3.2.b



Figures 1.4.3.a **<u>Lower bounding of $C_f(N)$ for N=5,6,…9</u>**

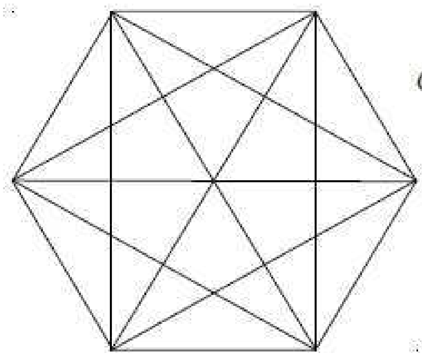
<u>N=5 :</u> $C_f(5) \geqslant \frac{\sqrt{5}-1}{2} \approx 0{,}6180$

<u>N=6 :</u>



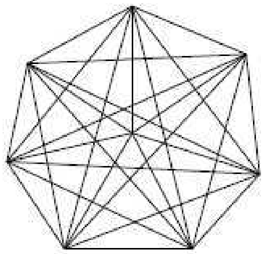
$C_f(6) \geqslant \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \approx 0{,}5257$

<u>N=7 :</u>



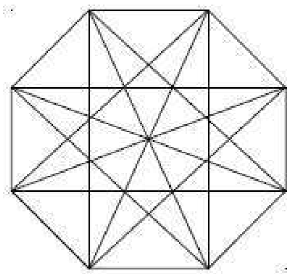
$C_f(7) \geqslant \frac{1}{2}$

N=8



$C_f(8) \geqslant 0{,}4451$

N=9 :



$C_f(9) \geqslant \frac{\sqrt{2-\sqrt{2}}}{2} \approx 0{,}3826$

Figure 1.4.3. b

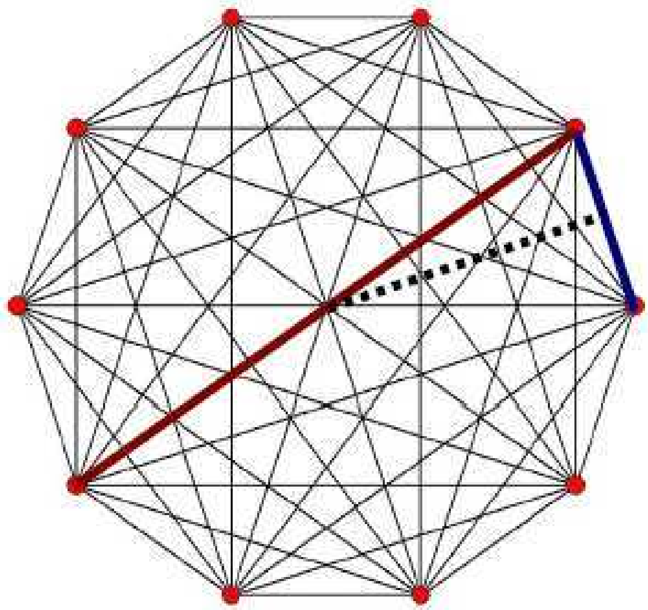


Figure 1.4.3. c

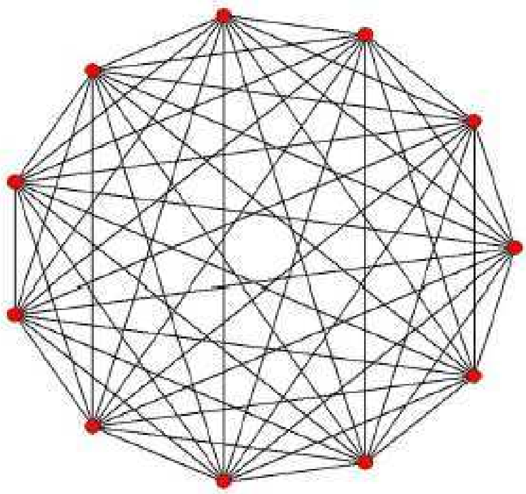


Figure 2.4.