Problem 5: A Strange Network

Team: France 2

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Abstract

For the whole problem, we will suppose $1 < k < N - 1$ to avoid impossible cases.

In the first question, we want to show that when all a_i's are distinct, then

$$T_{min} = 1 + \binom{N-2}{k} + 2\binom{N-1}{k-1}$$

We will show this with three lemmas :

- Knowing every different pairs (a_i, a_j) is enough to know the code.
  It becomes then obvious that the biggest number of sub-codes which don't allow us to be sure of the code doesn't give one pair or more.

- The biggest number of sub-codes which don't give a pair (a_u, a_v) is $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ for every $u < v$.

- There is only one way of choosing the $\binom{N-2}{k}) + 2\binom{N-1}{k-1}$ sub-codes which never give a particular pair (a_u, a_v) (the order of the sub-codes is not important) and they give every pair (a_i, a_j) except (a_u, a_v).
  Then, it seems obvious that with one more sub-code, we will have every pair (a_i, a_j) and so we will be sure of the code. We must now find $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ sub-codes which don't allow us to be sure of the code. By taking the $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ sub-codes which don't give the pair (a_u, a_{u+1}), we are not sure of the code. This allow us to conclude.

Question 1

We will take $2 \leq k \leq N-1$, so that every subcode gives at least one pair and that there is at least one pair (a_i, a_j) which is not in the subcode.

Lemma 1 *Knowing every different pairs (a_i, a_j) is enough to know the code.*

When we know every pair, we can order the integers because they are all different : we have all the pairs (a_1, a_i) and $\forall i, a_1 \neq a_i$ so we know a_1 is the first integer of the subcode. We do the same thing for a_2 , which is between a_1 and all the other a_i , etc.

So, knowing all the pairs (a_i, a_j) is enough to restore the code.

□

Lemma 2 *The biggest number of subcodes which don't give a pair (a_u, a_v) is $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ for every $u < v$.*

There are

$$\binom{N-2}{k}$$

different subcodes which don't give neither a_u nor a_v and

$$2\binom{N-1}{k-1}$$

different subcodes which give one of these two integers, and this for every $u < v$.

So, the biggest number of subcodes which never give a pair (a_u, a_v) is :

$$\binom{N-2}{k} + 2\binom{N-1}{k-1}$$

for every $u < v$.

□

Lemma 3 *There is only one way of choosing the $\binom{N-2}{k}+2\binom{N-1}{k-1}$ subcodes which never give a particular pair (a_u, a_v) (the order of the subcodes isn't important) and they give every pair (a_i, a_j) except (a_u, a_v).*

There are only $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ subcodes which never give the pair (a_u, a_v), so we can't choose other ones. The order of the subcodes isn't important, so there is only one way of choosing them.

We suppose there is an other pair (a_x, a_y) which is in no subcode. Then when we take one subcode with a_x and we replace one other integer by a_y. We obtain an other subcode, in which the pair (a_u, a_v) isn't and in which the pair (a_x, a_y) is. We had only $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ subcodes which never give the pair (a_u, a_v) and we found a new one : impossible.
So, the $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ subcodes which never give the pair (a_u, a_v) give every other pair (a_i, a_j).

□

We must now find the $\binom{N-2}{k} + 2\binom{N-1}{k-1}$ subcodes which never give a pair (a_u, a_v) and which don't allow us to be sure of the code. When we take the pair (a_u, a_{u+1}), it is obvious that we will never be able to say if the order between both integers is (a_u, a_{u+1}) or (a_{u+1}, a_u), and so we have two possible codes. We know that with one other subcode, we will always be sure to know the code because we will now the last pair (a_u, a_{u+1}). So, we have:

$$T_{min} = 1 + \binom{N-2}{k} + 2\binom{N-1}{k-1}$$

$\square$