Problem 3: A Cyclic Inequality

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Abstract

In the first question, we proved with an induction on n that, if $a \geq k-1$ then $C_{k,a}(x) \geq A(x)$ for any n-tuple x of positive real numbers.

In the second question, we proved that, if $n = 2$ and $0 < a < \frac{k-1}{k+1}$, we can find two 2-tuple x and y of positive real numbers, such that : $C_{k,a}(x) > A(x)$ but $C_{k,a}(y) < A(y)$. Then we extend the result for all $n > 2$.

In the third question, we have to investigate the case that $\frac{k-1}{k+1} \leq a < k-1$. We proved in the case that if $n = 2$ then we have $C_{k,a}(x) \geq A(x)$ for any 2-tuple x of positive real numbers. Then we find counterexamples that show that we can not use the same induction as in the first question.

Question 1 *Prove that if $a \geq k-1$ then $C_{k,a}(x) \geq A(x)$ for any n-tuple x of positive real numbers.*

We will prove this result with an induction on $n \in \mathbb{N}^*$, with x an n-tuple

$$(H_n) : C_{k,a}(x) \geq A(x)$$

For (H_1), we have $A(x) = x_1$ and $C_{k,a}(x) = (1+a)\left(\frac{x_1^{k+1}}{x_1^k + a x_1^k}\right) = x_1$, so $C_{k,a}(x) \geq A(x)$

We suppose that (H_n) is true and will prove (H_{n+1}) with an $(n+1)$-tuple $x = \{x_1, x_2, ..., x_n, x_{n+1}\}$. We can chose that $x_{n+1} = \max(x)$ (because we study a cyclic inequality). We name $x' = \{x_1, x_2, ..., x_n\}$. Then we have :

$$\begin{aligned}
C_{k,a}(x) &= \frac{1+a}{n+1}\left(\frac{x_1^{k+1}}{x_1^k + a x_2^k} + ... + \frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right) \\
&= \frac{1+a}{n+1}\left(\frac{n}{1+a}C_{k,a}(x') - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right) \\
&\geq \frac{1+a}{n+1}\left(\frac{n}{1+a}A(x') + \frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right) \\
&= \frac{x_1 + x_2 + ... + x_n}{n+1} + \frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right)
\end{aligned}$$

We will now prove $\frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right) \geq \frac{x_{n+1}}{n+1}$ (1)

If $x_{n+1} = x_n$, then :

$$\frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right)$$

$$= \frac{1+a}{n+1}\left(\frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_{n+1}^k} - \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right)$$

$$= \frac{1+a}{n+1} \cdot \frac{x_{n+1}}{1+a} = \frac{x_{n+1}}{n+1}$$

If $x_{n+1} = x_1$, then :

$$\frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_1^k}\right)$$

$$= \frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + a x_{n+1}^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + a x_{n+1}^k}\right)$$

$$= \frac{1+a}{n+1} \cdot \frac{x_{n+1}}{1+a} = \frac{x_{n+1}}{n+1}$$

We can now study the case that $x_{n+1} > x_1$ and $x_{n+1} > x_n$. We have :

$$(1) \Leftrightarrow (1+a)\left(\frac{x_n^{k+1}}{x_n^k + ax_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + ax_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + ax_1^k}\right) \geq x_{n+1}$$

$$\Leftrightarrow (1+a)\cdot\frac{x_n^{k+1}(x_n^k + ax_1^k) - x_n^{k+1}(x_n^k + ax_{n+1}^k)}{(x_n^k + ax_{n+1}^k)(x_n^k + ax_1^k)} + \frac{(1+a)x_{n+1}^{k+1} - x_{n+1}(x_{n+1}^k + ax_1^k)}{x_{n+1}^k + ax_1^k} \geq 0$$

$$\Leftrightarrow (1+a)\cdot x_n^{k+1}\cdot\frac{a(x_1^k - x_{n+1}^k)}{(x_n^k + ax_{n+1}^k)(x_n^k + ax_1^k)} + \frac{a(x_{n+1}^{k+1} - x_1^k x_{n+1})}{x_{n+1}^k + ax_1^k} \geq 0$$

We divide by $a(x_1^k - x_{n+1}^k) < 0$

$$(1) \Leftrightarrow (1+a)\cdot x_n^{k+1}\cdot(x_{n+1}^k + ax_1^k) - (x_n^k + ax_{n+1}^k)\cdot(x_n^k + ax_1^k)\cdot x_{n+1} \leq 0$$

$$\Leftrightarrow (1+a)\cdot x_n^{k+1}\cdot(x_{n+1}^k + ax_1^k) - (x_n^{2k}\cdot x_{n+1} + ax_n^k\cdot x_{n+1}^{k+1} + ax_n^k\cdot x_1^k\cdot x_{n+1} + a^2 x_1^k\cdot x_{n+1}^{k+1}) \leq 0$$

$$\Leftrightarrow ax_n^k\cdot x_{n+1}^k\cdot(x_n - x_{n+1}) + x_n^{k+1}\cdot x_{n+1}\cdot(x_{n+1}^{k-1} - x_n^{k-1}) + a^2 x_1^k\cdot(x_n^{k+1} - x_{n+1}^{k+1}) + ax_1^k x_n^k(x_n - x_{n+1}) \leq 0$$

We divide by $x_n - x_{n+1} < 0$ and use the factorization :

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^i b^{n-1-i}$$

$$(1) \Leftrightarrow ax_n^k\cdot x_{n+1}^k - x_n^{k+1}\cdot x_{n+1}\cdot\left(\sum_{j=0}^{k-2} x_n^j \cdot x_{n+1}^{k-2-j}\right) + a^2 x_1^k\cdot\left(\sum_{j=0}^{k} x_n^j\cdot x_{n+1}^{k-j}\right) + ax_1^k\cdot x_n^k \geq 0$$

But $x_{n+1} > x_n$ and $a \geq k-1$, so we can write :

$$ax_n^k\cdot x_{n+1}^k - x_n^{k+1}\cdot x_{n+1}\cdot\left(\sum_{j=0}^{k-2} x_n^j\cdot x_{n+1}^{k-2-j}\right) \geq ax_n^k\cdot x_{n+1}^k - x_n^k\cdot x_{n+1}^2\cdot(k-1)\cdot x_{n+1}^{k-2}$$

$$= (a-(k-1))\cdot x_n^k\cdot x_{n+1}^k \geq 0$$

And we have :

$$a^2 x_1^k\cdot\left(\sum_{j=0}^{k} x_n^j\cdot x_{n+1}^{k-j}\right) + ax_1^k\cdot x_n^k \geq 0$$

We proved (1) and we have $C_{k,a}(x) \geq A(x)$ so (H_{n+1}) is proved.
We can conclude that if $a \geq k-1$ then, $\forall n \in \mathbb{N}^*$, $C_{k,a}(x) \geq A(x)$

Question 2 *Prove that if* $0 < a < \frac{k-1}{k+1}$ *and* $n > 1$ *then there exist two n-tuples* x *and* y *of positive real numbers such that :* $C_{k,a}(x) > A(x)$ *but* $C_{k,a}(y) < A(y)$

First we study the case that $n = 2$. We want to find a 2-tuple $y = \{y_1, y_2\}$ such that $C_{k,a}(y) < A(y)$ (2). We choose that $y_1 > y_2$ and $ay_1 < y_2$ (it is possible because $a < \frac{k-1}{k+1} < 1$). ε is the positive real number that verifies $y_1 = y_2 + \varepsilon$

$$(2) \Leftrightarrow \frac{1+a}{2}\left(\frac{y_1^{k+1}}{y_1^k + ay_2^k} + \frac{y_2^{k+1}}{y_2^k + ay_1^k}\right) < \frac{y_1 + y_2}{2}$$

$$\Leftrightarrow \frac{(a+1)y_1^{k+1}}{y_1^k + ay_2^k} - y_1 + \frac{(a+1)y_2^{k+1}}{y_2^k + ay_1^k} - y_2 < 0$$

$$\Leftrightarrow \frac{ay_1^{k+1} + y_1^{k+1} - y_1^{k+1} - ay_1y_2^k}{y_1^k + ay_2^k} + \frac{ay_2^{k+1} + y_2^{k+1} - y_2^{k+1} - ay_2y_1^k}{y_2^k + ay_1^k} < 0$$

$$\Leftrightarrow ay_1(y_1^k - y_2^k)(y_2^k + ay_1^k) + ay_2(y_2^k - y_1^k)(y_1^k + ay_2^k) < 0$$

We divide by $a(y_1^k - y_2^k) > 0$

$$(2) \Leftrightarrow y_1(y_2^k + ay_1^k) - y_2(y_1^k + ay_2^k) < 0$$

$$\Leftrightarrow y_1y_2^k + ay_1^{k+1} - y_2y_1^k + ay_2^{k+1} < 0$$

$$\Leftrightarrow y_2^k(y_1 - ay_2) - y_1^k(y_2 - ay_1) < 0$$

$$\Leftrightarrow y_2^k(y_2 + \varepsilon - ay_2) + y_1^k(ay_2 + a\varepsilon - y_2) < 0$$

$$\Leftrightarrow y_2^k(y_2 + \varepsilon - ay_2) + y_2^k(ay_2 + a\varepsilon - y_2) + (y_1^k - y_2^k)(ay_2 + a\varepsilon - y_2) < 0$$

$$\Leftrightarrow y_2^k\varepsilon(1+a) + (y_1^k - y_2^k)(ay_2 + a\varepsilon - y_2) < 0$$

$$\Leftrightarrow y_2^k\varepsilon(1+a) + (y_1 - y_2)\left(\sum_{i=0}^{k-1} y_1^i y_2^{k-1-i}\right)(ay_1 - y_2) < 0$$

But $y_1 > y_2$ and $ay_1 - y_2 < 0$, so we can write :

$$(y_1 - y_2)\left(\sum_{i=0}^{k-1} y_1^i + y_2^{k-1-i}\right)(ay_1 - y_2) < \varepsilon k y_2^{k-1}(ay_1 - y_2)$$

We want that ε verifies :

$$y_2^k\varepsilon(1+a) + \varepsilon k y_2^{k-1}(ay_1 - y_2) < 0 \quad (3)$$

$$\Leftrightarrow y_2(1+a) + k(ay_1 - y_2) < 0$$

$$\Leftrightarrow y_2(1+a) + k(a(y_2 + \varepsilon) - y_2) < 0$$

$$\Leftrightarrow y_2(1 + a + ka - k) + ka\varepsilon < 0$$

$$\Leftrightarrow \varepsilon < \frac{y_2(k - ka - a - 1))}{ka}$$

So if we choose ε such that it verifies (3), y will verify (2)

But we must show that this choice of ε is compatible with the conditions $y_1 > y_2$ and $ay_1 < y_2$.

$y_1 > y_2 \Leftrightarrow \varepsilon > 0 \Leftarrow \frac{y_2(k-ka-a-1))}{ka} > 0 \Leftrightarrow k - ka - a - 1 > 0 \Leftrightarrow a < \frac{k-1}{k+1}$ so the first condition is verified.

$ay_1 = ay_2 + a\varepsilon < ay_2 + a\frac{y_2(k-ka-a-1))}{ka} = y_2\left(\frac{k-ka-a-1+ak}{k}\right) < y_2$ so the second condition is verified.

So, for $n = 2$ and $0 < a < \frac{k-1}{k+1}$, we can find a n-tuple y such that $C_{k,a}(y) < A(y)$.

Let prove it for all $n \geq 2$. We name Y_n the n-tuple $Y_n = \{y_1, y_2, ..., y_2\}$ (with $n-1$ times the real number y_2), such that Y_2 verifies (2)

$$\begin{aligned}
C_{k,a}(Y_n) = \frac{1+a}{n}\left(\frac{y_1^{k+1}}{y_1^k + ay_2^k} + \frac{y_2^{k+1}}{y_2^k + ay_2^k} + ... + \frac{y_2^{k+1}}{y_2^k + ay_2^k} + \frac{y_2^{k+1}}{y_2^k + ay_1^k}\right) \\
= \frac{1+a}{n}\left(\frac{2}{a+1}C_{k,a}(Y_2) + (n-2)\frac{y_2}{a+1}\right) \\
< \frac{1+a}{n}\left(\frac{2}{a+1}\cdot\frac{y_1+y_2}{2} + (n-2)\frac{y_2}{a+1}\right) \\
= \frac{y_1 + (n-1)y_2}{n} = A(Y_n)
\end{aligned}$$

So for all $n \geq 2$ and $0 < a < \frac{k-1}{k+1}$, we can find a n-tuple y of positive real numbers such that $C_{k,a}(y) < A(y)$.

We will now prove that we can find an other n-tuple x of positive reals numbers such that $C_{k,a}(x) > A(x)$.

First we study the case that $n = 2$, we want to find a 2-tuple $x = \{x_1, x_2\}$ such that $C_{k,a}(x) > A(x)$ (4). We choose that $x_1 > x_2$ and $ax_1 > x_2$. As for we have :

$$\begin{aligned}
(4) \Leftrightarrow \frac{1+a}{2}\left(\frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_1^k}\right) > \frac{x_1 + x_2}{2} \\
\Leftrightarrow x_2^k(x_1 - ax_2) - x_1^k(x_2 - ax_1) > 0 \\
\Leftrightarrow x_2^k(x_1 - ax_2) + x_1^k(ax_1 - x_2) > 0
\end{aligned}$$

But $ax_2 < x_2 < x_1$ and $x_2 < ax_1$ so it's true for all $\{x_1, x_2\}$ with $ax_1 > x_2$.

So, for $n = 2$ and $0 < a < \frac{k-1}{k+1}$, we can find a n-tuple x such that $C_{k,a}(x) > A(x)$.

Lets prove it for all $n \geq 2$. We name X_n the n-tuple $X_n = \{x_1, x_2, ..., x_2\}$ (with $n-1$ times the real number x_2), such that X_2 verifies (4)

$$C_{k,a}(X_n) = \frac{1+a}{n}\left(\frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_2^k} + ... + \frac{x_2^{k+1}}{x_2^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_1^k}\right)$$

$$= \frac{1+a}{n}\left(\frac{2}{a+1}C_{k,a}(X_2) + (n-2)\frac{x_2}{a+1}\right)$$

$$> \frac{1+a}{n}\left(\frac{2}{a+1}\cdot\frac{x_1+x_2}{2} + (n-2)\frac{x_2}{a+1}\right)$$

$$= \frac{x_1+(n-1)x_2}{n} = A(X_n)$$

So for all $n \geq 2$ and $0 < a < \frac{k-1}{k+1}$, we can find a n-tuple x of positive real numbers such that $C_{k,a}(x) > A(x)$.

Question 3 *Investigate the case that* $\frac{k-1}{k+1} \leq a < k-1$

If $n = 1$, then $C_{k,a}(x) = A(x)$ for all n-tuple x of positive real numbers.

We study the case that $n = 2$ and want to prove that, for all 2-tuple $x = \{x_1, x_2\}$ of positive real numbers, we have $C_{k,a}(x) \geq A(x)$. If $x_1 = x_2$, then we have $C_{k,a}(x) = A(x)$.

Now we study the case that $x_1 \neq x_2$. We can choose that $x_1 > x_2$. As for we have :

$$C_{k,a}(x) > A(x)$$

$$\Leftrightarrow \frac{1+a}{2}\left(\frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_1^k}\right) > \frac{x_1+x_2}{2}$$

$$\Leftrightarrow x_2^k(x_1 - ax_2) - x_1^k(x_2 - ax_1) > 0$$

$$\Leftrightarrow ax_1^{k+1} - x_2x_1^k + x_2^kx_1 - ax_2^{k+1} > 0 \quad (5)$$

We fix x_2 to any real number and we will study the function $f(x_1) = ax_1^{k+1} - x_2x_1^k + x_2^kx_1 - ax_2^{k+1}$ with $x_1 \in]x_2; +\infty[$.

$$f'(x_1) = a(k+1)x_1^k - x_2kx_1^{k-1} + x_2^k$$

$$f''(x_1) = a(k+1)kx_1^{k-1} - x_2k(k-1)x_1^{k-2} = kx_1^{k-2}(a(k+1)x_1 - x_2(k-1))$$

But we have $a \geq \frac{k-1}{k+1} \Leftrightarrow a(k+1) \geq (k-1)$ and $x_1 > x_2$, so $a(k+1)x_1 > x_2(k-1)$

We conclude that $\forall x_1 \in]x_2; +\infty[$, we have $f''(x_1) > 0 \Rightarrow f'$ increase strictly on $]x_2; +\infty[$

But $f'(x_2) = a(k+1)x_2^k - kx_2^k + x_2^k = x_2^k(a(k+1) - (k-1)) \geq 0$

So, $\forall x_1 \in]x_2; +\infty[$, we have $f'(x_1) > 0 \Rightarrow f$ increase strictly on $]x_2; +\infty[$

But $f(x_2) = ax_2^{k+1} - x_2^{k+1} + x_2^{k+1} - ax_2^{k+1} = 0$ So, $\forall x_1 \in]x_2; +\infty[$, we have $f(x_1) > 0$ and we proved (5)

In the case that $n = 2$, if $x_1 = x_2$, then $C_{k,a}(x) = A(x)$ and if $x_1 \neq x_2$, then $C_{k,a}(x) > A(x)$.

For $\frac{k-1}{k+1} \leq a < k-1$, we can show that we can not use the same induction as in the first question.

We had to prove $\frac{1+a}{n+1}\left(\frac{x_n^{k+1}}{x_n^k+ax_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k+ax_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k+ax_1^k}\right) \geq \frac{x_{n+1}}{n+1}$ (1) but here with $\frac{k-1}{k+1} \leq a < k-1$, we can find counterexamples whatever the choice of a minimal or maximal number among x_n, x_{n+1} and x_1. We name :

$$H(\{x_n, x_{n+1}, x_1\}) = \frac{x_n^{k+1}}{x_n^k + ax_{n+1}^k} - \frac{x_n^{k+1}}{x_n^k + ax_1^k} + \frac{x_{n+1}^{k+1}}{x_{n+1}^k + ax_1^k} - \frac{x_{n+1}}{a+1}$$

We want 3-tuples $\{x_n, x_{n+1}, x_1\}$ such that $H(\{x_n, x_{n+1}, x_1\}) < 0$. This counterexamples are valid for $k = 3$ with $\frac{k-1}{k+1} \leq a = 0.7 < 1$ and $1 \leq a = 1.5 < k-1$

With $\{x_n, x_{n+1}, x_1\} = \{7, 1, 2\}$
For $a = 0.7$, $H(\{7, 1, 2\}) \simeq -0.339$
For $a = 1.5$, $H(\{7, 1, 2\}) \simeq -0.117$
So we dealt with the cases that $x_n = max(x)$ and $x_{n+1} = min(x)$

With $\{x_n, x_{n+1}, x_1\} = \{4.5, 5, 1\}$
For $a = 0.7$, $H(\{4.5, 5, 1\}) \simeq -0.139$
For $a = 1.5$, $H(\{4.5, 5, 1\}) \simeq -0.015$
So we dealt with the cases that $x_{n+1} = max(x)$ and $x_1 = min(x)$

With $\{x_n, x_{n+1}, x_1\} = \{4, 5, 6\}$
For $a = 0.7$, $H(\{4, 5, 6\}) \simeq -0.178$
For $a = 1.5$, $H(\{4, 5, 6\}) \simeq -0.250$
So we dealt with the cases that $x_1 = max(x)$ and $x_n = min(x)$

But we must find examples such that $H(\{x_n, x_{n+1}, x_1\}) > 0$, too.

With $\{x_n, x_{n+1}, x_1\} = \{3, 5, 4\}$
For $a = 0.7$, $H(\{3, 5, 4\}) \simeq 0.319$
For $a = 1.5$, $H(\{3, 5, 4\}) \simeq 0.547$
So we dealt with the cases that $x_{n+1} = max(x)$ and $x_n = min(x)$

With $\{x_n, x_{n+1}, x_1\} = \{7, 5, 2\}$
For $a = 0.7$, $H(\{7, 5, 2\}) \simeq 0.534$
For $a = 1.5$, $H(\{7, 5, 2\}) \simeq 0.325$
So we dealt with the cases that $x_n = max(x)$ and $x_1 = min(x)$

With $\{x_n, x_{n+1}, x_1\} = \{4, 2, 6\}$
For $a = 0.7$, $H(\{4, 2, 6\}) \simeq 1.957$
For $a = 1.5$, $H(\{4, 2, 6\}) \simeq 1.413$
So we dealt with the cases that $x_1 = max(x)$ and $x_{n+1} = min(x)$

We conclude that we can not prove (1) or the contrary of (1) with $\frac{k-1}{k+1} \leq a < k-1$, and so we can not use a similar induction as in the first question.