Problem 9. **A Topological Problem**

France 1 — ITYM 2010

We'll start this problem by some considerations on topology, giving various results useful in the sequel. Then, we'll begin building up theorems that show equality of different operations on sets of $\mathbf{R}^n$. Perhaps the most important remark is that all operations are idempotent: the closure is closed, the interior is open, the convex hull is convex.

This way, there will be points were the questions can then be easily answered, introducing some breaks in the construction of the final results on numbers of sets we can achieve by the various operations we use.

Finally, we propose a generalization for including a further operation: the complement of a set. In this case, the main duality we get is whether $\text{conv}(A)$ or $\text{conv}(\text{comp}(A)) = \mathbf{R}^2$, the other model situation being "two complementary half-planes".

The main results are:

- In any dimension, interior and closure generate at most 7 sets; this limit is already atteined in dimension 1.
- In dimensions $n \geq 2$, interior, closure and convex hull generate at most 17 sets; this limit is also attained in dimension 2.
- The case $\text{int}(\text{conv}(A))$ is somewhat degenerated: we prove that this implies that the set is contained in a proper affine subset. This dramatically reduces all possibilities to 5 in dimension 2 and 6 in greater dimensions.
- The generalization gives at most 54 possible sets.

First remarks

We'll prove various propositions that establish relations among the operations we can do to sets. These will be very useful in the arguments that are used in answers.

The closure of a set A is closed, and the interior of a set A is open. Next, closure, interior and convex hull are monotone operations: if $X \subset Y$, then $Op(X) \subset Op(Y)$ for any of them as Op.

Then, we can use other definitions of closure and convex hull:

- The closure of a set S is the set of all the points x such that, for all $r > 0$, the open ball $B_r(x)$ contains at least one point of S.
- The closure of S is also the set of all the points x such that there is a sequence of points of S whose limit is x.
- The convex hull of a set $S \subset \mathbf{R}^n$ is the set of barycenters of $n+1$ points of S with nonnegative coefficients.

Theorem 1 (in $\mathbf{R}^n$). *For $A \subset \mathbf{R}^n$,*

1. *$int(cl(int(cl(A)))) = int(cl(A))$ and*
2. *$cl(int(cl(int(A)))) = cl(int(A))$.*

Proof.

$$\begin{aligned} \text{According to the definition,} \qquad & \text{int}(\text{cl}(A)) \subset \text{cl}(A), \\ \text{So} \qquad & \text{cl}(\text{int}(\text{cl}(A))) \subset \text{cl}(\text{cl}(A)) = \text{cl}(A), \\ \text{and then} \qquad & \text{int}(\text{cl}(\text{int}(\text{cl}(A)))) \subset \text{int}(\text{cl}(A)). \end{aligned}$$

Furthermore, $\text{int}(\text{cl}(A)) \subset \text{cl}(\text{int}(\text{cl}(A)))$, and as $\text{int}(\text{cl}(A))$ is open, and from the definiti1Gon of the interior $\text{int}(\text{cl}(A)) \subset \text{int}(\text{cl}(\text{int}(\text{cl}(A))))$, which implies the equality in 1.

For the second one, we use complements. Let $B = \mathbf{R}^n - A$. By the above result,

$$\begin{aligned} & \text{int}(\text{cl}(\text{int}(\text{cl}(B)))) = \text{int}(\text{cl}(B)), \\ \text{so} \qquad & \mathbf{R}^n - \text{int}(\text{cl}(\text{int}(\text{cl}(B)))) = \mathbf{R}^n - \text{int}(\text{cl}(B)), \end{aligned}$$

and finally, using repeatedly $\mathbf{R}^n - \text{int}(A) = \text{cl}(\mathbf{R}^n - A)$ to swap complements and (interior / closure), we get

$$\text{cl}(\text{int}(\text{cl}(\text{int}(A)))) = \text{cl}(\text{int}(A)).$$

□

Question I

Let $A = (-\infty, -2) \cup (-2; -1) \cup ([-1; 1) \cap \mathbf{Q}) \cup \{2\}$. Using repeatedly int and cl, and as $\mathrm{int} \circ \mathrm{int} = \mathrm{int}$ and $\mathrm{cl} \circ \mathrm{cl} = \mathrm{cl}$, we get

$$\begin{aligned} \mathrm{int}(A) &= (-\infty, -2) \cup (-2; -1) & \mathrm{cl}(A) &= (-\infty, 1] \cup \{2\} \\ \mathrm{cl}(\mathrm{int}(A)) &= (-\infty, -1] & \mathrm{int}(\mathrm{cl}(A)) &= (-\infty, 1) \\ \mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) &= (-\infty, -1) & \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) &= (-\infty, 1] \end{aligned}$$

in total 7 distinct sets with A.

According to Theorem 1, there is no possible additional set, for whichever A we choose. The answer is then 7.

More Theorems: the convex hull comes in

Theorem 2 (in $\mathbf{R}^n$, Case $\mathrm{int}(\mathrm{conv}(A)) \neq \emptyset$)**.** *For $A \subset \mathbf{R}^n$,*

1. *$int(cl(conv(A))) = int(conv(A))$ and*
2. *$cl(int(conv(A))) = cl(conv(A))$.*

For this, we state a very useful

Lemma 1. *Let C be a convex set. Let $x \in cl(C)$ and $y \in int(C)$. Then, $(x, y] \subset int(C)$.*

Proof. We will first prove that $(x, y] \subset C$.

Let $D = \mathrm{dist}(x, y)$. We have $y \in \mathrm{int}(C)$, so there is $R > 0$ such that the open ball $B_R(y)$ is in C. Additionally, $x \in \mathrm{cl}(C)$, so for all $r' > 0$, there is a point $z' \in C$ such that $d(x, z') < r'$.

We have to prove that, for all $r > 0$, there is a point $z \in [x, y]$, in C and such that $d(x, z) < r$. As C is convex, the segment $[z, y]$ will then be included in C, and taking the union over all such z, we get the half-open interval $(x, y]$.

For all z', let t be the point of $B_R(y)$ such that

$$\frac{d(y,t)}{R} = \frac{d(x,z')}{r'} \qquad \text{and} \qquad \widehat{xyt} = -\widehat{z'xy}.$$

By convexity of C, we have $[z't] \subset C$. According to the intercept theorem, $[z', t]$ and $[x, y]$ intersect in z such that

$$\frac{d(z,x)}{d(z,y)} = \frac{d(z',x)}{d(t,y)} = \frac{r'}{R}.$$

But $D = d(x, y) = d(x, z) + d(z, y)$, so we get

$$d(x,z) = \frac{Dr'}{R + r'},$$

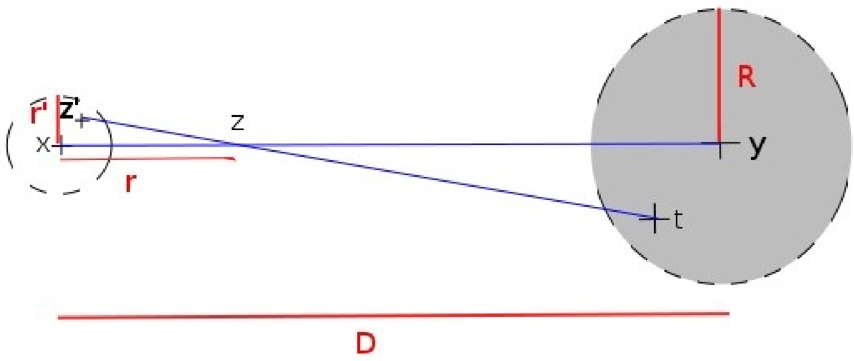

Figure 1: Construction of t and z from z' for lemma 2

so that, when r' goes to zero, we have points in $[x, y]$ arbitrarily near x, which shows that $(x, y] \subset C$.

Now, for each $z \in (x, y]$, consider the cone C_z whose base is $B_R(y)$ and vertex z. As C is convex, $C_z \subset C$. Then all interior points of C_z are also in the interior of C. In particular, all points in $(z, y]$. But for every z in $(x, y]$, there is another $zz \in (x, y]$ such that $z \in (zz, y]$. So, considering the cone C_{zz}, we have $z \in \text{int}(C)$. $\square$

Proof of the theorem. (1) Let $C = \text{conv}(A)$, hence C is convex. We have $C \subset \text{cl}(C)$, then $\text{int}(C) \subset \text{int}(\text{cl}(C))$, which is the easy inclusion.

We still have to show that $\text{int}(\text{cl}(C)) \subset \text{int}(C)$. Let $x \in \text{int}(\text{cl}(C))$ and $y \in \text{int}(C)$. These two sets are indeed non-empty: the last one by hypothesis, and the second one because $C \subset \text{cl}(C) \Rightarrow \text{int}(C) \subset \text{int}(\text{cl}(C))$.

As $x \in \text{int}(\text{cl}(C))$, there is an $r > 0$ such that we $B_r(x) \subset \text{cl}(C)$. Let now $z \in \overrightarrow{y, x}$ (the half-line starting from y passing through x) such that $d(y, z) = d(y, x) + (r/2)$. We have $z \in \text{cl}(C)$, $y \in \text{int}(C)$ and $x \in (z, y]$. Therefore, according to the previous lemma, $x \in \text{int}(C)$.

(2) It is enough to show that $\text{cl}(\text{conv}(A)) \subset \text{cl}(\text{int}(\text{conv}(A)))$, the other inclusion following from $\text{int}(X) \subset X$.

Write as before $C = \text{conv}(A)$ and let $x \in \text{cl}(C)$ and $y \in \text{int}(C)$ be two points. According to the lemma, $(x, y] \subset \text{int}(C)$, which implies $x \in \text{cl}(\text{int}(C))$. $\square$

Question 2.a)

The result we need is exactly part (2) of the following

Theorem 3. *For all $A \subset \mathbf{R}^n$,*

1. *$cl(conv(cl(A))) = cl(conv(A))$ and*
2. *if $int(conv(A)) \neq \emptyset$, then $int(conv(cl(A))) = int(conv(A))$.*

Proof. (1) We will prove that $\mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A))) \subset \mathrm{cl}(\mathrm{conv}(A))$, the reciprocal inclusion being obvious. For the sake of notational simplicity, we write the proof for $\mathbf{R}^2$; it is easy to adapt it to $\mathbf{R}^n$. In fact, we just have to use $n+1$ points for the barycenters.

Let x be a point of $\mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A)))$. (The case where $A = \emptyset$ is trivial) By definition, x is the limit of $x_1, x_2, \ldots$, who are themselves (weighted) barycenters of, respectively,

$$[(l_1; \alpha_1), (m_1; \beta_1), (n_1; \gamma_1)] \quad [(l_2; \alpha_2), (m_2; \beta_2), (n_2; \gamma_2)] \quad \ldots$$

with all the α_i, β_i and γ_i nonnegative reals such that $\alpha_i + \beta_i + \gamma_i = 1$ and l_i, m_i, n_i in $\mathrm{cl}(A)$.

Thus, we can find sequences $l_{i,n}$, $m_{i,n}$ and $n_{i,n}$ of points in A converging to l_i, m_i, n_i. We can even demand that

$$d(l_i, l_{i,n}) < \frac{1}{n}.$$

The important point is not the exact speed of convergence of the double-sequences, but rather that they all converge with at least the same speed.

Let now x'_1 be the weighted barycenter of $(l_{1,1}; \alpha_1)$, $(m_{1,1}; \beta_1)$ and $(n_{1,1}; \gamma_1)$; x'_2 of $(l_{2,2}; \alpha_2)$, $(m_{2,2}; \beta_2)$ and $(n_{2,2}; \gamma_2)$; and so on. All the x'_i are in $\mathrm{conv}(A)$, so all we still have to prove is that x is also the limit of the sequence x'_i.

$$\begin{aligned}
\text{But} \qquad & x_i - x'_i = \alpha_i(l_i - l_{i,i}) + \beta_i(m_i - m_{i,i}) + \gamma_i(n_i - n_{i,i}) \\
\text{so} \qquad & \|x_i - x'_i\| \leq \frac{\alpha_i + \beta_i + \gamma_i}{i} = \frac{1}{i} \\
& d(x, x'_i) \leq d(x, x_i) + d(x_i, x'_i) \to_{i \to \infty} 0
\end{aligned}$$

because $x_i \to x \Rightarrow d(x, x_i) \to 0$.

(2)

$$\begin{aligned}
\text{According to the previous part,} \qquad & \mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{conv}(A)), \\
\text{so} \qquad & \mathrm{int}(\mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(\mathrm{conv}(A))). \\
\text{Now, by Theorem 2,} \qquad & \mathrm{int}(\mathrm{conv}(\mathrm{cl}(A))) = \mathrm{int}(\mathrm{conv}(A)).
\end{aligned}$$

□

Three theorems

We start with two results showing that the operations are somewhat independent as they respect some properties of one another.

Theorem 4. *If C is a convex set of $\mathbf{R}^n$,*

- *$int(C)$ is convex, and*
- *$cl(C)$ is convex.*

Proof. (1) Let x and y be two points of $\text{int}(C)$, so there is $r > 0$ such that the open balls $B_r(x)$ and $B_r(y)$ are contained in C. Let z be the weighted barycenter of (x, α) and (y, β), for α and β nonnegative numbers.

For all vectors $\vec{u}$ whose norm is smaller than r, the points x' and y', images of x and y by the $\vec{u}$-vector translation, are in C. As C is convex, $[x', y']$ is in C. So z', image of z, is the barycenter of (x', α) and (y', β), which gives $z' \in C$.

All this implies that the open ball $B_r(z)$ is in C, therefore $z \in \text{int}(C)$.

(2) Let x be the limit of the sequence $x_1, x_2, x_3, \ldots$, and y, limit of $y_1, y_2, y_3, \ldots$, be two points of $\text{cl}(C)$ (x_i and $y_i \in C$). Let $z \in [x, y]$; it is a barycenter of (x, α) and (y, β) with α and β nonnegative numbers. Therefore, if z_i is the barycenter of (x_i, α) and (y_i, β), by convexity of C, $z_i \in C$, and z, being the limit of z_i, is in $\text{cl}(C)$. □

Theorem 5. *For $A \subset \mathbf{R}^n$, $int(conv(int(A))) = conv(int(A))$. Or, without symbols: if O is open, $conv(O)$ is open.*

Proof. Let x be the barycenter of $[(l; \alpha), (m; \beta), (n; \gamma)]$, with α, β and γ three nonnegative numbers and with l, m and n in $O = \text{int}(A)$. As l, m and n are in an open set, there is $r > 0$ such that the open balls $B_r(l)$, $B_r(m)$ and $B_r(n)$ are in $\text{int}(A)$. For all vectors $\vec{u}$ whose norm is less than r, let l', m' and n' be the images of l, m and n by the $\vec{u}$-vector translation. x', barycenter of $[(l', \alpha)(m', \beta)(n', \gamma)]$ is thus in $\text{conv}(\text{int}(A))$, and is the image of x by the $\vec{u}$-translation.

This shows that the open ball $B_r(x)$ is in $\text{conv}(\text{int}(A))$, which implies the equality. □

Theorem 6. *Let $A \subset \mathbf{R}^n$. If $int(conv(A)) \neq \emptyset$, then $conv(int(cl(int(A)))) = conv(int(A))$.*

Proof. Let $X = \text{cl}(\text{int}(A))$.

From theorem 5	$\text{conv}(\text{int}(X)) = \text{int}(\text{conv}(\text{int}(X)))$.
By theorem 3 with $\text{int}(X)$	$\text{int}(\text{conv}(\text{int}(X))) = \text{int}(\text{conv}(\text{cl}(\text{int}(X))))$.
By theorem 1, $\text{cl}(\text{int}(X)) = X$, so	$\text{int}(\text{conv}(\text{cl}(\text{int}(X)))) = \text{int}(\text{conv}(X))$.

Unpacking X and using theorem 3 the other way around, we have $\text{int}(\text{conv}(\text{cl}(\text{int}(A)))) = \text{int}(\text{conv}(\text{int}(A)))$, which is equal to $\text{conv}(\text{int}(A))$ by theorem 5 again. □

Summary of theorems

Theorem 1: $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) = \mathrm{int}(\mathrm{cl}(A))$ and $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) = \mathrm{cl}(\mathrm{int}(A))$.
Theorem 2: $\mathrm{int}(\mathrm{cl}(\mathrm{conv}(A))) = \mathrm{int}(\mathrm{conv}(A))$ and $\mathrm{cl}(\mathrm{int}(\mathrm{conv}(A))) = \mathrm{cl}(\mathrm{conv}(A))$.
Theorem 3: $\mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{conv}(A))$ and $\mathrm{int}(\mathrm{conv}(\mathrm{cl}(A))) = \mathrm{int}(\mathrm{conv}(A))$.
Theorem 4: $\mathrm{conv}(\mathrm{int}(\mathrm{conv}(A))) = \mathrm{int}(\mathrm{conv}(A))$ and $\mathrm{conv}(\mathrm{cl}(\mathrm{conv}(A))) = \mathrm{cl}(\mathrm{conv}(A))$.
Theorem 5: $\mathrm{int}(\mathrm{conv}(\mathrm{int}(A))) = \mathrm{conv}(\mathrm{int}(A))$.
Theorem 6: $\mathrm{conv}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) = \mathrm{conv}(\mathrm{int}(A))$.

Question 2.b)

Case where int(conv(A)) $\neq \emptyset$

To make the greatest possible number of different sets, we must take pieces which change a lot with the operations. So, we'll take closed, open, and dense-but-with-no-interior sets, for open and closed. Then, for the convex case, we take a line (which also makes $\mathrm{cl}(\mathrm{conv}(A)) \neq \mathrm{conv}(\mathrm{cl}(A))$ in many situations), and a point far from it, who disappears in the interior, but remains in the closure.

For instance, let A be the union of disjoint sets T, U, V and W.

T is the set which contains only the point with coordinates $(0, 2)$;

U contains all the points of ordinate between -1 and 1, with the exception of the origin;

V is the open disc with center on $(0, -4)$ and radius 1;

W contains all the points with abscissa between -1 and 1, ordinates between -6 and -8 (-8 being excluded), and with rational coordinates.

It is a matter of time to verify that we can obtain 17 distinct sets from set A. Furthermore, from the diagram at the end, the 6 previous theorems limit the generation of sets by the 3 operations at a maximum of 17 sets.

The case for int(conv(A)) $= \emptyset$

If $\mathrm{conv}(A)$ has no interior points, it must be contained in a line. Indeed, if it contains other points that those of a single line, as it is convex, it contains a triangle, and its interior. But A is contained in $\mathrm{conv}(A)$, so it is also in an affine line of $\mathbf{R}^2$.

Theorem 7 (A in a line identified to $\mathbf{R}$)**.** *If $A \subset \mathbf{R}$, $\mathrm{cl}(\mathrm{conv}(A)) = \mathrm{conv}(\mathrm{cl}(A))$.*

Proof. According to theorem 3, $\mathrm{cl}(\mathrm{conv}(A)) = \mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A)))$. We will then prove that $\mathrm{conv}(\mathrm{cl}(A))$ is closed.

$\mathrm{conv}(\mathrm{cl}(A))$ is an interval. If it has an open and finite bound x, x cannot be in $\mathrm{cl}(A)$. Without loss of generality, we can assume x is a lower bound for $\mathrm{conv}(\mathrm{cl}(A))$. Then, for all $r \in \mathbf{R}^+$, there is one point y_r of $(x, x + r)$ in $\mathrm{conv}(\mathrm{cl}(A))$, and therefore one point z_r of $\mathrm{cl}(A)$ in $(x, y_r]$. But then, as r goes to zero, z_r goes to x, and so we have that $\mathrm{cl}(A)$ is not closed, which gives a contradiction. □

If three points or more in A are not in the same line, the triangle which has three corners among them is in $\mathrm{conv}(A)$, and then $\mathrm{int}(\mathrm{conv}(A)) \neq \emptyset$. So, in case $\mathrm{int}(\mathrm{conv}(A)) = \emptyset$, all the points of A are in the same line.

Now let A be the following subset of the abscissa line. Points in A will be characterized by their abscissa.

$$\begin{aligned} A &= \{0\} \cup (1,2) \\ \mathrm{int}(A) &= \emptyset \\ \mathrm{cl}(A) &= \{0\} \cup [1,2] \\ \mathrm{conv}(A) &= [0,2) \\ \mathrm{cl}(\mathrm{conv}(A)) &= [0;2] \end{aligned}$$

Moreover, for all A we have

$$\begin{aligned} \mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{conv}(A)) = \mathrm{int}(\mathrm{cl}(\mathrm{conv}(A))) &= \emptyset = \mathrm{int}(A) \\ \mathrm{conv}(\mathrm{cl}(A)) &= \mathrm{cl}(\mathrm{conv}(A)) \end{aligned}$$

The two last lines imply there's no possible additional sets. The answer is then 5, from set A as above.

Question 2.b) generalized to $\mathbf{R}^n$

Case 1: int(conv(A)) $\neq \emptyset$

Put A_2 the subset of $\mathbf{R}^2$ which generates 17 different subsets by the three operations. In $n+1$ dimensions, define the set A_{n+1} as the one which contains all the lines perpendicular in the n previous dimensions, and which contains one point of A_n.

The 17 sets are still different, as the product by $\mathbf{R}$ "factors" outside the operations. Therefore, 17 is the answer for all two-or-more dimension.

Case 2: int(conv(A)) $= \emptyset$

Like in the Question 2.b), the set A will be a subset of a $(n-1)$-dimensional affine subspace.

In 3 dimensions, choose the set B containing a line and a parallel open interval. We effectively have B, $\mathrm{conv}(B)$, $\mathrm{cl}(B)$, $\mathrm{int}(B)$, $\mathrm{conv}(\mathrm{cl}(B))$ and $\mathrm{cl}(\mathrm{conv}(B))$ all distinct. Moreover, $\mathrm{cl}(\mathrm{conv}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{conv}(A))$ (Theorem 3), $\mathrm{conv}(\mathrm{cl}(\mathrm{conv}(A))) = \mathrm{cl}(\mathrm{conv}(A))$ (Theorem 4) and $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{conv}(A)) = \mathrm{int}(\mathrm{cl}(\mathrm{conv}(A))) = \mathrm{int}(\mathrm{conv}(\mathrm{cl}(A))) = \emptyset = \mathrm{int}(A)$. So we get 6 different sets.

For $n > 3$ dimensions, the recurrence is the same as the case $\mathrm{int}(\mathrm{conv}(A)) \neq \emptyset$.

Generalization

We will study the previous problem in R^2 with one additional function: the complement (denoted comp()).

We first look at these two sets: $\text{conv}(A)$ and $\text{conv}(\text{comp}(A))$. Let's make a definition: a set $A \subset \mathbf{R}^2$ is a **quasi-half plane** if its contained in a closed half plane and its interior is an open half plane. Then we have

Theorem 8. *If $conv(A) \neq \mathbf{R}^2$ and $conv(comp(A)) \neq \mathbf{R}^2$, then all of them are quasi-half planes.*

Proof. As $A \cup \text{comp}(A) = \mathbf{R}^2$, for each direction and orientation, there is a sequence which goes to infinity in this direction, either contained in A or in $\text{comp}(A)$.

If only two opposite orientations in a single direction admit infinite-going sequences in the same set, the convex hull of its complement is the plane.

If the two opposite orientations in a direction and a third admit infinite-going sequences in the same set, then its convex hull contains an open half plane.

Then, if, in all directions, one orientation admits infinite-going sequences in each set, it's obvious that the two convex hulls are the hole plane or two quasi-half planes. □

Some equalities with complements

Let's study the possibilities for $\text{conv}(A)$ and their relations to other operations.

If $\text{conv}(A)$ is the all plane, $\text{comp}(\text{conv}(A)) = \emptyset$. More, $\text{cl}(\emptyset) = \text{int}(\emptyset) = \text{conv}(\emptyset) = \emptyset$ and $\text{comp}(\emptyset) = \mathbf{R}^2$.

If $\text{conv}(A)$ is a quasi-half plane, because it is convex, the intersection with its border can be nothing (when $\text{conv}(A)$ is a open half plane), a segment, a half line or a line (in which case $\text{conv}(A)$ is a closed half plane). Then $\text{conv}(\text{comp}(\text{conv}(A)))$ is an opposite quasi-half plane, which can be closed (which implies $\text{conv}(\text{comp}(\text{conv}(A))) = \text{cl}(\text{comp}(\text{conv}(A)))$), open, or contains the complementary half line, in which case $\text{conv}(\text{comp}(\text{conv}(A))) = \text{comp}(\text{conv}(A)))$.

Otherwise, $\text{conv}(\text{comp}(\text{conv}(A)))$ is the plane, and $\text{comp}(\text{conv}(\text{comp}(\text{conv}(A)))) = \emptyset$.

Then, if $\text{conv}(A)$ is a half plane, $\text{comp}(\text{int}(\text{conv}(A)))$ and $\text{comp}(\text{cl}(\text{conv}(A)))$ are convex.

Finally, if $\text{conv}(\text{comp}(\text{conv}(A))) = \mathbf{R}^2$,

$$\mathbf{R}^2 = \text{conv}(\text{comp}(\text{int}(\text{conv}(A)))) = \text{conv}(\text{comp}(\text{cl}(\text{conv}(A)))).$$

Answer to the generalization

From the definition of open and closed, we have $\text{int}(\text{comp}(A)) = \text{comp}(\text{cl}(A))$ and $\text{cl}(\text{comp}(A)) = \text{comp}(\text{int}(A))$.

Start with A giving us 17 sets. We have to search, for each B of the 17 previous sets, what sets we can add passing to the complement.

We will first study the case when $\text{conv}(A)$ is a quasi half plane.

If $B = A$, then:

$$\begin{array}{rr} \text{comp}(B) & \\ \text{conv}(\text{comp}(B)) & \text{comp}(\text{conv}(\text{comp}(B))) \\ \text{int}(\text{conv}(\text{comp}(B))) & \text{comp}(\text{int}(\text{conv}(\text{comp}(B)))) \\ \text{cl}(\text{conv}(\text{comp}(B))) & \text{comp}(\text{cl}(\text{conv}(\text{comp}(B)))) \end{array}$$

Now, if $B = \text{int}(A)$ or $B = \text{int}(\text{cl}(A))$ or $B = \text{int}(\text{cl}(\text{int}(A)))$, because $\text{comp}(\text{int}(B')) = \text{cl}(\text{comp}(B'))$ and because of theorem 3, we get the different sets

$$\begin{array}{rr} \text{comp}(B) & \\ \text{conv}(\text{comp}(B)) & \text{comp}(\text{conv}(\text{comp}(B))) \end{array}$$

Next, if $B = \text{cl}(A)$ or $B = \text{cl}(\text{int}(A))$, because $\text{comp}(\text{cl}(B')) = \text{int}(\text{comp}(B'))$ and because of theorem 5,

$$\begin{array}{rr} \text{comp}(B) & \\ \text{conv}(\text{comp}(B)) & \text{comp}(\text{conv}(\text{comp}(B))) \\ \text{cl}(\text{conv}(\text{comp}(B))) & \text{comp}(\text{cl}(\text{conv}(\text{comp}(B)))) \end{array}$$

If $B = \text{cl}(\text{int}(\text{cl}(A))), \text{comp}(B) = \text{int}(\text{cl}(\text{int}(\text{comp}(A))))$. Then, by Theorem 6, we have $\text{conv}(\text{comp}(B)) = \text{conv}(\text{int}(\text{comp}(A)))$. The only set the new operation add is $\text{comp}(B)$.

In the other cases, B is convex, and the only set the new operation adds is $\text{comp}(B)$.

So we have $17 + 7 + 3 \times 3 + 2 \times 5 + 1 + 10 = 54$ sets at most.

With A the following set, we effectively have the 54 sets.

Let A be the set such that, if $y \neq 0$, $(x, y) \in A \Leftrightarrow (x, -y) \notin A$ (A contains the abscissa line, and is anti-symetric w.r.t. it) and which contains the sets:

T All the points whose ordinates are between 0 and 1, and abscissas between -1 and 1.

U The points whose ordinate is in $[1, 2]$ and abscissa in $[-1, 1]$ and with rational coordinates.

V The open segment (x, y), with $x = (-1, 3)$ and $y = (-1, 4)$

W The closed segment $[x', y']$ with $x' = (1, 3)$ and $y' = (1, 4)$

And nothing else in positive half plan.

In case $\text{conv}(A)$ is not a quasi half plane, as A and $\text{comp}(A)$ produce the same sets, we will suppose $\text{conv}(\text{comp}(A)) = \mathbf{R}^2$.

Then, for all such A, we only can add, for $B = A$:

$\text{comp}(B)$

$\text{conv}(\text{comp}(B)) = \mathbf{R}^2$

$\text{comp}(\text{conv}(\text{comp}(B))) = \emptyset$

If $\mathrm{conv}(\mathrm{comp}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))))) = \mathbf{R}^2$ or $\mathrm{conv}(\mathrm{comp}(\mathrm{conv}(B'))) = \mathbf{R}^2$, we can add two sets ($\mathbf{R}^2$ and $\emptyset$). But we can only count these as new once. Indeed, if, for any other B, $\mathrm{conv}(\mathrm{comp}(B)) = \mathbf{R}^2$, that removes at least 2 from the number of different sets.

Then, as

$$\begin{aligned} \mathrm{int}(\mathrm{cl}(A)) &\subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))), \\ \mathrm{comp}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) &\subset \mathrm{comp}(\mathrm{int}(\mathrm{cl}(A))), \\ \mathbf{R}^2 = \mathrm{conv}(\mathrm{comp}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))))) &\subset \mathrm{conv}(\mathrm{comp}(\mathrm{int}(\mathrm{cl}(A)))) \end{aligned}$$

Similarly, if $\mathrm{conv}(\mathrm{comp}(\mathrm{conv}(B'))) = \mathbf{R}^2$,

$$\mathbf{R}^2 = \mathrm{conv}(\mathrm{comp}(\mathrm{conv}(B'))) \subset \mathrm{conv}(\mathrm{comp}(B')).$$

Then, 54 is the global answer.

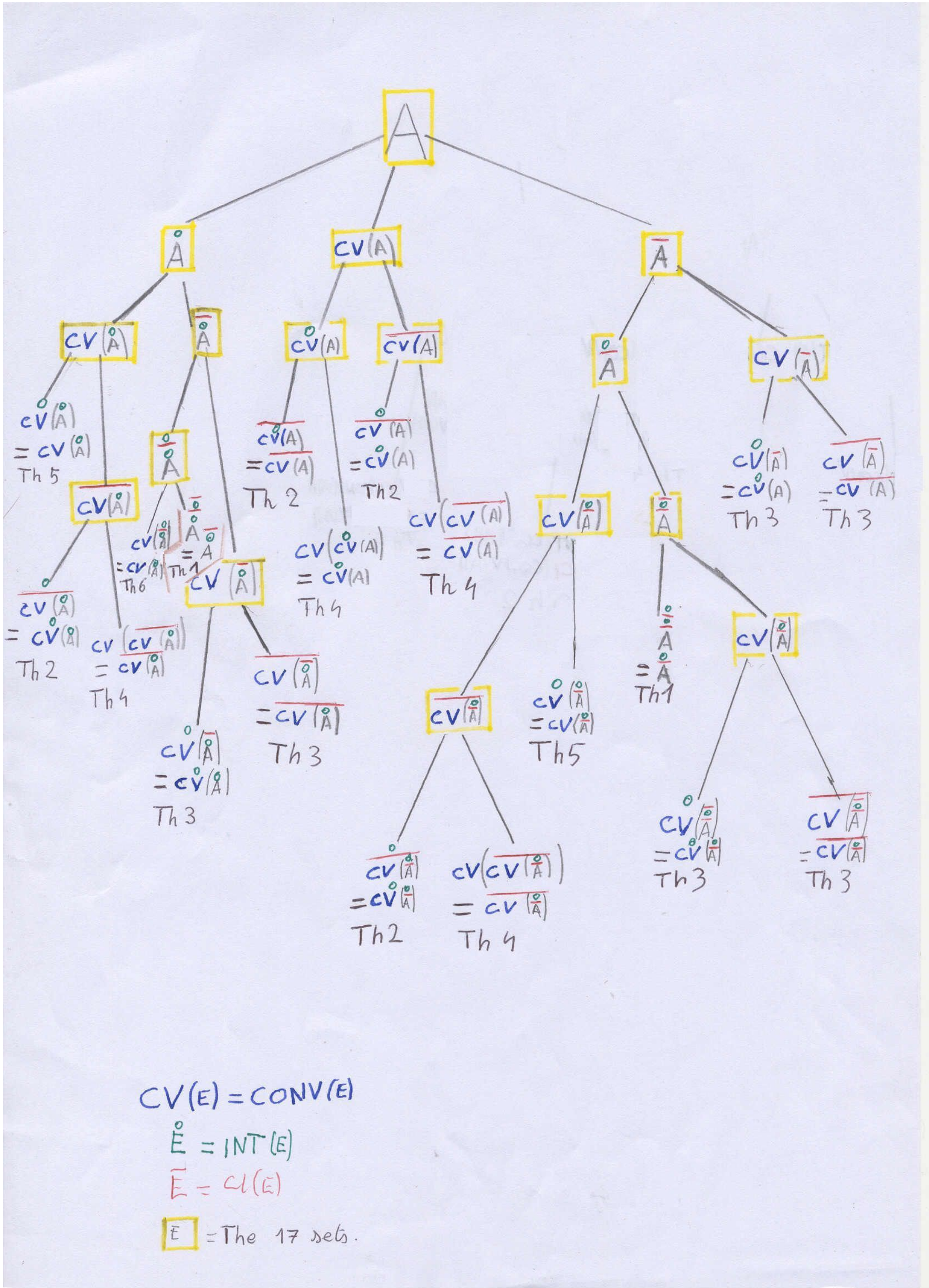

Figure 2: Diagram showing no more than 17 different sets can exist in any dimension