# Problem 3. **A Cyclic Inequality**

France 1 — ITYM 2010

**Abstract**

In this problem we exploit a very useful technique to answer the different questions: instead of considering the inequality in its globality, we will prove a stronger inequality on each term. More precisely, we will find two real numbers $\alpha$ and $\beta$ such that $\alpha+\beta = \frac{1}{a+1}$ and we will compare $\frac{x_i^{k+1}}{x_i^k+ax_{i+1}^k}$ to $\alpha x_i + \beta x_{i+1}$. In so doing, we can more easily "estimate-telescope" one side of the sum and compare to the arithmetic mean. Note that with $\lambda = \frac{x_{i+1}}{x_i}$, this becomes a problem in 1 variable: we have to compare $\frac{1}{1+a\lambda^k}$ and $\alpha + \beta\lambda$.

We completely solved questions 1 and 2.

In the case of the first question, we will prove that $\frac{x_i^{k+1}}{x_i^k+ax_{i+1}^k} \geq \alpha x_i + \beta x_{i+1}$ for some $\alpha$ and $\beta$ satisfying the conditions above. Then we just have to sum these inequalities to conclude.

In the second question, we will easily find $x$ such that $C_{k,a}(x) > A(x)$. To find $y$ such that $C_{k,a}(y) < A(y)$, we will use again the technique of $\alpha$ and $\beta$ : we will find x such that $\frac{x_i^{k+1}}{x_i^k+ax_{i+1}^k} = \alpha x_i + \beta x_{i+1}$ for some i and $\frac{x_i^{k+1}}{x_i^k+ax_{i+1}^k} < \alpha x_i + \beta x_{i+1}$ for some others i.

In the third question, we will prove that, if $n$ is sufficiently large, then it exists $x$ such that $A(x) > C_{a,k}(x)$. The sequence $(x_i)$ we choosed is a geometric progression with ratio near to 1.

Finally, in the fourth question, similarly as in the first question, we find $\alpha$ and $\beta$ such that $\alpha + \beta = \frac{1}{a+1}$ and we compare $\frac{1}{1+a\lambda^k}$ and $\alpha + \beta\lambda^l$. Note that with $y = \lambda^l$, the situation is similar as in question 1. We proved that if $a > \frac{k}{l} - 1$ then $C_{k,l,a}(x) \geq A(x)^l$, $\forall x$, and a partial analogue to question 2.

## Introduction

## 1st question: the case where $a \geq k-1$

Here, we must prove that, for any $n$, and for any $n$-tuple of positive real numbers,

$$\frac{x_1 + x_2 + \cdots + x_n}{1+a} \leq \left( \frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_3^k} + \ldots + \frac{x_n^{k+1}}{x_n^k + ax_1^k} \right). \qquad (1)$$

Let's search two real numbers $\alpha$ and $\beta$ such that

$$\begin{cases} \alpha + \beta = \frac{1}{a+1} \\ \dfrac{x_i^{k+1}}{x_i^k + ax_{i+1}^k} \geq \alpha x_i + \beta x_{i+1} \quad \forall i \end{cases}$$

Indeed, if two such real numbers exist, then we get, summing the inequality above on $i$:

$$\frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_3^k} + \ldots + \frac{x_n^{k+1}}{x_n^k + ax_1^k} \geq (\alpha + \beta)(x_1 + x_2 + \ldots + x_n)$$

which, as $\alpha + \beta = \frac{1}{a+1}$, implies the inequality 1.

Moreover, the conditions on $\alpha$ and $\beta$ do not really depend on $x_i$, but rather on the ratio $\lambda = \frac{x_{i+1}}{x_i}$:

$$\frac{x_i^{k+1}}{x_i^k + ax_{i+1}^k} \geq \alpha x_i + \beta x_{i+1} \Leftrightarrow \frac{x_i^k x_i}{x_i^k(1 + a\frac{x_{i+1}^k}{x_i^k})} \geq \alpha x_i + \beta x_{i+1}$$

simplifying $x_i > 0$:

$$\Leftrightarrow \frac{1}{1 + a\lambda^k} \geq \alpha + \beta\lambda$$

So, it remains to prove the following

**Lemma 1.** *For all $a > 0$, $k \in \mathbb{N}$, there exist constants $\alpha$ and $\beta$ such that, $\alpha + \beta = \frac{1}{1+a}$ and for all real $\lambda > 0$,*

$$\frac{1}{1 + a\lambda^k} \geq \alpha + \beta\lambda.$$

*Proof.* We start studying $f : \lambda \mapsto \frac{1}{1+a\lambda^k}$. It is obvious that $f$ is differentiable on $[0, +\infty[$ and that

$$f'(\lambda) = -\frac{ka\lambda^{k-1}}{(1 + a\lambda^k)^2}.$$

Next, we consider the line tangent to the graph of $f$ in $(1, f(1))$. The slope of this tangent is

$$\beta = f'(1) = -\frac{ka1^{k-1}}{(1 + a1^k)^2} = -\frac{ak}{(1+a)^2}$$

and its constant term

$$\alpha = f(1) - 1 \times f'(1) = \frac{1}{1+a} + \frac{ak}{(1+a)^2}.$$

We get $\alpha + \beta = \frac{1}{1+a}$, which explains our choice of notation.

Consider the difference $d : \lambda \mapsto f(\lambda) - (\alpha + \beta\lambda)$, that is

$$d(\lambda) = \frac{1}{1+a\lambda^k} + \frac{a\lambda k}{(1+a)^2} - \frac{1}{1+a} - \frac{ak}{(1+a)^2},$$

which we want to prove positive for all $\lambda > 0$.

Reducing the denominators (that are strictly positive), there remains to study the sign of the polynomial

$$(a^2k)\lambda^{k+1} + (-a - a^2 - a^2k)\lambda^k + (ak)\lambda + (a + a^2 - ak)$$

As in the case $a = 0$ we get equality, we can assume $a > 0$ and simplify to prove the positivity of

$$(ak)\lambda^{k+1} + (-1 - a - ak)\lambda^k + k\lambda + (1 + a - k) \quad ,$$

which, after rearranging, is an affine function of $a$:

$$P(a, \lambda) = a\big(k\lambda^{k+1} - (k+1)\lambda^k + 1\big) + (k\lambda - \lambda^k + 1 - k).$$

The slope equals 0 when $\lambda = 1$. The derivative of this slope with respect to $\lambda$ is $k(k+1)\lambda^{k-1}(\lambda - 1)$, thus the slope decreases when $\lambda$ goes from 0 to 1 and increases thereafter, so is always positive.

This shows that the polynomial $P(a, \lambda)$ increases with $a$. It will be minimal when $a$ is minimal, so when $a = k - 1$:

$$\begin{aligned} k(k-1)\lambda^{k+1} + (-1 - (k-1) - (k-1)k)\lambda^k + k\lambda + ((k-1) + 1 - k) \\ = k\lambda((k-1)\lambda^k - k\lambda^{k-1} + 1). \end{aligned}$$

The case $\lambda = 0$ is clearly good for our positivity needs. If $\lambda \neq 0$, $\lambda > 0$ and this polynomial has the same sign as:

$$(k-1)\lambda^k - k\lambda^{k-1} + 1$$

already studied (the slope of $P(a, \lambda)$ in $a$) and shown to be positive. □

## 2nd question: the case where $0 < a < \frac{k-1}{k+1}$

### Exihibit an $n$-tuple $x$ such that $C_{k,a}(x) > A(x)$

**Proposition 2.** *For all $n > 1$, the $n$-tuple $x(\varepsilon) = (1, \varepsilon, \varepsilon, \ldots, \varepsilon)$, with $\varepsilon$ sufficiently small, satisfies the desired inequality.*

*Proof.* We have $\lim_{\varepsilon\to 0} A(x(\varepsilon)) = \frac{1}{n}$ and after a simple limit calculation

$$\lim_{\varepsilon\to 0} C_{k,a}(x(\varepsilon)) = \frac{1+a}{n} \quad ,$$

because

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{k+1}}{\varepsilon^k + a\varepsilon^k} = \lim_{\varepsilon\to 0} \frac{\varepsilon}{1+a} = 0 \quad , \quad \lim_{\varepsilon\to 0} \frac{1}{1+a\varepsilon} = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\varepsilon^{k+1}}{\varepsilon^k + a} = 0.$$

So $\lim\limits_{\varepsilon\to 0} C_{a,k}(x(\varepsilon)) - A(x(\varepsilon)) = \dfrac{a}{n} > 0$ and so, if $\varepsilon$ is sufficiently small, according to the definition of the limit:

$$C_{k,a}(x(\varepsilon)) - A(x(\varepsilon)) > 0.$$

□

## Exihibit an $n$-tuple $x$ such that $C_{k,a}(y) < A(y)$

Let $y = (y_0 + \delta, y_0, y_0, \ldots, y_0)$ with $\delta$ a positive real as small as we like and for all $i$ let $\lambda_i = \frac{y_i}{y_{i+1}}$.

Keep the notations of the first question for the functions $g$ and $f$, and for the real $\alpha$ and $\beta$.

Let's prove that, if $\lambda_i = 1$:

$$\frac{1}{1 + a\lambda_i^k} = \alpha + \beta\lambda_i,$$

and, if $\lambda_i = 1 + \epsilon$ with $|\epsilon|$ sufficiently small:

$$\frac{1}{1 + a\lambda_i^k} < \alpha + \beta\lambda_i.$$

The first result is obvious because we have $\alpha + \beta = \frac{1}{1+a}$. In order to prove the second one, let's study again $g(\lambda) - (\alpha + \beta\lambda)$ and so the polynomial of same sign:

$$\begin{aligned}
P(\lambda) &= (ak)\lambda^{k+1} + (-1 - a - ak)\lambda^k + k\lambda + (1 + a - k)\\
P'(\lambda) &= k(a(k+1)\lambda^k + (-1 - a - ak)\lambda^{k-1} + 1)\\
P''(\lambda) &= k(ak(k+1)\lambda^{k-1} + (-1 - a - ak)(k-1)\lambda^{k-2})\\
P''(1) &= k(ak(k+1) + (-1 - a - ak)(k-1)\\
&= k(k(k+1) - (k-1) - k(k-1)) - (k-1))\\
&= k(a(k+1) - (k-1))
\end{aligned}$$

Thus $P(1) = P'(1) = 0$ and if $a < \frac{k-1}{k+1}$, then $P''(1) < 0$.

Using Taylor's expansion we have:

$$P(1+\epsilon) = P(1) + \epsilon P'(1) + \frac{\epsilon^2}{2} P''(1) + \epsilon^2\tau(\epsilon)$$

with:

$$\lim_{\epsilon \to 0} \tau(\epsilon) = 0$$

So:

$$P(1+\epsilon) = \epsilon^2 \left( \frac{P''(1)}{2} + \tau(\epsilon) \right)$$

And so if $\epsilon$ is sufficiently small, $P(1+\epsilon) < 0$ because $\frac{P''(1)}{2}$ is a negative constant and $\tau(\epsilon)$ goes to 0 with $\epsilon$, and so

$$\frac{1}{1+a(1+\epsilon)^k} < \alpha + \beta(1+\epsilon)$$

Moreover for all $\epsilon > 0$ and for all $y_0 > 0$ it exists $\delta > 0$ such that:

$$1+\epsilon > \frac{y_0+\delta}{y_0} > 1$$

and

$$1 > \frac{y_0}{y_0+\delta} > 1-\epsilon$$

Therefore, as all $\lambda_i$ are equals 1 but $\lambda_1 = \frac{y_0+\delta}{y_0}$ and $\lambda_n = \frac{y_0}{y_0+\delta}$, choosing $\delta$ sufficiently small:

- if $i \neq 1$ and $i \neq n$:
$$\frac{1}{1+a\lambda_i^k} = \alpha + \beta\lambda_i$$
- if $i = 1$ or $i = n$:
$$\frac{1}{1+a\lambda_i^k} < \alpha + \beta\lambda_i$$

If we sum we obtain:

$$C_{k,a}(y) < A(y)$$

## 3rd question: the intermediate case $\frac{k-1}{k+1} < a < k-1$

Let's prove that if $n$ sufficiently large, then it exists $x$ such that $A(x) > C_{a,k}(x)$.
We take the $x_i$ in geometrical progression, for all $i$: $x_i = r^{i-1}$, $r > 1$.
So:

$$A(x) = \frac{1+r+r^2+\cdots+r^{n-1}}{n} = \frac{r^n-1}{n(r-1)}$$

and:

$$\begin{aligned}
C_{a,k}(x) &= \frac{1+a}{n} \left( \frac{1^{k+1}}{1^k+ar^k} + \frac{r^{k+1}}{r^k+ar^{2k}} + \cdots + \frac{r^{(n-1)(k+1)}}{r^{(n-1)k}+a} \right) \\
&= \frac{1+a}{n} \left( \frac{1}{1+ar^k} + r\frac{1}{1+ar^k} + \cdots + r^{n-2}\frac{1}{1+ar^k} + \frac{r^{(n-1)(k+1)}}{r^{(n-1)k}+a} \right) \\
&= \frac{1+a}{n(1+ar^k)} \times \frac{r^{n-1}-1}{r-1} + \frac{(1+a)r^{(n-1)(k+1)}}{n(r^{(n-1)k}+a)}
\end{aligned}$$

Then $A(x) - C_{k,a}(x)$ equals:

$$\frac{r^n - 1}{n(r-1)} - \frac{1+a}{n(1+ar^k)} \times \frac{r^{n-1}-1}{r-1} - \frac{(1+a)r^{(n-1)(k+1)}}{n(r^{(n-1)k}+a)},$$

and so, as $n > 0$, it is of the same sign of:

$$\frac{r^n - 1}{r-1} - \frac{1+a}{1+ar^k} \times \frac{r^{n-1}-1}{r-1} - \frac{(1+a)r^{(n-1)(k+1)}}{r^{(n-1)k}+a}.$$

Simplifying by $r^{n-1} > 0$:

$$\frac{r - \frac{1}{r^{n-1}}}{r-1} - \frac{1+a}{1+ar^k} \times \frac{1 - \frac{1}{r^{n-1}}}{r-1} - \frac{(1+a)r^{(n-1)k}}{r^{(n-1)k}(1+\frac{a}{r^{(n-1)k}})}$$

When $n$ goes to $+\infty$ it becomes:

$$\frac{r}{r-1} - \frac{1+a}{1+ar^k}\frac{1}{r-1} - (1+a).$$

As $r - 1 > 0$ and $1 + ar^k > 0$, it is of the sign of:

$$\begin{aligned} r(1+ar^k) - (1+a) - (1+a)(r-1)(1+ar^k) \\ &= r + ar^{k+1} - 1 - a - r - ar^{k+1} + 1 + ar^k - ar - a^2r^{k+1} + a + a^2r^k \\ &= -a^2r^{k+1} + a(1+a)r^k - a \\ &= a(-ar^{k+1} + (1+a)r^k - 1). \end{aligned}$$

Its derivative with respect to $r$ is:

$$a(-a(k+1)r^k + k(a+1)r^{k-1} - 1).$$

For $r = 1$:

$$a(-a(k+1) + k(a+1) - 1) = a(-ak - a + ak + k) = a((k-1) - a) > 0$$

So this polynomial is increasing when $r = 1$ and equals $-a^2 + a^2 + a - a = 0$ for $r = 1$.

So there is $\epsilon > 0$ sufficiently small such that if $r = 1 + \epsilon$, then:

$$a(-ar^{k+1} + (1+a)r^k - 1) > 0\,,$$

consequently such that:

$$\frac{r}{r-1} - \frac{1+a}{1+ar^k}\frac{1}{r-1} - (1+a) > 0.$$

Thus, for a such real number $r$, when $n$ goes to infinity, $A(x) - C_{k,a}(x)$ goes to a positive value. So, according to the definition of limit, we can find $N \in \mathbb{N}$ such that $n > N$ then $A(x) - C_{k,a}(x) > 0$. So, if $n$ is sufficiently large, it exists $x$ such that $A(x) > C_{k,a}(x)$.

Moreover, using the same method than in the 2.1, we can prove that it exist $x$ such that $A(x) < C_{a,k}(x)$.

## Question 4

By analogy to question 1, we search $\alpha$ and $\beta$ such that :

$$\begin{cases} \alpha+\beta = \frac{1}{a+1} \\ \dfrac{1}{1+a\lambda^k} \geq g(\lambda) = \alpha + \beta\lambda^l \quad \forall \lambda > 0 \end{cases}$$

Indeed, if two such real numbers exist, we get, summing the inequality for $\lambda = \frac{x_{i+1}}{x_i} \geq 0$ :

$$\frac{x_1^{k+l}}{x_1^k + a x_2^k} + \cdots + \frac{x_n^{k+l}}{x_n^k + a x_1^k} \geq (\alpha+\beta)(x_1^l + \cdots + x_n^l) = \frac{1}{1+a}(x_1^l + \cdots + x_n^l)$$

and then $C_{k,l,a}(x) \geq \dfrac{\sum_{i=1}^n x_i^l}{n} \geq (A(x))^l$, where the last inequality is the generalized mean inequality.

But at this point, there is nothing more to do. In fact, take the substitution $y = \lambda^l$ into the inequality we want to prove. It then amounts to find $\alpha$ and $\beta$ such that

$$\frac{1}{1+ay^{k/l}} \geq \alpha + \beta y \quad \forall y > 0$$

which has a well-established solution replacing $k$ by $k/l$ in the previous formulæ.

We believe that this way one can rewrite questions 1–3 in this generalized setting, by replacing all numbers with their $l$-th root. This sure involves estimating also the difference between the generalized mean and the arithmetic mean, in case of the reverse inequality.

Next, we give what we have been able to prove by means of this technique.

### The easy case: $a \geq k/l - 1$

We can choose $\alpha$ and $\beta$ as in question 1, namely

$$\alpha = \frac{(1+a)l + ak}{(1+a)^2 l} \quad \text{and}$$
$$\beta = \frac{-ak}{(1+a)^2 l}$$

because all we have done is clearly independent from the fact that $k$ was integer. As here the generalized mean is bigger than the arithmetic mean, there is nothing left to do, and we conclude $C_{k,l,a}(x) > A(x)$.

One single caveat: if $k/l \leq 1$, we impose $a > 0$. Indeed, finding $\alpha$ and $\beta$ is just a matter of imposing tangence at $\lambda = 1$. But now, after the substitution $y = \lambda^l$, the function we get is $\frac{1}{1+ay^{k/l}}$, which is convex, because its second derivative is (with $t = k/l$)

$$\frac{2a^2 y^{2t} t^2}{(1+ay^t)^3 y^2} + \frac{ay^t(t-t^2)}{(1+ay^t)^2 y^2}$$

which is positive ($t > t^2$!), and so the function is always above its tangent.

### The second easy case: $a < \frac{k-l}{k+l}$

Here once we have shown the equivalent of 2.1: there is $x$ such that $C(x) > A(x)$.

Taking the $n$-tuple $x(\varepsilon)$, $A(x(\varepsilon))^l = \left(\frac{1+(n-1)\varepsilon}{n}\right)^l$, while $C_{k,l,a}(x(\varepsilon)) = \frac{1+a}{n}(1+O(\varepsilon))$. It is clear that $A$ goes much faster to 0, so, for a sufficiently small $\varepsilon$, $C > A$.