International tournament of Young Mathematicians

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University Paris-Sud 11, Orsay, France

Team:
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Problem:
9. A Topological Problem

Abstract

Problem 9. is examined in details. The following results are reached:

9.1. Answer: 3

9.2. a) It is shown that $int(conv(A)) = int\left(conv(cl(A))\right)$ holds true. 9.2. b) Answer: 8

9.3. Answer: 8.

We used a method for the solution of this problem which exhausts all possibilities. Therefore, this method exhausts all cases of the problem.

Theoretical Part

1. Closed set, without the points on the boundary, is an open set.
2. $Interior$ – the operation removes the boundary of the set (these objects from the set, which have less than n dimensions disappear).
3. $Closure$ – adds the boundary of the set.
4. $Convex\ cover$ – for a set of points A is the minimal convex set containing A.

Solution

Problem 1.

A consists closed and also opened sets (and some more, which are neither opened, nor closed).

$int(A)$ and $cl(A)$ change only the boundary of A.

But A, $int(A)$ and $cl(A)$ are different.

$$int\big(int(A)\big) = int(A)$$
$$int\big(cl(A)\big) = int(A)$$
$$cl\big(int(A)\big) = cl(A)$$
$$cl\big(cl(A)\big) = cl(A)$$

No matter how the operations are arranged, we obtain $int(A)$ or $cl(A)$ (only if we have made at least one operation).

I.e. the sets are A, $int(A)$ i $cl(A)$ - 3 sets.

Example: $A = [0; 1)$

Problem 2.

a)

$int(conv(A)) \neq \emptyset \Rightarrow$

We can contend that the convex cover of A has area because $int(A)$ removes the boundary and we obtain something more than empty set.

$conv(A)$ and $conv(cl(A))$ [eventually] are different only by their boundary (because if A is closed, $cl(A)$ doesn't change it) and when we apply $int(conv(A))$ and $int(conv(cl(A))$, we remove the boundary (if it exists) and consequently obtain the same set, i.e. $int(conv(A)) = int\left(conv(cl(A))\right)$ – Q.E.D.

b)

If $int(conv(A)) = \emptyset$,
Then, since int removes the boundary, we have that $conv(A) \in \mathbb{R}$, i.e. we return the case to the previous problem, but with the operation $conv$.

Then, the possible sets are A, $int(A)$, $cl(A)$

If $int(conv(A)) \neq \emptyset$, we have:
(the operations only with int and cl follow directly by the previous problem)

1) $int\left(int(int(A))\right) = int(A)$ [from the previous problem];

2) $int\left(int(cl(A))\right) = int(A)$ [from the previous problem];

3) $int\left(int(conv(A))\right) = int(conv(A))$, because $int(int(B)) = int(B)$;

4) $int\left(cl(int(A))\right) = int(A)$ [from the previous problem];

5) $int\left(cl\left(cl(A)\right)\right) = int(A)$ [from the previous problem];

6) $int\left(cl\left(conv(A)\right)\right) = int\left(conv(A)\right)$, because $int\left(cl(B)\right) = int(B)$;

7) $int\left(conv\left(int(A)\right)\right) = conv\left(int(A)\right)$, because after $int(A)$ the convex cover $cinv\left(int(A)\right)$ don't have a boundary to be removed;

8) $int\left(conv\left(cl(A)\right)\right) = int\left(conv(A)\right)$, because even we add the boundary with cl, after that int removes it;

9) $int\left(conv\left(conv(A)\right)\right) = int\left(conv(A)\right)$, because $conv\left(conv(B)\right) = conv(B)$);

10) $cl\left(cl\left(int(A)\right)\right) = cl(A)$ [from the previous problem];

11) $cl\left(cl\left(cl(A)\right)\right) = cl(A)$ [from the previous problem];

12) $l\left(cl\left(conv(A)\right)\right) = cl(conv(A))$, because $cl\left(cl(B)\right) = cl(B)$;

13) $cl\left(int\left(cl(A)\right)\right) = cl(A)$ [from the previous problem];

14) $cl\left(int\left(int(A)\right)\right) = cl(A)$ [from the previous problem];

15) $cl\left(int\left(conv(A)\right)\right) = cl\left(conv(A)\right)$, because $cl\left(int(B)\right) = cl(B)$;

16) $cl\left(conv\left(int(A)\right)\right) = cl\left(conv\left(int(A)\right)\right)$ [from the previous problem];

17) $cl\left(conv\left(cl(A)\right)\right) = cl\left(conv(A)\right)$, because never mind, in the end of the operations we take the boundary of A;

18) $cl\left(conv\left(conv(A)\right)\right) = cl\left(conv(A)\right)$, because $conv\left(conv(B)\right) = conv(B)$;

19) $conv\left(conv\big(int(A)\big)\right) = conv\big(int(A)\big)$, because $conv\big(conv(B)\big) = conv(B)$;

20) $conv\left(conv\big(cl(A)\big)\right) = conv\big(cl(A)\big)$, because $conv\big(conv(B)\big) = conv(B)$;

21) $conv\left(conv\big(conv(A)\big)\right) = conv(A)$, because $conv\big(conv(B)\big) = conv(B)$;

22) $conv\left(int\big(int(A)\big)\right) = conv\big(int(A)\big)$, because $int\big(int(A)\big) = int(A)$;

23) $conv\left(int\big(cl(A)\big)\right) = conv\big(int(A)\big)$, because $int\big(cl(A)\big) = int(A)$;

24) $conv\left(int\big(conv(A)\big)\right) = int\big(conv(A)\big)$, because $conv$ dava gives convex cover, int removes the boundary and after that we take the convex cover, i.e. after $conv$ and int one more time $conv$ doesn't change the set;

25) $conv\left(cl\big(int(A)\big)\right) = conv\big(cl(A)\big)$, because $cl\big(int(A)\big) = cl(A)$;

26) $conv\left(cl\big(cl(A)\big)\right) = conv\big(cl(A)\big)$, because $cl\big(cl(A)\big) = cl(A)$;

27) $conv\left(cl\big(conv(A)\big)\right) = cl\big(conv(A)\big)$ – analogical to 24);

Consequently the possible outcomes are the following:

$$int(A)$$
$$int\big(conv(A)\big)$$
$$conv\big(int(A)\big)$$
$$cl(A)$$
$$cl(conv(A))$$
$$cl\left(conv\big(int(A)\big)\right)$$
$$conv\big(cl(A)\big)$$

$$conv(A)$$

Answer: 8.

Problem 3.

The answers for $n > 2$ are analogical to the previous problem (9.2.b) because the operations int, cl and c $conv$ work only with the boundary. Whatever operations we apply (it doesn't matter the order) we get the same sets as in the previous problem (9.2.b).

Answer: 8.