International tournament of Young Mathematicians

24th June – 1st July 2010
University Paris-Sud 11, Orsay, France

Team:
Bulgaria

Problem:
8. Points on Curves

Abstract

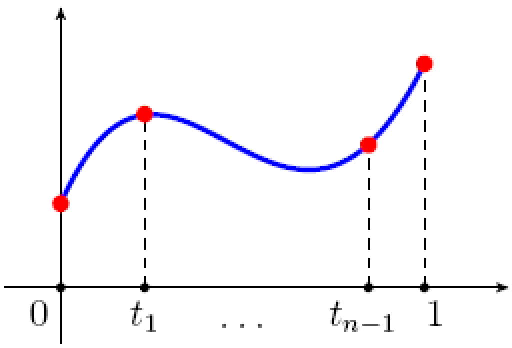
In the problem 8.1. we use two famous theorems: Weierstrass theorem and Bolzano-Koshi theorem. The problem is considered in his general form and there are given examples for the cases a), b) and c). In the second part, the problem 8.2. is considered in his general from and the given cases a), b) and c) are shown.

The Problem

1. Let be $f: [0,1] \to R$ be a continuous function. Find all positive integers n such that there exist a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ for which the expression

$$|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$$

does not depend on $i = 0, 1, \dots, n-1$.



Consider the case that f is

a) a polygonal chain (that is, piecewise linear);

b) a polynomial of degree $k = 2,3, \dots$;

c) a trigonometric function

Theoretical Part

1. Weierstrass theorem [1]
   Every continuous function, which is defined in closed interval, has maximum and minimum and also reaches them.

2. Bolzano-Koshi theorem [2]
   If the function f is continuous in particular interval and if 2 points from the interval a and b it takes different values ($f(a) \neq f(b)$), then for every number y between $f(a)$ and $f(b)$ we can find at least one point c between a and b, for which $f(c) = y$.

3. From [1] and [2] follows that every number between the minimum and the maximum of the given function is a value of this function. [3]

Solution

Problem 1.

$$f: [0; 1] \to \mathbb{R}$$

We can assume that $f(0) = 0$, because if $f(0) = c$, we can take the function

$$p(t) = f(t) - c$$

and the condition

$$|p(t_{i+1}) - p(t_i)| + |t_{i+1} - t_i| = |f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$$

stands.

Let's cosider the function

$$g(x) = f(x) + x.$$

This function is continuous in the interval [0; 1].

Let $g(1) = a > 0$ $(g(z_n = 1) = a)$.

Then, we are going to examine the numbers

$$0;\ \frac{a}{n};\ \frac{2a}{n};\ \frac{3a}{n};\ \dots;\ \frac{n-1}{n}a;\ \ a\ \in [0; a].$$

Then, according to [3] exists a number $t_{n-1} \in [0; 1]$, for which

$$g(t_{n-1}) = \frac{n-1}{n}a.$$

Applying [3] we obtain that there is a number $t_{n-2} \in [0;\ t_{n-1}]$, such that

$$g(t_{n-2}) = \frac{n-2}{n}a.$$

Continue applying [3], while reaching to $t_1 \in [0;\ t_2]$ and $g(t_1) = \frac{a}{n}$

$$g(t_0) = g(0) = f(0) + 0 = 0.$$

Thereby we obtain:

$$g(t_0) = 0$$
$$g(t_1) = \frac{a}{n}$$
$$g(t_2) = \frac{2a}{n}$$
$$\dots$$

$$g(t_i) = \frac{ia}{n}$$

$$\dots$$

$$g(t_{n-1}) = \frac{n-1}{n}a$$

$$g(t_n) = g(1) = a.$$

Then, for every i is true that

$$g(t_{i+1}) - g(t_i) = \tfrac{i+1}{n}a - \tfrac{i}{n}a = \tfrac{a}{n}.$$

Consequently the value of $g(t_{i+1}) - g(t_i)$ is a constant and doesn't depend from i.

But because of the choice of construction for $g(t_i)$, i.e. $g(t_i) = f(t_i) + t_i$, we have that

$$g(t_{i+1}) - g(t_i) = f(t_{i+1}) + t_{i+1} - g(t_i) - t_i = f(t_{i+1}) - f(t_i) + t_{i+1} - t_i.$$

Consequently $f(t_{i+1}) - f(t_i) + t_{i+1} - t_i$ is constant.

a) Polynomial chain

In this case we have $f(t) = at$ (because $f(0) = 0$).

Consequently we have

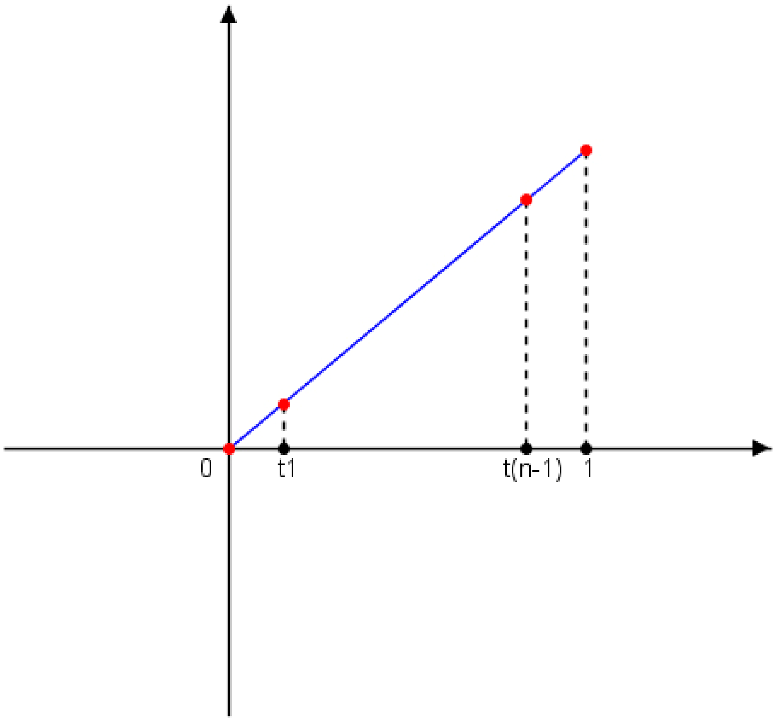


Figure 1

And the described method can be applyed.

b) Polynomial

In this case we have: $f(t) = (at + c)^k$, i.e.

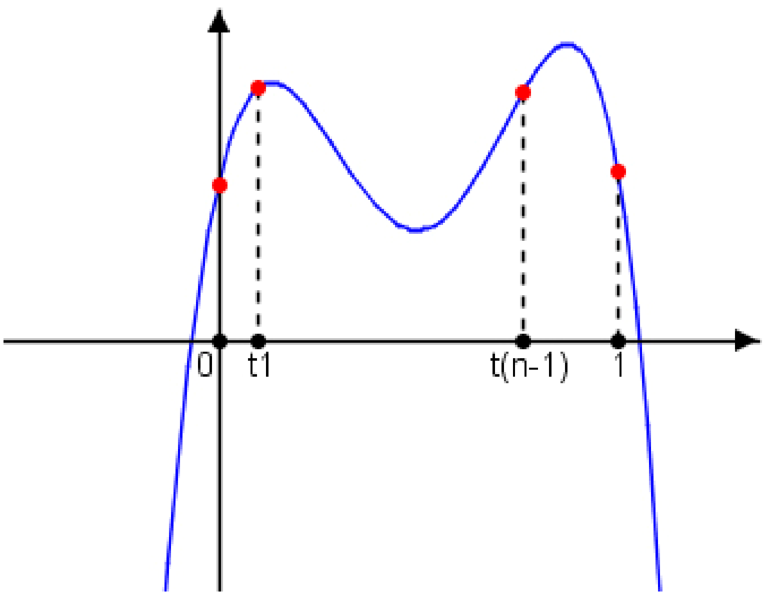


Figure 2

Trigonometric function

$$f(t) = \sin t \text{ i } f(t) = \cos t$$

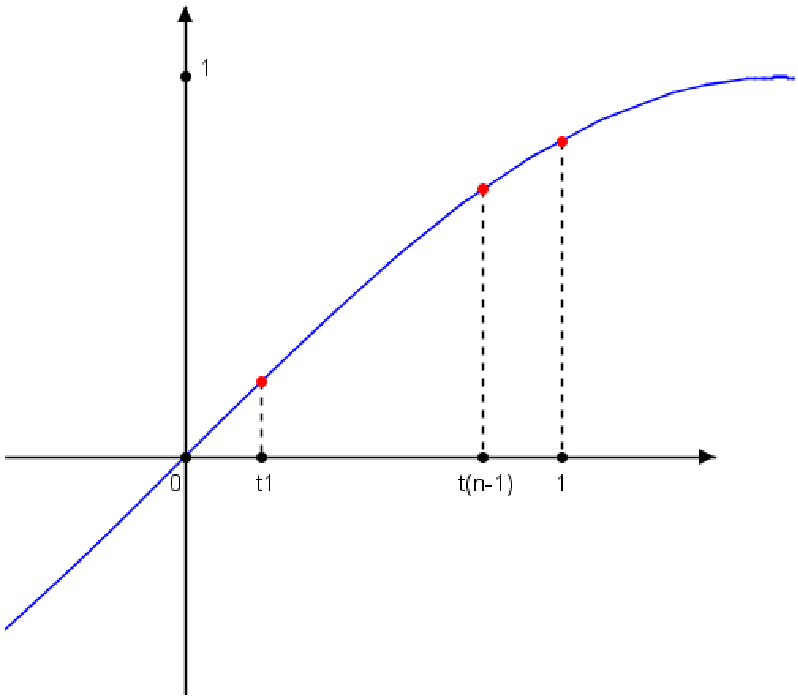


Figure 3

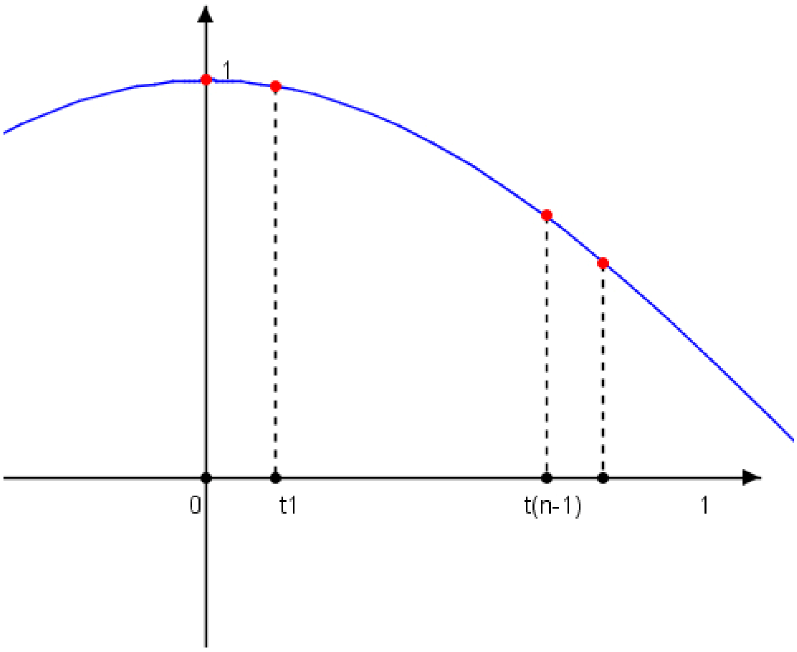


Figure 4

Problem 2.

$$F: \mathbb{R}^2 \to \mathbb{R}$$

For facility's sake, we assume that the function F is constant.

Then $F(A_j) - F(A_i) = 0$.

Let's consider the inequality

$$|x_i y_j - x_j y_i| \leq C.\frac{j-i}{n}.$$

But $\left(|x_i y_j - x_j y_i|\right)^2 = (x_i y_j - x_j y_i)^2 \leq (x_i^2 + y_i^2)(x_j^2 + y_j^2)$,

because $x_i^2 y_j^2 - 2x_i x_j y_i y_j + x_j^2 y_i^2 \leq x_i^2 x_j^2 + x_j^2 y_i^2 + x_i^2 y_j^2 + y_i^2 y_j^2$,

which is always true, because $(x_i x_j + y_i y_j)^2 \geq 0$.
Consequently

$$|x_i y_j - x_j y_i| \leq \sqrt{x_i^2 + y_i^2}.\sqrt{x_j^2 + y_j^2},$$

but $d(A_i; A_j) = \sqrt{x_i^2 + y_i^2}$ pri $j = 0, i \neq 0$

and $d(A_i; A_j) = \sqrt{x_j^2 + y_j^2}$ pri $i = 0, j \neq 0$.

Consequently

$$d(A_i; A_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2},$$

from where

$$|x_i y_j - x_j y_i| \leq \sqrt{x_i^2 + y_i^2}.\sqrt{x_j^2 + y_j^2} \leq \left(\frac{j-i}{n}\right)^\alpha.\left(\frac{j-i}{n}\right)^\alpha \leq \left(\frac{j-i}{n}\right)^{2\alpha}.$$

According to the condition $d(A_i; A_j) \leq \left(\frac{j-i}{n}\right)^\alpha$.

If it must true for every n then:
$\left(\frac{j-i}{n}\right)^{2\alpha} \leq C.\frac{j-i}{n}$, from where $C \geq \left(\frac{j-i}{n}\right)^{2\alpha-1}$.

1) When $2\alpha - 1 > 0$, i.e. $\alpha > \frac{1}{2}$ we have:

$C \geq 1^{2\alpha-1} \geq \left(\frac{j-i}{n}\right)^{2\alpha-1}$ (when $(i;j) = (0;n), (n;0)$).
Consequently $C \geq 1$.

2) When $2\alpha - 1 < 0$, i.e. $\alpha < \frac{1}{2}$ we have:
$C \geq \left(\frac{1}{n}\right)^{2\alpha-1} \geq \left(\frac{j-i}{n}\right)^{2\alpha-1}$ (when $(i;j) = (0;1), (1;0)$).
Consequently $C \geq \left(\frac{1}{n}\right)^{2\alpha-1}$.

3) When $\alpha = \frac{1}{2}$ follows that $C \geq \left(\frac{j-i}{n}\right)^{0}$, i.e. $C \geq 1$.

a) $\alpha = 1$
It's the same as the first case, i.e. $C \geq 1$.

b) $\alpha = \frac{2010}{2011}$
Again we are in the first case, i.e. $C \geq 1$.

c) $\alpha = \frac{1}{2}$
We are in the third case, where $C \geq 1$.