International tournament of Young Mathematicians

24th June – 1st July 2010
University Paris-Sud 11, Orsay, France

Team:
Bulgaria

Problem:
7. Friendly Polynomials

Abstract

Both of the problem 7.1. and problem 7.2. are considered in the general case. In both it is used the hypothesis of Casas-Alvero. A hypothesis about the multiplicity of the roots has been made and proved.

Theoretical Part

1. A polynomial, which is not coprime with its derivatives, we call friendly. Consequently, friendly polynomial is this one, which has a common divisor with the last non-zero derivative.

Solution

Problem 1.

Consider the field $F_P = \{0, 1, \dots, p-1\}$ of residue classes modulo a prime p. Try to find all friendly polynomials $P \in F_P[x]$ of degree $n \in N$.

Hypothesis: Let $P(x) \in \mathbb{F}_p[x]$.
Let α is a zero of $P(x)$ with multiplicity k.
Then α is $(k-1)$ – aliquot zero of $P'(x)$.
Proof:

$$P(x) = (x-\alpha)^k . F(x)$$

Consequently $F(\alpha) \neq 0$, because α is k-aliquot zero.
Then

$$P'(x) = k.(x-\alpha)^{k-1}.F(x) + (x-\alpha)^k.F'(x)$$

$$P'(x) = (x-\alpha)^{k-1}\big(k.F(x) + (x-\alpha).F'(x)\big).$$

But

$$(x-\alpha) \nmid (k.F(x) + (x-\alpha).F'(x)),$$

since

$$(x-\alpha) \nmid F(x).$$

Consequently $P'(x) \equiv 0 (mod\ (x-\alpha)^{k-1})$, but $P'(x) \not\equiv 0 (mod\ (x-\alpha)^k)$.

If x_0 is a root of $P(x)$, but x_0 is not a root of $P^{(l)}(x)$, then x_0 is not a root of $P^{(l+1)}(x)$. (Hypothesis of Casas-Alvero).

Now, let consider a zero x_0 with highest multiplicity k, where $k < n$

Let differentiate $n-1$ times, but $k < n$, consequently for every zero x_i of $P(x)$ $P^{(n-1)}(x) \neq 0$.

Consequently $\left(P^{(n-1)}(x); P(x)\right) = 1$.

But $P(x)$ is friendly, i.e. $P(x)$ i $P^{(n-1)}(x)$ are not coprime.

Consequently we have contradiction.

Consequently $k \geq n$, but the polynomial is from exponent n.

Consequently $k = n$.

Consequently $P(x) = c(x-a)^n$, where $c, a \in \mathbb{F}_p$

Problem 2.

Let C be the field of complex numbers. It is true that if polynomial $P \in C[x]$ of degree n is friendly then $P(x) = c(x-a)^n$ for some $a, c \in C$? Study this question for particular values of n $(e.g., 2, 3, 4, 5)$, and also when n is a prime number, a power of a prime number, etc.

Hypothesis: Let $P(x) \in \mathbb{C}_p$. Let α is a zero of $P(x)$ with multiplicity k. Then α is $(k-1)$ - aliquot zero of $P'(x)$.
Proof:

$$P(x) = (x-\alpha)^k . F(x) \text{ and again } F(\alpha) \neq 0.$$

Then

$$P'(x) = k.(x-\alpha)^{k-1}.F(x) + (x-\alpha)^k.F'(x)$$

$$P'(x) = (x-\alpha)^{k-1}(k.F(x) + (x-\alpha).F'(x)).$$

But

$$(x-\alpha) \nmid (k.F(x)+(x-\alpha).F'(x)),$$
since
$$(x-\alpha) \nmid F(x).$$

Consequently $P'(x) \equiv 0(mod\ (x-\alpha)^{k-1})$, but $P'(x) \not\equiv 0(mod\ (x-\alpha)^{k})$.

If x_0 is a root of $P(x)$, but x_0 is not a root of $P^{(l)}(x)$, then x_0 is not a root of $P^{(l+1)}(x)$. (Hypothesis of Casas-Alvero).

Now, let consider a zero x_0, wit highest multiplicity k and $k < n$ (the degree of the polynomial).

Differentiate $n-1$ times, but $k < n$.

Consequently for every zero x_i of $P(x)$ $P(x_i) \neq 0$.

Consequently $\left(P(x);\ P^{(n-1)}(x)\right) = 1$, but $P(x)$ is friendly.

Consequently $P(x)$ and $P^{(n-1)}(x)$ are not coprime.

Now we have contradiction.

And consequently $k \geq n$.

But the polynomial $P(x)$ is from exponent n.

Consequently $k = n$, and $P(x) = c(x-a)^n$.