International tournament of Young Mathematicians

24th June – 1st July 2010
University Paris-Sud 11, Orsay, France

Team:
Bulgaria

Problem:
6. Min/Max Questions

Abstract:

We have examined the problem about different shapes and conditions.

We have solved and proved: 6.1.a), 6.1.b) and 6.1.f)

The problem

1.

a) S is a straight line. Let the points are $A_1, A_2, \dots, A_N$ in this order on the line and let the minimal distance between two adjacent points is that between the points A_i and A_{i+1}, i.e. $l_{min} = A_i A_{i+1}$.

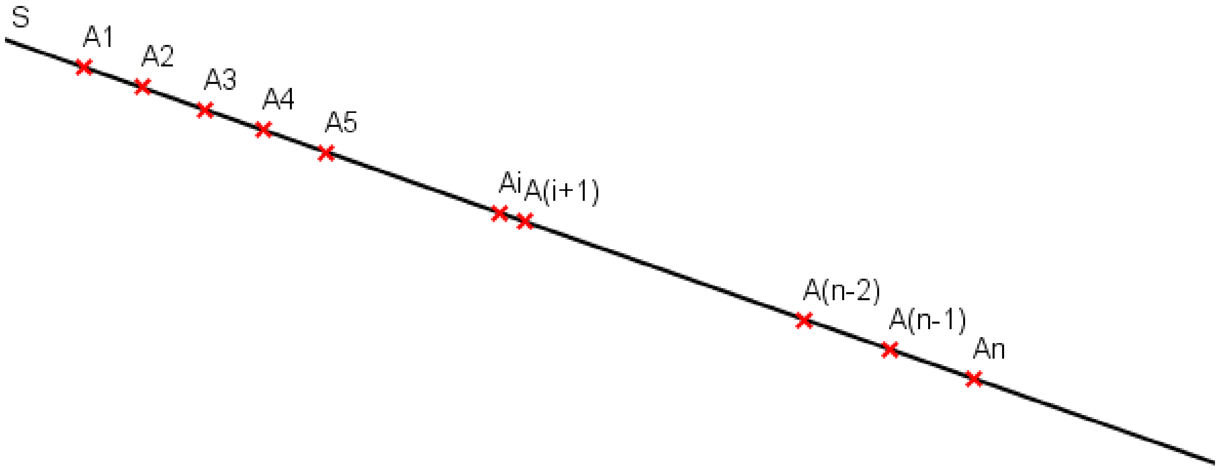

Figure 1

That means that $A_i A_{i+1} \le A_j A_{j+1}$ for every $j = 1, 2, \dots, N-1$.
Summarizing by j and obtain:

$(N-1).\, l_{min} \le A_1A_2 + A_2A_3 + \cdots + A_{N-1}A_N = A_1A_N = l_{max}$, i.e.
$l_{max} \ge (N-1).\, l_{min}$ - Q.E.D.

b) S is a circumference.
Let this circumference has radius R and center O.
We have two cases in the dependence of N:

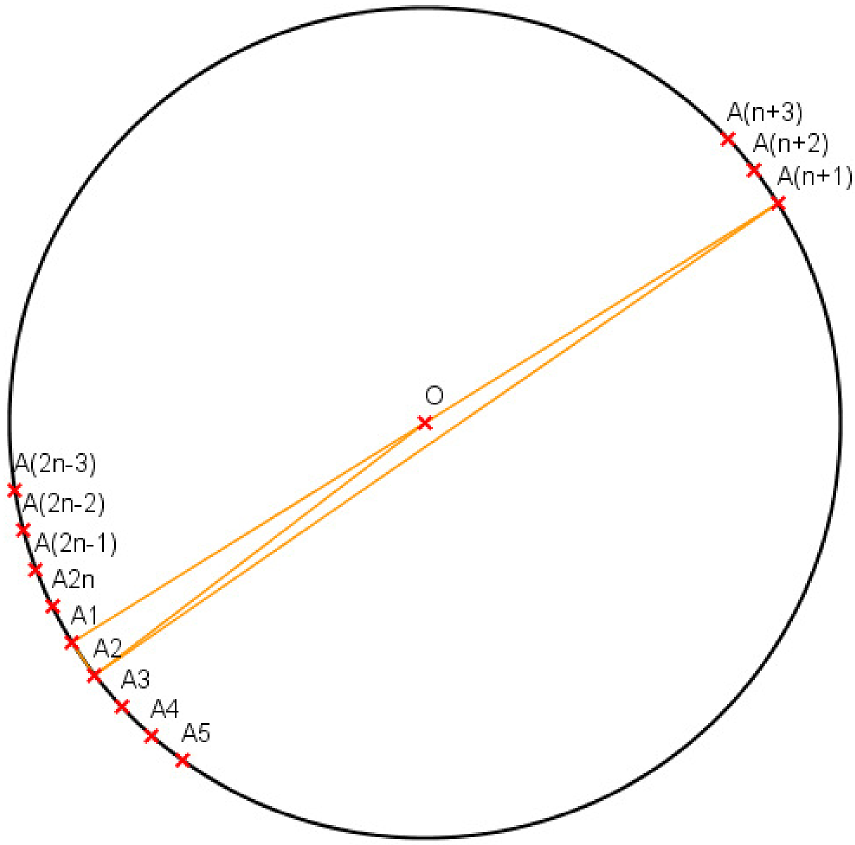

Figure 2

1) If $N = 2n$:

We take the points $A_1, A_2, \dots, A_{2n}$ in this order on the circumference. Then the maximal value of l_{min} will be reached when the points are on equal distances, i.e. $A_1A_2 = A_2A_3 = \cdots = A_{2n-1}A_{2n} = A_{2n}A_1$ i $A_1A_{n+1} = 2R = l_{max}$ is diameter.

Then:

$$\frac{l_{min}}{l_{max}} = \frac{A_1A_2}{A_1A_{n+1}} = \sin \sphericalangle A_1A_{n+1}A_2 = \sin\frac{1}{2}\sphericalangle A_1OA_2 = \sin\left(\frac{1}{2}.\frac{360^o}{2n}\right) = \sin\frac{90^o}{n}.$$

Consequently we have the following inequality:

$$\frac{l_{min}}{l_{max}} \le \sin\frac{90^o}{n} \text{ ili } l_{min} \le l_{max}.\sin\frac{90^o}{n}$$

2) If $N = 2n + 1$:

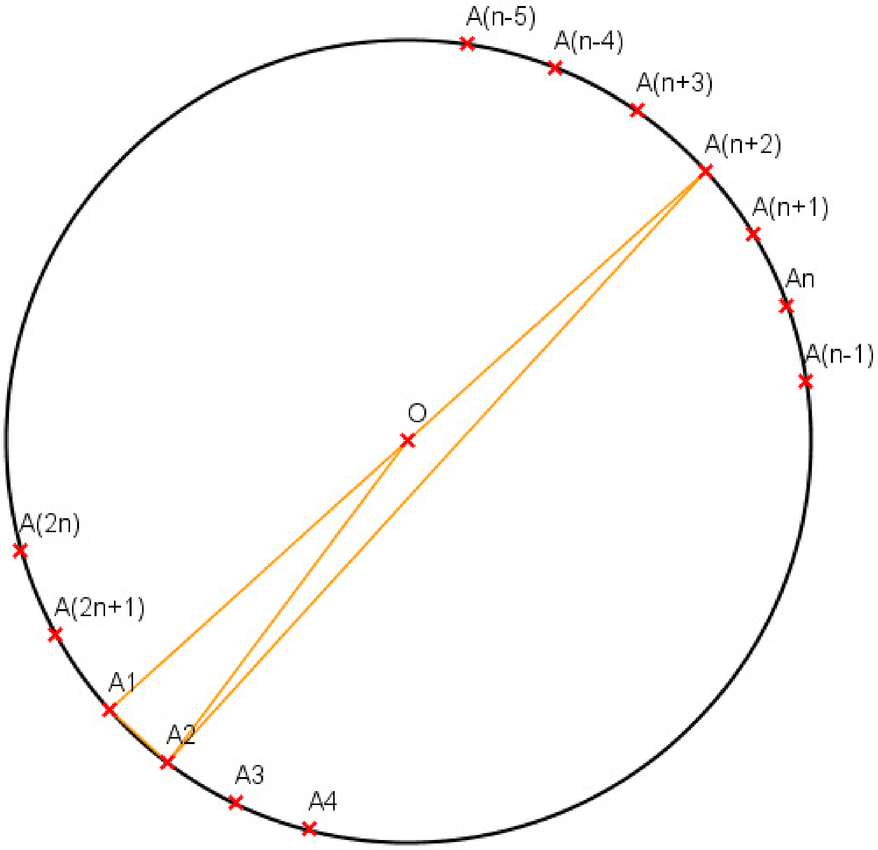

Figure 3

If two from the points are diametrical opposite, then on the one semicircle there are $n - 1$ points, and on the other – n points. Analogically to the previous case, this time we obtain $l_{min} \leq l_{max}. \sin\frac{90^o}{n+1}$, because on the first semicircle there is one more point.

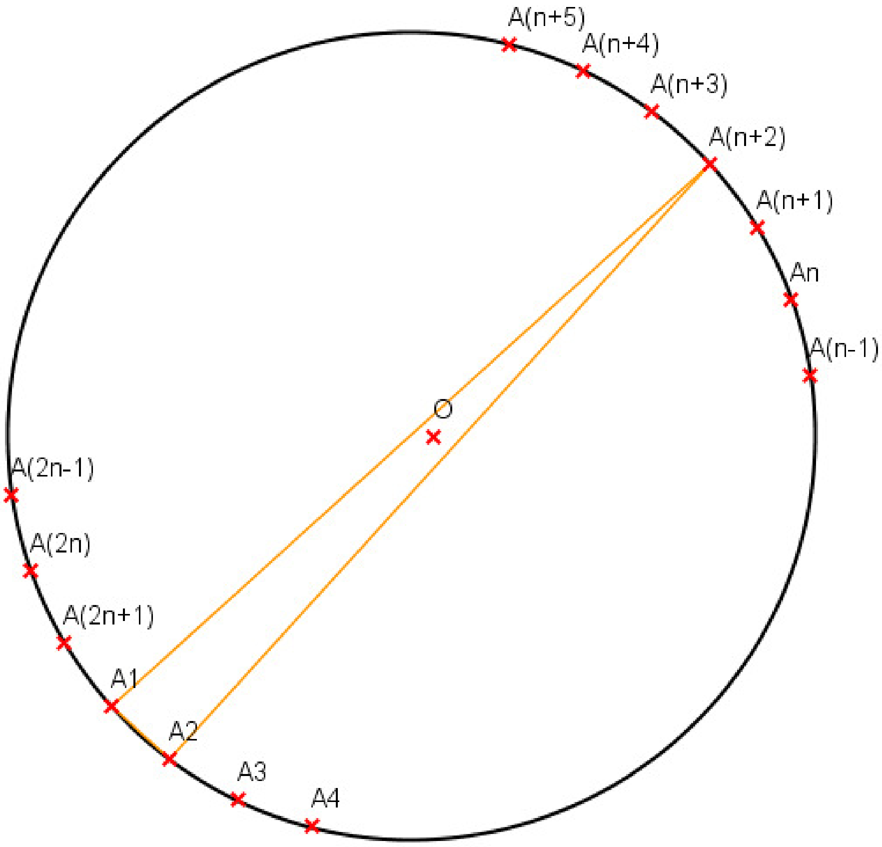

Figure 4

If we dispose the points on equal distances on the circumference, then we are going to look at $\Delta A_1A_2A_{n+2}$:

$$\frac{l_{min}}{l_{max}} = \frac{A_1A_2}{A_1A_{n+2}} = \frac{\sin \sphericalangle A_1A_{n+2}A_2}{\sin \sphericalangle A_1A_2A_{n+2}} = \frac{\sin \sphericalangle A_1A_{n+2}A_2}{\sin\left(90^o - \frac{1}{2}\sphericalangle A_1A_{n+2}A_2\right)} =$$

$$\frac{2.\sin\frac{1}{2}\sphericalangle A_1A_{n+2}A_2 . \cos\frac{1}{2}\sphericalangle A_1A_{n+2}A_2}{\cos\frac{1}{2}\sphericalangle A_1A_{n+2}A_2} = 2.\sin\frac{1}{2}\sphericalangle A_1A_{n+2}A_2 = 2.\sin\frac{1}{2}.\frac{360^o}{2n+1} =$$

$2.\sin\left(\frac{180^o}{2n+1}\right)$, from where $l_{min} \leq l_{max}.\sin\frac{180^o}{2n+1}$.

f) S is the whole plane, $N = 4$

We want to prove that $l_{max} \geq \sqrt{2}.l_{min}$. Assuming the contrary, i.e. that there are four points in the plane, for which $l_{min} < \sqrt{2}.l_{max}$.

Let the two poits, which are on the minimal distance from the one to the other, are A and B, i.e. $l_{min} = AB$ and without loss of generality, we assume that $AB = 1$.

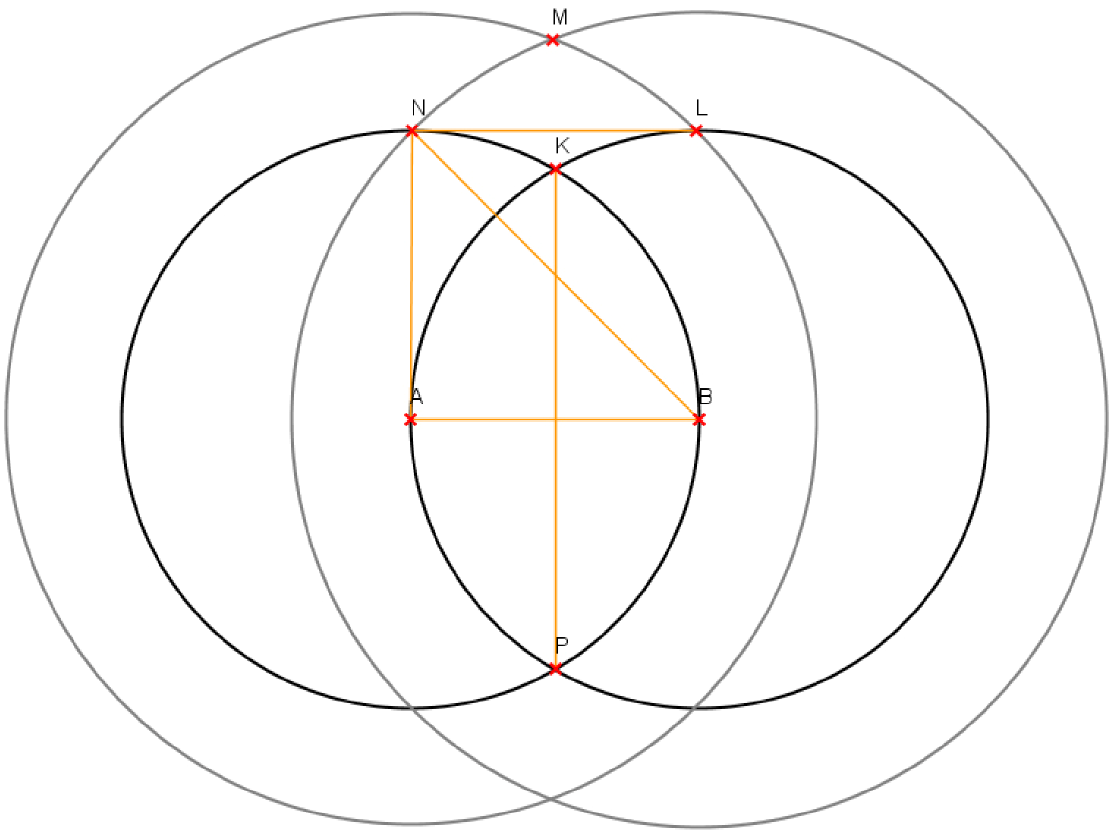

Figure 5

We construct the following four circumferences: $k_1(A;1), k_2(B;1), k_3(A;\sqrt{2}), k_4(B;\sqrt{2})$.
Let $k_1 \cap k_2 = K, k_2 \cap k_3 = L, k_3 \cap k_4 = M, k_1 \cap k_4 = N$, as the points K, L, M and N are in a same semi-plane towards AB.

Then we must choose two points in the area, defined by the points K, L, M, N and the other on the circumferences, or one point in this area and one point in the symmetrical area toward the straight line AB (because of the assumption that $l_{min} < \sqrt{2}.l_{max}$ and $AB = l_{min}$).

Let the second point of intersection of k_1 and k_2 is the point P.
If we choose the two points in different areas, then $l_{max} \geq KP = \sqrt{3}$ (because in ΔAKP, $AK = AP = 1$ and $\sphericalangle KAP = 120^o$). Although $\sqrt{3} > \sqrt{2}$. Consequently we reach a contradiction with the assumption.

I.e. the two points must be places in the area, defined by K, L, M, N and the arcs of the circumferences.

But in this area the maximal distance between points is LN, moreover, it can't be reached, because the other two points can't lay on k_3 or k_4 (because of the assumption).

ΔABN has $AB = AN = 1$ and $BN = \sqrt{2}$, i.e. ΔABN is isosceles and rectangular with $\sphericalangle BAN = 90^o$

Analogically for ΔABL. Then $ABLN$ is square, i.e. $LN = 1$.This means that the maximal distance between two points from the area, defined by K, L, M, N and the arcs of the circumferences, is smaller than 1, i.e. it is smaller than AB.

Consequently we have a contradiction with the assumption, i.e. $l_{min} \geq \sqrt{2}. l_{max}$ - Q.E.D.