International tournament of Young Mathematicians

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University Paris-Sud 11, Orsay, France

Team:
Bulgaria

Problem:
2. Separating functions

Abstract:

We have given a solution of problem 2.2. in 2 different ways;

We have proved 2.3.a) with an easy computing method and we have given a “beautiful” generalization in 2.4.)

We have used the Euclidean algorithm, full system of remainders by module and other methods for solving the problems.

Solution

1. If among the n numbers there are two co-prime, we go to problem 2.2. If none two numbers are co-prime, for all $i, j; i \neq j$ is true that $\left(a_i; a_j\right) = d_{i,j} > 1$, when $d_{i,j} = d_{j,i}$. But maximum $n-1$ from the numbers can have a same common divisor. Then we are going to look at two from the numbers – a and b, for which $(a; b) = d > 1$ and the number c, for which $(c; d) = 1$ (exists at least one such number, because the n numbers are co-prime, i.e. they haven't got same common divisor). Applying the idea of problem 2.3.b (without the condition that $c \geq F\left(\frac{a}{d}; \frac{b}{d}\right)$, written with these symbols).

2. Case $\boldsymbol{n} = 2$

Let the numbers be a and b and their coefficients – respectively x and y.

First, we are going to prove that the number $(a-1).(b-1)-1$ can't be represented as $xa + yb$.

Assume the contrary: the equation $(a-1).(b-1)-1 = xa + yb$ has a solution in natural numbers a and b and non-negative x and y.

$$(a-1).(b-1)-1 = xa + yb$$
$$ab - a - b + 1 - 1 = xa + yb$$
$$ab - a - b = xa + yb$$

Let consider the two sides of the inequality by module a:

$$ab - a - b \equiv xa + yb \ (mod\ a)$$
$$-b \equiv yb \ (mod\ \ a)$$

According to the condition $(a; b) = 1$, from where

$$-1 \equiv y \ (mod\ a)$$
$$\Rightarrow y = q.a - 1$$
$$y \in N_0 \ \Rightarrow q \in N$$

Analogically, examine by module b and obtain:

$$x = p.b - 1$$
$$x \in N_0 \ \Rightarrow p \in N$$

Replace it in the first condition:

$$(a-1)(b-1)-1=(pb-1)a+(qa-1)b$$
$$ab-a-b+1-1=pab-a+qab-b$$
$$ab-a-b=(p+q)ab-a-b$$
$$ab=(p+q)ab$$
$$p+q=1$$

But $p,q \in N \Rightarrow p+q \geq 1+1=2>1$

$\Rightarrow$ Contradiction with the assumption $\Rightarrow$ the number $(a-1)(b-1)-1$ can't be represented in the wanted way.

This means that

$$F(a;b)>(a-1)(b-1)-1,$$
i.e. $F(a;b) \geq (a-1)(b-1)$.

We are going to prove that $F(a;b)=(a-1)(b-1)$.

Let's take $(a-1).(b-1)+i$, where $i=0,1,\ldots$.

Now we have to prove that there are x and y such that:

$(a-1).(b-1)+i=x.a+y.b$

for some non-negative $x,y \in \mathbb{Z}$. In order to prove this, we are going to look at the residues (which are given from the system of the numbers:

$c-i.b \ (mod\ a)$ for $i=0,1,2,\ldots,a-2$.

(We consider WLOG that $b \geq$a), where $c=(a-1).(b-1)+i$.

Then $c>(a-2)b$, and we have positive numbers for $i=0,1,2,\ldots,a-2$. Now since $\gcd(a;b)=1$ we have that there are no i_1 and i_2, such that $c-i_1.b \equiv c-i_2.b \ (mod\ a)$. And also $|c-(a-1).b|=a-1$ is not divisible by a, so there is i, such that $c-i.b$ is divisible by a, and exists the desirable x,y.

There is an idea about another solution. For the purpose we have to prove that the equation $(a-1)(b-1)+i=xa+yb$ has a solution in natural numbers a and b and non-negative integers x and y for every $i = 0, 1, 2, \dots, ab - 1$. This will be enough, because ab in number consecutive natural numbers are going to be represented in the wanted way, i.e. all the numbers, which are bigger than them will be represented in the wanted way (by adding ab aliquot numbers).

WLOG we can consider that $a < b$.

Let $m = b - a$.

Then, if the equation $(a-1)(b-1) + i = xa + yb$ is realized for some $x, y \in N_0$, than the next equation is true:

$$(a-1)(m-1) + i = x_*.a + y_*.m,$$

because

$$\begin{gathered}(a-1)(m-1) + i = (a-1)(b-a-1) + i = \\ = (a-1)(b-1) - (a-1)a + i = \\ = (a-1)(b-1) + i - a(a-1) = \\ = xa + yb - a(a-1) = \\ = xa - (a-1)a + y(a+m) = \\ = (x - a + 1 + y)a + ym = \\ = (x + y + 1 - a)a + ym\end{gathered}$$

i.e. $x_* = x + y + 1 - a$ and $y_* = y$

Applying Euclidean algorithm for the co-prime numbers a and b:

$$(a; b) = (a; b - a) = (a; m) = \cdots = (1; \alpha) = 1$$

Consequently, it is enough to prove out hypothesys about the numbers 1 and α, i.e. to prove that $F(1; \alpha) = 0$.

It is obviously true, because all non-negative integers can be represented like a product of one with the respectively non-negative integer.

3. Case $n = 3$

a) Wether 2 from the numbers are even, prove that F is also even.

Let the numbers are a, b and c and let $F(a; b; c) = F$.

Let a and b are even and c – odd.

Assume that F is odd.

Then $F - 1$ is even and there is no presentment like $k.a + m.b + n.c$.

But c is odd $\Rightarrow F - 1 + c \geq F$, i.e. $F - 1 + c$ is odd and has presentment like $x.a + y.b + z.c$.

$$F - 1 + c = x.a + y.b + z.c$$

a – even, b – even $\Rightarrow x.a + y.b$ is even $\Rightarrow z.c$ is odd $\Rightarrow z$ is odd $\Rightarrow z \geq 1 \Rightarrow z - 1 \geq 0$, i.e. $z - 1 \in N_0$

Then $F - 1 = x.a + y.b + (z - 1).c$, i.e. there is such a presentment.

Contradiction with the assumption.
$\Rightarrow F(a; b; c)$ is even.

b) Let the numbers are again a, b and c and $\gcd(a; b) = d$.

Then the condition is the following:

$$c \geq F\left(\frac{a}{d}; \frac{b}{d}\right) = \left(\frac{a}{d} - 1\right)\left(\frac{b}{d} - 1\right).$$

And we want to prove that:

$$F(a; b; c) = d.F\left(\frac{a}{d}; \frac{b}{d}\right) + F(d; c).$$

4. Generalization of 2.3.a)

Let m is the number of odd numbers and let $F(a_1; a_2; \dots; a_n) = F$. From the definition of F follows that $F - 1$ can't be represented as a sum like $x_1.a_1 + \cdots + x_n.a_n$.

We are going to prove that if m is odd, than F is even.

Assuming the contrary: i.e., that F is odd.

m is odd. Consequently from among the a_i-s there is at least one odd number.

Let arrange the numbers in a way that in the beginning are the even numbers and after that are the odd, i.e. $a_1, a_2, \ldots, a_{n-m}$ are the even numbers; $a_{n-m+1}, a_{n-m+2}, \ldots, a_n$ are the odd.

Analogically to 2.3.a):

$F - 1$ doesn't have presentment

$F - 1 + a_i$ has presentment, where $n - m < i \leq n$, i.e. a_i is an odd number.

$F - 1 + a_i$ is an even number, i.e. at least one from the $x_j a_j$ must be odd.

If a_j is even, then $x_j a_j$ is also even.

Then a_j must be odd and x_j also must be odd, i.e. $x_j \geq 1$, which means that $x_j - 1 \in \mathbb{N}_0$.

Let a_i is exactly that a, for which x_i is odd.

Then $F - 1 + a_i$ has presentment, where $x_i \geq 1$, i.e. $F - 1$ also has presentment: the coefficient before a_i is $x_i - 1$.

Contradiction with the assumption, consequently F is even – Q.E.D.