International tournament of Young Mathematicians

24th June – 1st July 2010
University Paris-Sud 11, Orsay, France

Team:
Bulgaria

Problem:
1. Blocking sets

Abstract:

We have included some basic concepts, which are used in the investigation of the problem.

In 1.1. is founded an example for blosking set with 16 elements. Using the conditions for connecting two points, we have proved that doesn't exists smaller set.

In 1.4. we have proved that always exists a diagonal path.

Theoretical Part

1. A minimal point by abscissa X toward another point Y we call such a point X, which abscissa is bigger than the abscissa of Y, but also don't exist another point U, for which $x_U - x_Y < x_X - x_Y$. Analogical we definite minimal point by ordinate.

2. *A one's square* – a square with apexes $(0;0)$, $(1;0)$, $(1;1;)$, $(0;1)$.

3. *An interior rectangle* – a rectangle, which is fully covered by the one's square.

4. Blocking – one point is blocked if we can't connect it with another point. This happends when at least one point is an obstacle by abscissa and at least one point is an obstacle by ordinate (according to the condition)
   i)$a_i < a_j$ and $b_i < b_j$
   ii) either there is no point $(a_k; b_{k)}$ $\in S$ such that $a_i < a_k < a_j$, or here is no point $(a_k; b_k) \in S$ such that $b_i < b_k < b_j$

Solution

Since the set $S = \{(a_1;b_1);(a_2;b_2);\dots;(a_n;b_n)\}$ is defined in such a way that $a_i \neq a_j$ i $b_i \neq b_j$ for $i \neq j$, then follows that there are no points with equal abscissas or ordinates.

Assume that the minimal points by abscissa and ordinate are identical and note this point with $N\,(x;y)$. Then without restriction of the without loss of generality we can assume that in the place of a square with apexes $(0;0)$, $(0;1)$, $(1;1)$ and $(1;0)$ we can examine the rectangle, bounded by N and $(1;1)$. This leads to an equivalent to the given problem and augmentation the point's number with one, i.e. contradiction, because we are looking for the minimal number.

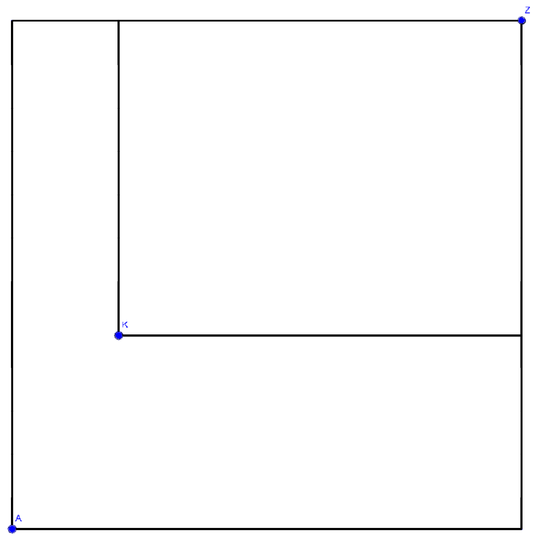

Figure 1

That’s why we take the pints C and F, $C \not\equiv F$, where C is with minimal abscissa, F is with minimal ordinate. We can always connect C with another point L, minimal by ordinate toward $C$and F with another point G, minimal by abscissa toward F. About decreasing the number of points with one, we can assume that G and L coincide

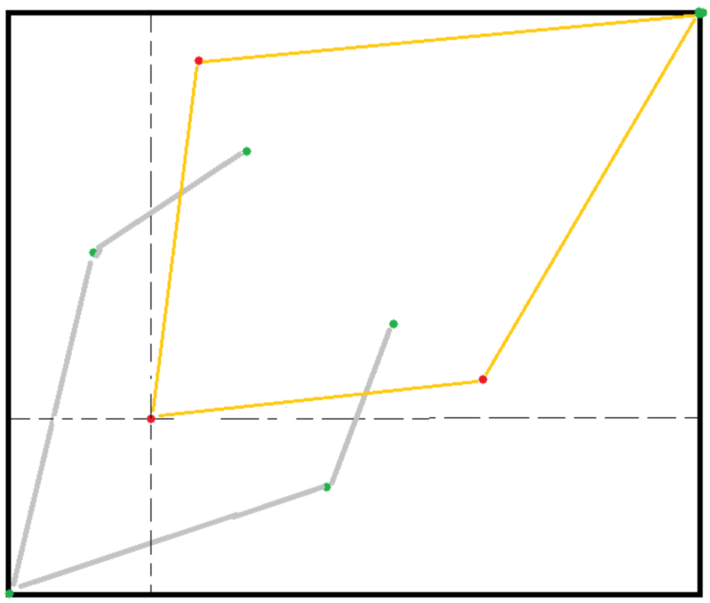

Figure 2

and let this point is D. That's how we decrease the neaded number by 1.

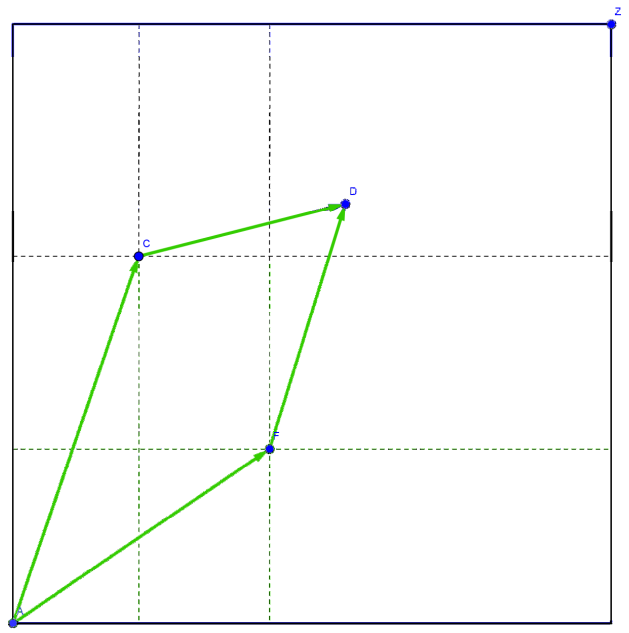

Figure 3

These points can't be connected with $(1;1)$. We need at least 2 points about the impossible connecting of D with $(1;1)$ and G with $(1;1)$, which leads to diagonal path. Let's see whether we need only one point, with which we can connect both C, and F.

No, because it leads to the case where in the place of one's square, we examined an interior rectangle.

Consequently the minimal number is 2.

Let these points are E and H, as E blocks D and G by ordinate, whereas H blocks D and G by abscissa.

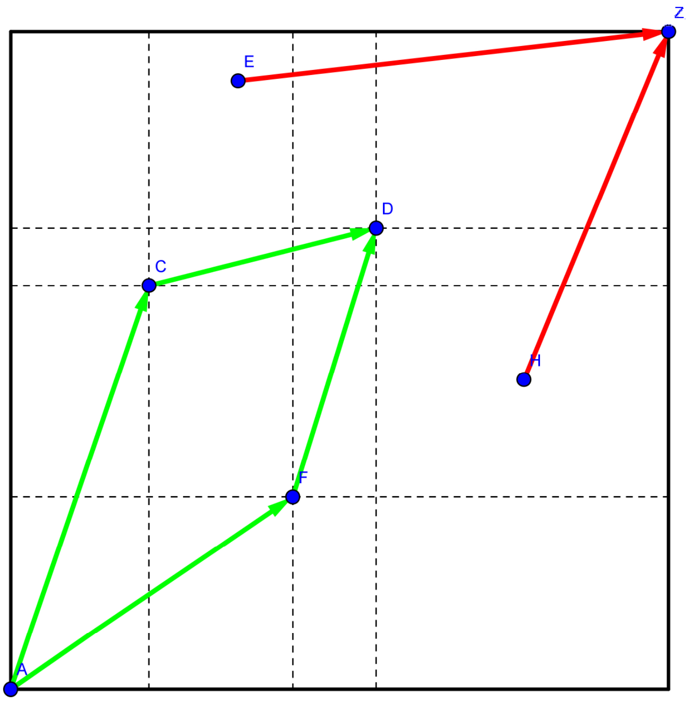

Figure 4

Despite all, the question "*Why we can't connect C with E and/or F with H?*" stays open. If this is possible, we create a diagonal path, that's why we take B, such that $x_C < x_B < x_E$ and $y_F < y_B < y_H$. It is obvious that such a point exists.

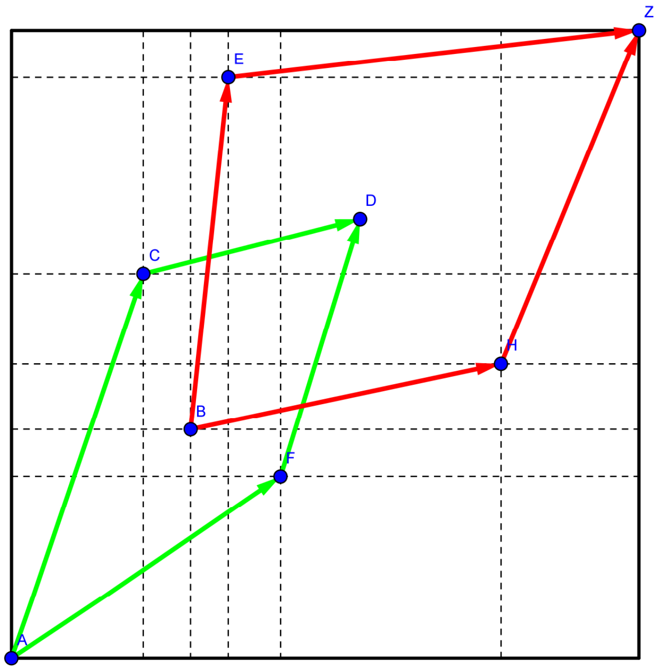

Figure 5

We are going to ask ourselves the next question: "*If we increase the number of points, whether we can decrease the number of edges, which will lead to a set with smaller number elements?*".

Wherever we put a point, in such a way to block another point, towards the point, we have already put, compulsory there is a rib, i.e. the number of elements increases with one. Consequently, we can't increase the number of points at the expense of the number of ribs.

4.

Always exists a diagonal path

Proof:

We are going to show that it is possible to construct a rib from any point, which is enough for the existence of diagonal path.

Assume the contrary and let the point A is such that we can not construct a path to any point. In the interior rectangle, definated bt A and $(1;1)$ we definitely have endmost number of points. And let B is such a point, for which the distinction of the abscissas of B and A is minimal (because of the endmost of the number, such a point exists). If we assume that we can't connect A with B, then exists a point C in the interior rectangle, determinated by A and $(1;1)$, for which the distinction of the abscissas of C

and A is smaller than the distinction of the abscissas of B and A, which is contradiction with the choice of B.

Consequently we can construct a path from A to B, which is contradiction with the assumption that we can’t.