Problem 9: A Topological Problem

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Abstract

Initial statement of the problem is solved completely.

In point one the answer is six (without initial *A*) and the example for the such set *A* ($A \subset R^n$ in Euclidian space) is found; point 2a) is proved; in point 2b) the suggested cases are solved: in the case *int*(*conv*(*A*))=ø the answer is two for *n*=1, four for *n*=2 and five in the other cases; in the case *int*(*conv*(*A*))≠ø the answer is 13 (without initial *A*) for *n*=1 and 16 (without initial *A*) in the other cases. For all cases the example of such set was found.

Also, +the following generalizations were studied:

In point one we considered arbitrary topological spaces and proved, that in any of them the maximal number of sets, that can be obtained from initial *A* is not more, than *7*.

We considered so-called relative operations *rint*, *rcl*, *rconv*, investigated some natural generalizations and estimated the maximal number of sets that can be obtained in those cases.

Part 1: open and closed sets

1.1 Properties of open sets:

Definition. A set $X\subset R^n$ is called *open* if, for any point $p \in X$, there exist a real number r>0 such that $B_r(p)\subset X$. For example, any open ball is open set.

- Union of any number of the open sets is the open set.
- Intersection of finite number of the open sets is the open set.

1.2 Properties of operation *int*:

Definition. $\mathrm{int}(A) = \bigcup_{open\ X\subset A} X$

- $\mathrm{int}(A) = A \Leftrightarrow A$ is open set.
- $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$
- If $A\subset B$, then $\mathrm{int}(A) \subset \mathrm{int}(B)$

1.3 Properties of closed sets:

Definition. A set $Y\subset R^n$ is called *closed* if its complement $C(Y)= R^n\backslash Y$ is open. For example, a complement of the any open ball is closed set.

- Union of finite number of the closed sets is the closed set.
- Intersection of any number of the closed sets is the closed set.

1.4 Properties of operation *cl*:

Definition. $cl(A) = \bigcap_{closed\ Y\supset A} Y$

- $cl(A) = (A) \Leftrightarrow A$ is closed set.
- $cl(cl(A)) = cl(A)$
- If $A\subset B$, then $cl(A) \subset cl(B)$

Point1

Consider all integer $n\geq 1$ instead of a case for $n=1$.

Theorem. Number of distinct sets that can be obtained from A using the operations *int* and *cl* equals six (without initial A).

Proof. First prove two equalities:

1) $int(cl(int(cl(A))))=int(cl(A))$:

$cl(int(cl(A)))\supset int(cl(A))$ by definition of operation *cl*. So, $int(cl(int(cl(A))))\supset int(int(cl(A)))=int(cl(A))$.

$int(cl(A))\subset cl(A)$ by definition of operation *int*. So, $cl(int(cl(A)))\subset cl(cl(A))=cl(A)$. So, $int(cl(int(cl(A))))\subset int(cl(A))$.

The result follows. Q.E.D.

2) $cl(int(cl(int(A))))=cl(int(A))$:

$cl(int(A))\supset int(cl(int(A)))$ by definition of operation *int*. So, $cl(cl(int(A)))=cl(int(A))\supset cl(int(cl(int(A))))$.

$cl(int(A))\supset int(A)$, so, $int(cl(int(A)))\supset int(int(A))=int(A)$, so, $cl(int(cl(int(A))))\supset cl(int(A))$.

The result follows. Q.E.D.

Now construct an example of set A, from which six distinct sets can be obtained:

Let $A=([0;1]\cup(2;3)\cup(3;4)\cup([5,6]\cap Q)\cup\{7\})\times R^{n-1}$

On a picture (see appendix) you can see this example for $A\subset\mathbb{R}^1$ and $A\subset\mathbb{R}^2$.

From such set the following sets can be obtained (you can see examples for $A\subset\mathbb{R}^1$ below):

№	set	formula for $n=1$
1	int(A)	$((0;1)\cup(2;3)\cup(3;4))\times R^{n-1}$
2	cl(A)	$([0;1]\cup[2;4]\cup[5,6]\cup\{7\})\times R^{n-1}$

3	int(cl(A))	$((0;1)\cup(2;4)\cup(5;6))\times R^{n-1}$
4	cl(int(A))	$([0;1]\cup[2;4])\times R^{n-1}$
5	int(cl(int(A)))	$((0;1)\cup(2;4))\times R^{n-1}$
6	cl(int(cl(A)))	$([0;1]\cup[2;4]\cup[5,6])\times R^{n-1}$

If we apply operations *int* and *cl* to first four sets, we will obtain one of sets 1-6. If we apply operations *int* and *cl* to sets five and six, we will obtain the following sets: *int(cl(int(cl(A))))=int(cl(A)), cl(int(cl(int(A))))=cl(int(A)), int(int(cl(int(A))))= int(cl(int(A))), cl(cl(int(cl(A))))= cl(int(cl(A))).* So, we will obtain sets that were obtained before. Thereby no more sets can be obtained from *A* by sequences of these two operations.

The theorem is proved.

So, the maximal number of distinct sets that can be obtained from *A* using the operations *int* and *cl* equals six (without initial *A*).

Remark: *during the proofs of this part we used only topological (and not metric) structure of $\mathbb{R}^n$. Therefore all of our proofs remain correct for an arbitrary topological space, although the total number of obtainable sets may vary depending on topological structure of set being studied (for example, in discreet topology, where any set is open, one can obtain only the initial set).*

Part 2: convex sets

From now on we consider R^n as a topological vector space.

Definition. For any $A, B \in \mathrm{R}^n$ the set of points *X* satisfying the equality $X = (1-\lambda)A + \lambda B$ for some $\lambda \in [0,1]$ is called *line segment* between *A* and *B*.

2.1. Properties of convex sets:

Definition. A set $C \subset R^n$ is called *convex* if, for any two points $p, q \in C$, the line segment connecting *p* and *q* is in *C*. For example, any ball is convex set.

- Intersection of any number of the convex sets is the convex set.
- Linear combination of convex sets is convex set
- $((1-\lambda)C + \lambda C) \subset C$, where *C* is convex set and $\forall \lambda \in [0,1]$.

2.2. Properties of operation *conv*:

Definition. $conv(A) = \bigcap\limits_{convex\ Z \supset A} Z$

- $conv(C) = C \Leftrightarrow C$ is a convex set.
- *conv(conv(C))=conv(C)*
- If $A \subset B$, then $conv(A) \subset conv(B)$

Theorem 1. Let *C* be a convex set and *int(C) ≠ø*, $x \in \mathrm{int}(C)$, $y \in cl(C)$. Then $\forall \lambda$, $0 \le \lambda < 1$, the point $((1-\lambda)x + \lambda y) \in \mathrm{int}(C)$, i.e. $[x, y) \subset \mathrm{int}(C)$.

Proof. Notice that $cl(A) = \bigcap\limits_{\forall \varepsilon > 0} (A + \varepsilon B_1(0))$. Therefore we need to prove that

$((1-\lambda)x + \lambda y + \varepsilon B_1(0)) \subset C$ for certain $\varepsilon > 0$ and $\forall \lambda \in [0,1)$.

$y \in cl(C) \Rightarrow y \in (C + \varepsilon B_1(0)),\ \forall \varepsilon > 0$.

$((1-\lambda)x + \lambda y + \varepsilon B_1(0)) \subset ((1-\lambda)x + (C + \varepsilon B_1(0)) + \varepsilon B_1(0)) = (1-\lambda)x + \varepsilon(1+\lambda)B_1(0) + \lambda C =$

$= \left((1-\lambda)(x+\varepsilon(1+\lambda)(1-\lambda)^{-1}B_1(0)) + \lambda C\right) \subset \left((1-\lambda)C + \lambda C\right) \subset C$, because for sufficiently small $\varepsilon > 0$, $\left(x+\varepsilon(1+\lambda)(1-\lambda)^{-1}B_1(0)\right) \subset C$.

Theorem 2. Let *C* be a convex set and *int(C)≠ø*. Then *cl(C)* and *int(C)* are also convex sets.

Proof. $cl(C) = \bigcap_{\forall \varepsilon>0}(C+\varepsilon B_1(0))$ is a convex set, because it is the intersection of convex sets $(C+\varepsilon B_1(0))$. These sets are convex, because they are a linear combination of convex sets.

int(C) is a convex set by the theorem 1, because if $x \in \text{int}(C)$, $y \in \text{int}(C) \subset cl(C)$, then $[x,y] \subset \text{int}(C)$.

Theorem 3. If *C* is a convex set and int(C)≠ø, then

a) *cl(int(C))=cl(C)*

b) *int(cl(C))=int(C)*

Proof. a) $\text{int}(C) \subset C \Rightarrow cl(\text{int}(C)) \subset cl(C)$. On the other hand, if $y \in cl(C)$, $x \in \text{int}(C)$, then by theorem 1 $[x,y) \subset \text{int}(C) \Rightarrow y \in cl(\text{int}(C))$, i.e. $cl(C) \subset cl(\text{int}(C))$, so $cl(\text{int}(C)) = cl(C)$.

b) $C \subset cl(C) \Rightarrow \text{int}(C) \subset \text{int}(cl(C))$. On the other hand, let $z \in \text{int}(cl(C))$. Show, that $z \in \text{int}(C)$. Let $x \in \text{int}(C)$, $x \neq z$. Consider a straight line *l*, that passes through points *x* and *z*. Point $y = \left(x + \lambda(z-x)\right) \in l$, $\forall \lambda$. For sufficiently small $\lambda > 0$, $y \in \text{int}(cl(C)) \subset cl(C)$. By theorem 1 $(y,x] \subset \text{int}(C)$ and as $z = \frac{1}{1+\lambda}y + \frac{\lambda}{1+\lambda}x$, then $z \in (y,x)$ and so $z \in \text{int}(\tilde{N}) \Rightarrow \text{int}(cl(C)) \subset \text{int}(C)$. Therefore, *int(cl(C))=int(C)*.

Corollary. if *int(conv(A))≠ø*, then: *conv(cl(conv(A)))=cl(conv(A)), conv(int(conv(A)))=int(conv(A)), cl(int(conv(A)))=cl(conv(A)), int(cl(conv(A)))=int(conv(A))*.

Theorem 4. If *int(conv(A))≠ø,* then *int(conv(int(A)))=conv(int(A))*

Proof. $\text{int}(conv(\text{int}(A))) \subset conv(\text{int}(A))$ by definition of operation *int;*

$\text{int}(A) \subset conv(\text{int}(A)) \Rightarrow \text{int}(\text{int}(A)) = \text{int}(A) \subset \text{int}(conv(\text{int}(A))) \Rightarrow$

$\Rightarrow conv(\text{int}(A)) \subset conv(\text{int}(conv(\text{int}(A)))) = \text{int}(conv(\text{int}(A)))$ by the theorem 2.

So, *int(conv(int(A)))=conv(int(A)).*

Theorem 5. If *int(conv(A))≠ø,* then *cl(conv(cl(A)))=cl(conv(A))*

Proof. $A \subset cl(A) \Rightarrow conv(A) \subset conv(cl(A)) \Rightarrow cl(conv(A)) \subset cl(conv(cl(A)))$;

$A \subset conv(A) \Rightarrow cl(A) \subset cl(conv(A)) \Rightarrow conv(cl(A)) \subset conv(cl(conv(A))) = cl(conv(A)) \Rightarrow$

$\Rightarrow cl(conv(cl(A))) \subset cl(cl(conv(A))) = cl(conv(A))$ by the theorem 2.

So, *cl(conv(cl(A)))=cl (conv (A)).*

Corollary. *conv(cl(A))⊂cl(conv(A)).*

Point2

a) int(conv(cl(A)))=int(conv(A)), int(conv(A))≠ø

Theorem 6. i*nt(conv(cl(A)))=int(conv(A))*.

Proof. By the corollary from theorem 5, $conv(cl(A))\subset cl(conv(A))$, so, $int(conv(cl(A)))\subset int(cl(conv(A)))=int(conv(A))$. On the other hand, $A\subset cl(A)\Rightarrow conv(A)\subset conv(cl(A))\Rightarrow \mathrm{int}(conv(A))\subset \mathrm{int}(conv(cl(A)))$.

So, i*nt(conv(cl(A)))=int(conv(A)),* QED.

b) int(conv(A))=ø

Consider an arbitrary integer $n\geq 1$ instead of for $n=2$.

Prove, that in this case *A* is contained in *(n-1)-dimensional* subspace. Really, select *(n+1)* points from *A*, that aren't situated in any *(n-1)-dimensional* subspace (if we can't do it, then *A* is contained in *(n-1)-dimensional* subspace). Consider *n-dimensional* simplex with vertexes in these points. It is contained in *conv(A)* and interior of it isn't empty. Contradiction. So, *A* is contained in *(n-1)-dimensional* subspace.

Consider the cases for $n=1$ and $n=2$ separately.

1) $n=1$. In this case *A* is single point or empty set. When we apply any of operations to empty set, we obtain itself, and if we apply operations *cl* and *conv* for single point we will get itself, but after using operation *int* we will get an empty set.

So, for this case, the maximal number of distinct sets that can be obtained from *A* is two (it's no matter with *A* or without it).

2) $n=2$. Number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals four (without initial *A*). Firstly construct an example of such *A*, and then prove that it is impossible to obtain more sets.

Suppose $A=\{0\}\cup(1,2)\cup(2,3)$. Then:

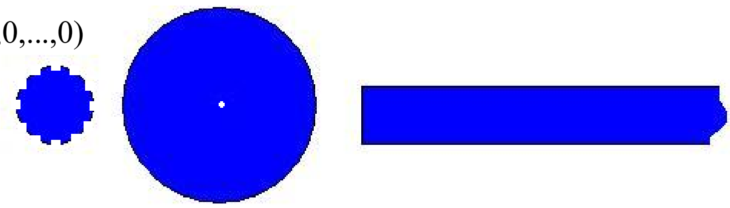
1) *int(A)=ø*
2) *cl(A)={0}*$\cup$*[1,3]*
3) *conv(A)=[0,3)*
4) *cl(conv(A))=[0,3]*

Prove that *conv(cl(A))* is a closed set. A closed set is either union of disjoint segments and closed rays, or R^1 itself. Thus obviously *conv(cl(A))* is a closed set. Therefore, *conv(cl(A))=cl(conv(cl(A)))=cl(conv(A))* by the theorem 5, if $A\subset P^1$ (**theorem 7**).

3) $n\geq 3$

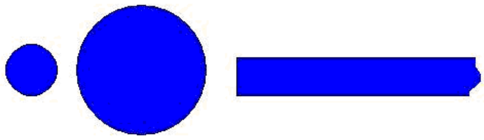
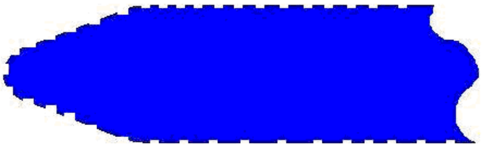
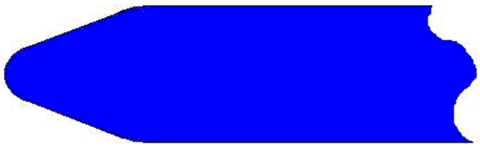
Number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals five (without initial *A*). At first let's make an example of such *A*, and then prove that it is impossible to obtain more sets.

$$\text{Let } A=\begin{cases}\left[\begin{array}{l}-1\leq x_2\leq 1,\ x_1\geq 0\\ (x_1+4)^2+x_2^2\leq 9,\ (x_1,x_2,x_3,...,x_n)\neq(-4,0,0,...,0)\\ (x_1+12)^2+x_2^2<4\end{array}\right.\\ x_3=x_4=x_5=...=x_n=0\end{cases}$$



So, the following sets can be obtained (coordinates of them you can see in appendix):

№	set	Picture
1	*int(A)*	Ø

2	*cl(A)*	
3	*conv(A)*	
4	*cl(conv(A))*	

Now prove that when we apply to the initial set other sequences of these three operations, we will take sets that were taken before. For this prove that *cl(conv(cl(A)))=cl(conv(A)).* Without loss of generality int(conv(A))≠ø, then the result follows from theorem 5.

So, when we apply operation *int* for any of taken sets or A we will take an empty set and when we apply other two operations for any of taken sets or A we will take one of already taken sets: necessary equalities were proved before.

b) int(conv(A))≠ø

Consider an arbitrary *n≥1* instead of *n=2*.

Consider the case for *n=1* separately:

Number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals 13 (without initial *A*). At first let's make an example of such *A*, and then prove that it is impossible to obtain more sets.

Let $A=\{0\}\cup(1,2)\cup(2,3)\cup[4,5)\cup((6,7)\cap Q)$.

Then:

1) $\mathrm{int}(A)=(1,2)\cup(2,3)\cup(4,5)$
2) $cl(A)=\{0\}\cup[1,,3]\cup[4,5]\cup[6,7]$
3) *conv(A)=[0,7)*
4) *int(conv(A))=(0,7)*
5) *conv(int(A))=(1,5)*
6) *cl(conv(A))=[0,7]*
7) $int(cl(A))=(1,3)\cup(4,5)\cup(6,7)$
8) $cl(int(A))=[1,3]\cup[4,5]$
9) $int(cl(int(A)))=(1,3)\cup(4,5)$
10) $cl(int(cl(A)))=[1,3]\cup[4,5]\cup[6,7]$
11) *conv(int(cl(A)))=(1,7)*
12) *conv(cl(int(A)))=[1,5]*
13) *conv(cl(int(cl(A))))=[1,7]*

To prove that no more sets cannot be obtained it's necessary to prove that sets that are obtained from A by next sequences of these three operations will be equal to some of already obtained sets:

1) *int(conv(cl((int(A))))=int(conv(int(A)))=conv(int(A))* by theorems 4 and 6.
2) *cl(conv(cl(int(A))))=cl(conv(int(A)))=conv(cl(int(A)))* by theorems 5 and 7.
3) *int(conv(cl(int(cl(A)))))=int(conv(int(cl(A))))=conv(int(cl(A)))* by theorems 6 and 4.
4) *cl(conv(cl(int(cl(A)))))=cl(conv(int(cl(A))))=conv(cl(int(cl(A))))* by theorems 5 and 7.
5) *int(conv(int(cl(A))))=conv(int(cl(A)))* by theorem 4.
6) *cl(conv(int(cl(A))))=conv(cl(int(cl(A))))* by theorem 7

7) *int(cl(int(cl(A))))=int(cl(A))* by point 1.
8) *cl(int(cl(int(A))))=cl(int(A))* by point 1.
9) *conv(int(cl(int(A))))=conv(int(A))* **(theorem 8)**

$$\mathrm{int}(A) \subset cl(\mathrm{int}(A)) \Rightarrow \mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A) \subset \mathrm{int}(cl(\mathrm{int}(A))) \Rightarrow conv(\mathrm{int}(A)) \subset conv(\mathrm{int}(cl(\mathrm{int}(A))))$$

$$\mathrm{int}(X) \subset X \Rightarrow conv(\mathrm{int}(X)) \subset conv(X) \Rightarrow conv(\mathrm{int}(X)) = \mathrm{int}(conv(\mathrm{int}(X))) \subset \mathrm{int}(conv(X)) \Rightarrow$$

$$conv(\mathrm{int}(cl(\mathrm{int}(A)))) \subset \mathrm{int}(conv(cl(\mathrm{int}(A)))) = \mathrm{int}(conv(\mathrm{int}(A))) = conv(\mathrm{int}(A))$$

Thereby, *conv(int(cl(int(A))))=conv(int(A))*.

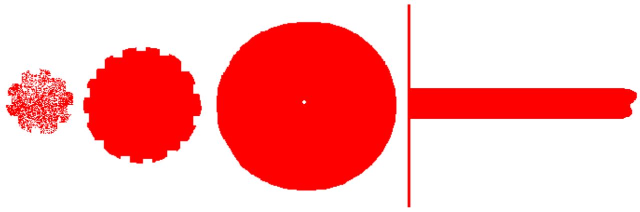
So, it's impossible to obtain more, than 13 sets.

So, the maximal number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals 13 (without initial *A*).

The case for *n≥2*:

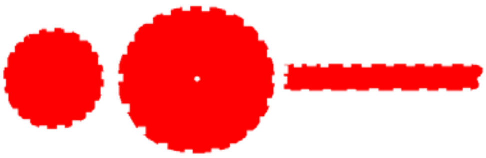
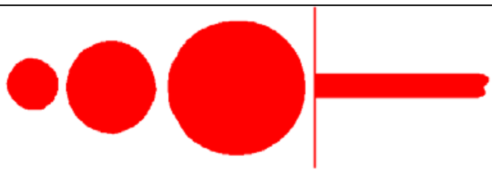
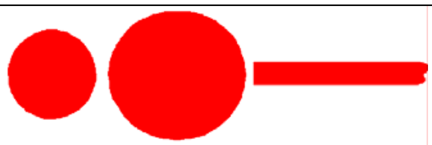
Unknown number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals 16 (without initial *A*). At first let's make an example of such *A*, and then prove that it is impossible to obtain more sets.

On the picture you can see example for $A \subset \mathbb{R}^2$:



$$\text{Let } A = \left[\begin{array}{l} \begin{cases} x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le 49 \\ x_1 = 0 \end{cases} \\ \begin{cases} x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le 1 \\ x_1 \ge 0 \end{cases} \\ (x_1 + 7)^2 + x_2^2 + \ldots + x_n^2 \le 36, \ (x_1, x_2, x_3, \ldots, x_n) \ne (-7, 0, 0, \ldots, 0) \\ (x_1 + 18)^2 + x_2^2 + \ldots + x_n^2 < 16 \\ (x_1 + 25)^2 + x_2^2 + \ldots + x_n^2 < 4, \ \textit{where } x_1, x_2, \ldots, x_n \textit{ are rational} \end{array} \right.$$

Then the next distinct sets can be obtained (on the pictures you can see examples for $A \subset \mathbb{R}^2$):

№	set	Picture
1	*int(A)*	
2	*cl(A)*	
3	*conv(A)*	
4	*cl(int(A))*	

5	*int(cl(A))*	
6	*cl(conv(A))*	
7	*conv(cl(A))*	
8	*conv(int(A))*	
9	*int(conv(A))*	
10	*int(cl(int((A)))*	
11	*cl(int(cl(A)))*	
12	*conv(cl(int(A)))*	
13	*conv(int(cl(A)))*	
14	*cl(conv(int(A)))*	
15	*conv(cl(int(cl(A))))*	
16	*cl(conv(int(cl(A))))*	

To prove that more sets cannot be obtained it's necessary to prove that sets that are obtained from A by next sequences of these three operations will be equal to some of already obtained sets:

1. *int(cl(conv(int(cl(A)))))=int(conv(int(cl(A))))=conv(int(cl(A)))* by theorems 3 and 4.
2. *conv(cl(conv(int(cl(A)))))=cl(conv(int(cl(A))))* by theorem 2.
3. *int(conv(cl(int(cl(A)))))=int(conv(int(cl(A))))=conv(int(cl(A)))* by theorems 6 and 4.
4. *cl(conv(cl(int(cl(A)))))=cl(conv(int(cl(A))))* by theorem 5.
5. *int(cl(conv(int(A))))=int(conv(int(A)))=conv(int(A))* by theorems 3 and 4.
6. *conv(cl(conv(int(A))))=cl(conv(int(A)))* by theorem 2.

7. *int(conv(int(cl(A))))=conv(int(cl(A)))* by theorem 4.
8. *int(conv(cl(int(A))))=int(conv(int(A)))=conv(int(A))* by theorems 6 and 4.
9. *cl(conv(cl(int(A))))=cl(conv(int(A)))* by theorem 5.
10. *cl(int(cl(int(A))))=cl(int(A))* by point 1.
11. *int(cl(int(cl(A))))=int(cl(A))* by point 1.
12. *conv(int(cl(int(A))))=conv(int(A))* by theorem 8.
13. *int(cl(conv(A)))=int(conv(A))* by theorem 3.
14. *int(conv(cl(A)))=int(conv(A))* by theorem 6.
15. *cl(int(conv(A))=cl(conv(A))* by theorem 3.
16. *conv(int(conv(A)))=int(conv(A))* by theorem 2.
17. *conv(cl(conv(A)))=cl(conv(A))* by theorem 2.
18. *cl(conv(cl(A)))=cl(conv(A))* by theorem 5.
19. *int(conv(int(A)))=conv(int(A))* by theorem 4.

If we apply other sequences of these three operations, we will get sets obtained before.
Therefore, it's impossible to obtain more sets using these three operations.
Thus, the maximal number of distinct sets that can be obtained from *A* using the operations *int, cl* and *conv* equals 16 (without initial *A*).

Sketch of generalization

1. Relative operations.

Affine hull of set A is the set $affA = \left\{\sum \lambda_i x_i \middle| x_i \in A, \sum \lambda_i = 1, \{x_i\} - \text{finite set}\right\}$, *dimension* of affine hull is by definition $\dim(affA - a), a \in A$. Suppose dim*affA* = *m*. One can easily show that *affA* with topology induced from $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$. Therefore one can introduce relative operation *rint(A)* = $int_{affA}A$, which means interior in *affA*. Similarly *rcl(A)* = $cl_{affA}A$, *rconv(A)* = $conv_{affA}A$. Obviously *rcl*(*A*) = *cl*(*A*), *rconv*(*A*) = *conv*(*A*). Consider the initial problem with operations *rint*, *rcl*, *rconv*. Due to the properties mentioned above, the problem for such operations reduces to the initial problem with condition *int*(*conv*(*A*)) ≠ *ø*, and the answer does not change.

2. Projectors.

Since $int^2 = int$, $cl^2 = cl$, $conv^2 = conv$, it's natural to study other idempotent operations.
Consider such linear operators *T* that $T^2 = T$, which are called *projectors*. It's a well-known fact that after applying *T* on some set one'll obtain subset of some *k*-dimensional hyperplane [1]. Consider the initial problem with operations *rint*, *rcl*, *rconv*, *T*, where *T* is some projector. Hence first three operations obviously do not change affine hull of the set *A*, there's no point in applying *T* on *A* more than once. Therefore the following simple estimation can be obtained.

Proposition. The number of sets obtainable from *A* using operations *rint*, *rcl*, *rconv*, *T* is not more than 17^2.

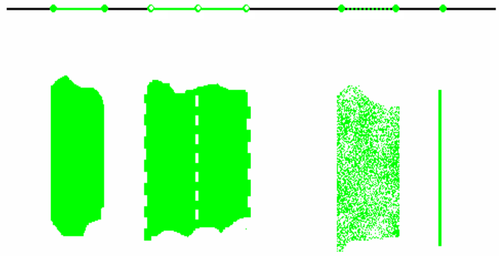
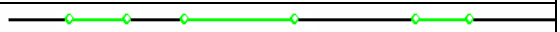
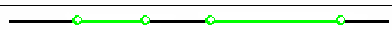
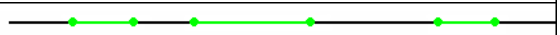
Sketch of proof. Every obtainable set can be represented as *U*(*T*(*V*(*A*))), where *U* and *V* are compositions of the operations *rint*, *rcl*, *rconv*. By point 2b one can obtain not more than 17 sets from *A*. After using *T* on these sets the number of sets won't increase. From each of 17 sets not more than 17 new sets can be obtained. Therefore one can obtain not more, than 17^2 sets.

References:

1. http://en.wikipedia.org/wiki/Projector_(mathematics)
2. Makarov. Additional chapter of calculus.

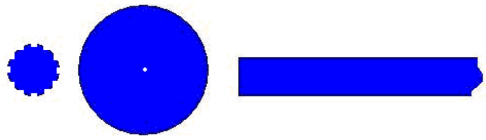
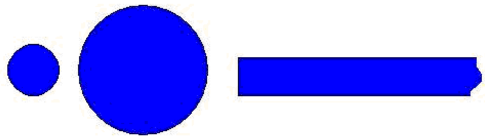
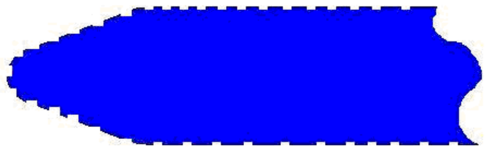
Appendix

Point 1

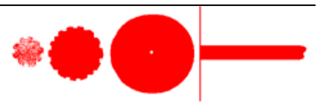
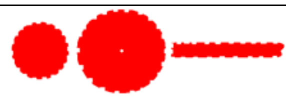
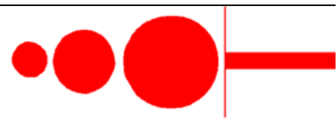
№	set	coordinates of this set	picture
1	*A*	$\begin{cases}\left[\begin{array}{l} 0\le x_1\le 1 \\ 2< x_1< 3 \\ 3< x_1< 4 \\ 5\le x_1\le 6, \text{ where } x_1 \text{ is rational} \\ x_1=7 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
2	int(A)	$\begin{cases}\left[\begin{array}{l} 0< x_1< 1 \\ 2< x_1< 3 \\ 3< x_1< 4 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
3	cl(A)	$\begin{cases}\left[\begin{array}{l} 0\le x_1\le 1 \\ 2\le x_1\le 4 \\ 5\le x_1\le 6 \\ x_1=7 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
4	int(cl(A))	$\begin{cases}\left[\begin{array}{l} 0< x_1< 1 \\ 2< x_1< 4 \\ 5< x_1< 6 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
5	cl(int(A))	$\begin{cases}\left[\begin{array}{l} 0\le x_1\le 1 \\ 2\le x_1\le 4 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
6	int(cl(int(A)))	$\begin{cases}\left[\begin{array}{l} 0< x_1< 1 \\ 2< x_1< 4 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	
7	*cl(int(cl(A)))*	$\begin{cases}\left[\begin{array}{l} 0\le x_1\le 1 \\ 2\le x_1\le 4 \\ 5\le x_1\le 6 \end{array}\right. \\ x_2, x_3, x_4, \dots, x_n \text{ take all real values}\end{cases}$	

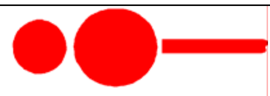
Point 2

int(conv(A))=ø

№	set	coordinates of this set	picture
1	*A*	$\begin{cases} \left[\begin{array}{l} -1\le x_2\le 1,\ x_1\ge 0 \\ (x_1+4)^2+x_2^2\le 9,\ (x_1,x_2,x_3,\dots,x_n)\ne(-4,0,0,\dots,0) \\ (x_1+12)^2+x_2^2<4 \end{array}\right. \\ x_3=x_4=x_5=\dots=x_n=0 \end{cases}$	
2	*int(A)*	ø	
3	*cl(A)*	$\begin{cases} \left[\begin{array}{l} \begin{cases} -1\le x_2\le 1 \\ x_1\ge 0 \end{cases} \\ (x_1+4)^2+x_2^2\le 9 \\ (x_1+12)^2+x_2^2\le 4 \end{array}\right. \\ x_2=x_3=\dots=x_n=0 \end{cases}$	
4	*conv(A)*	$\begin{cases} \left[\begin{array}{l} \begin{cases} x_1\ge -4 \\ -3<x_2<3 \end{cases} \\ (x_1+4)^2+x_2^2\le 9 \\ (x_1+12)^2+x_2^2<4 \\ \begin{cases} -\frac{49}{4}\le x_1\le -\frac{35}{8} \\ -\frac{4\sqrt{7}}{3}-\frac{x_1}{3\sqrt{7}}<x_2<\frac{4\sqrt{7}}{3}+\frac{x_1}{3\sqrt{7}} \end{cases} \end{array}\right. \\ x_3=x_4=x_5=\dots=x_n=0 \end{cases}$	
5	*cl(conv(A))*	$\begin{cases} \left[\begin{array}{l} \begin{cases} x_1\ge -4 \\ -3\le x_2\le 3 \end{cases} \\ (x_1+4)^2+x_2^2\le 9 \\ (x_1+12)^2+x_2^2\le 4 \\ \begin{cases} -\frac{49}{4}\le x_1\le -\frac{35}{8} \\ -\frac{4\sqrt{7}}{3}-\frac{x_1}{3\sqrt{7}}\le x_2\le \frac{4\sqrt{7}}{3}+\frac{x_1}{3\sqrt{7}} \end{cases} \end{array}\right. \\ x_3=x_4=x_5=\dots=x_n=0 \end{cases}$	

int(conv(A))≠ø

№	set	coordinates of it	picture
1	*A*	$\left[\begin{array}{l} \begin{cases} x_2^2+x_3^2+x_4^2+\ldots+x_n^2\le 49 \\ x_1=0 \end{cases} \\ \begin{cases} x_2^2+x_3^2+x_4^2+\ldots+x_n^2\le 1 \\ x_1\ge 0 \end{cases} \\ (x_1+7)^2+x_2^2+\ldots+x_n^2\le 36,\ (x_1,x_2,x_3,\ldots,x_n)\ne(-7,0,0,\ldots,0) \\ (x_1+18)^2+x_2^2+\ldots+x_n^2<16 \\ (x_1+25)^2+x_2^2+\ldots+x_n^2<4,\ \mathit{where}\ x_1,x_2,\ldots,x_n\ \mathit{are\ rational} \end{array}\right.$	
2	*int(A)*	$\left[\begin{array}{l} \begin{cases} x_2^2+x_3^2+x_4^2+\ldots+x_n^2< 1 \\ x_1>0 \end{cases} \\ (x_1+7)^2+x_2^2+\ldots+x_n^2< 36,\ (x_1,x_2,x_3,\ldots,x_n)\ne(-7,0,0,\ldots,0) \\ (x_1+18)^2+x_2^2+\ldots+x_n^2<16 \end{array}\right.$	
3	*cl(A)*	$\left[\begin{array}{l} \begin{cases} x_2^2+x_3^2+x_4^2+\ldots+x_n^2\le 49 \\ x_1=0 \end{cases} \\ \begin{cases} x_2^2+x_3^2+x_4^2+\ldots+x_n^2\le 1 \\ x_1\ge 0 \end{cases} \\ (x_1+7)^2+x_2^2+\ldots+x_n^2\le 36 \\ (x_1+18)^2+x_2^2+\ldots+x_n^2\le 16 \\ (x_1+25)^2+x_2^2+\ldots+x_n^2\le 4 \end{array}\right.$	

4	*conv(A)*	$\left[\begin{array}{l} \begin{cases} x_1 \ge 0 \\ x_2^2+x_3^2+x_4^2+...+x_n^2 \le 49 \end{cases} \\ \begin{cases} -\frac{202}{11} < x_1 \le -\frac{83}{11} \\ x_2^2+x_3^2+x_4^2+...+x_n^2 < \left(\frac{\sqrt{30}}{16}x_1+\frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ \begin{cases} \frac{25-3\sqrt{62}}{2317-294\sqrt{62}} \le x_1 \le 0 \\ x_2^2+x_3^2+x_4^2+...+x_n^2 \le \left((49-6\sqrt{62})x_1+7\right)^2 \end{cases} \\ (x_1+7)^2+x_2^2+...+x_n^2 \le 36 \\ (x_1+18)^2+x_2^2+...+x_n^2 < 16 \\ (x_1+25)^2+x_2^2+...+x_n^2 < 4 \\ \begin{cases} -\frac{179}{7} < x_1 < -\frac{134}{7} \\ x_2^2+x_3^2+x_4^2+...+x_n^2 < \left(\frac{2\sqrt{5}}{15}x_1+\frac{64\sqrt{5}}{15}\right)^2 \end{cases} \end{array}\right.$	
5	*cl(int(A))*	$\left[\begin{array}{l} \begin{cases} x_2^2+x_3^2+x_4^2+...+x_n^2 \le 1 \\ x_1 \ge 0 \end{cases} \\ (x_1+7)^2+x_2^2+...+x_n^2 \le 36,\ (x1,x2,x3,...,xn) \ne (-7,0,0,...,0) \\ (x_1+18)^2+x_2^2+...+x_n^2 \le 16 \end{array}\right.$	
6	*int(cl(A))*	$\left[\begin{array}{l} \begin{cases} x_2^2+x_3^2+x_4^2+...+x_n^2 < 1 \\ x_1 > 0 \end{cases} \\ (x_1+7)^2+x_2^2+...+x_n^2 < 36 \\ (x_1+18)^2+x_2^2+...+x_n^2 < 16 \\ (x_1+25)^2+x_2^2+...+x_n^2 < 4 \end{array}\right.$	

7	*cl(conv(A))*	$\left[\begin{array}{l} \begin{cases} x_1 \ge 0 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le 49 \end{cases} \\ \begin{cases} -\frac{202}{11} \le x_1 \le -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8} \right)^2 \end{cases} \\ \begin{cases} \frac{25 - 3\sqrt{62}}{2317 - 294\sqrt{62}} \le x_1 \le 0 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le \left((49 - 6\sqrt{62}) x_1 + 7 \right)^2 \end{cases} \\ (x_1 + 7)^2 + x_2^2 + \ldots + x_n^2 \le 36 \\ (x_1 + 18)^2 + x_2^2 + \ldots + x_n^2 \le 16 \\ (x_1 + 25)^2 + x_2^2 + \ldots + x_n^2 \le 4 \\ \begin{cases} -\frac{179}{7} \le x_1 \le -\frac{134}{7} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \le \left(\frac{2\sqrt{5}}{15} x_1 + \frac{64\sqrt{5}}{15} \right)^2 \end{cases} \end{array} \right.$	

8	*conv(cl(A))*	$\left[\begin{array}{l} \begin{cases} x_1 \ge 0 \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<49 \end{cases} \\ \begin{cases} -\frac{202}{11} \le x_1 \le -\frac{83}{11} \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2 \le \left(\frac{\sqrt{30}}{16}x_1+\frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ \begin{cases} \frac{25-3\sqrt{62}}{2317-294\sqrt{62}} \le x_1 \le 0 \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2 \le \left((49-6\sqrt{62})x_1+7\right)^2 \end{cases} \\ (x_1+7)^2+x_2^2+\ldots+x_n^2 \le 36 \\ (x_1+18)^2+x_2^2+\ldots+x_n^2 \le 16 \\ (x_1+25)^2+x_2^2+\ldots+x_n^2 \le 4 \\ \begin{cases} -\frac{179}{7} \le x_1 \le -\frac{134}{7} \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2 \le \left(\frac{2\sqrt{5}}{15}x_1+\frac{64\sqrt{5}}{15}\right)^2 \end{cases} \end{array}\right.$	
9	*conv(int(A))*	$\left[\begin{array}{l} \begin{cases} x_1 \ge -7 \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<36 \end{cases} \\ \begin{cases} -\frac{202}{11} \le x_1 \le -\frac{83}{11} \\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2 < \left(\frac{\sqrt{30}}{16}x_1+\frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ (x_1+7)^2+x_2^2+\ldots+x_n^2 < 36 \\ (x_1+18)^2+x_2^2+\ldots+x_n^2 < 16 \end{array}\right.$	

10	*int(conv(A))*	$\left[\begin{array}{l}\begin{cases}x_1 \ge 0\\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<49\end{cases}\\ \begin{cases}-\frac{202}{11}<x_1\le-\frac{83}{11}\\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<\left(\frac{\sqrt{30}}{16}x_1+\frac{15\sqrt{30}}{8}\right)^2\end{cases}\\ \begin{cases}\frac{25-3\sqrt{62}}{2317-294\sqrt{62}}\le x_1\le 0\\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<\left((49-6\sqrt{62})x_1+7\right)^2\end{cases}\\ (x_1+7)^2+x_2^2+\ldots+x_n^2<36\\ (x_1+18)^2+x_2^2+\ldots+x_n^2<16\\ (x_1+25)^2+x_2^2+\ldots+x_n^2<4\\ \begin{cases}-\frac{179}{7}<x_1<-\frac{134}{7}\\ x_2^2+x_3^2+x_4^2+\ldots+x_n^2<\left(\frac{2\sqrt{5}}{15}x_1+\frac{64\sqrt{5}}{15}\right)^2\end{cases}\end{array}\right.$	
11	*int(cl(int((A)))*	$\left[\begin{array}{l}\begin{cases}x_2^2+x_3^2+x_4^2+\ldots+x_n^2<1\\ x_1>0\end{cases}\\ (x_1+7)^2+x_2^2+\ldots+x_n^2<36\\ (x_1+18)^2+x_2^2+\ldots+x_n^2<16\end{array}\right.$	
12	*cl(int(cl(A)))*	$\left[\begin{array}{l}\begin{cases}x_2^2+x_3^2+x_4^2+\ldots+x_n^2\le 1\\ x_1\ge 0\end{cases}\\ (x_1+7)^2+x_2^2+\ldots+x_n^2\le 36\\ (x_1+18)^2+x_2^2+\ldots+x_n^2\le 16\\ (x_1+25)^2+x_2^2+\ldots+x_n^2\le 4\end{array}\right.$	

13	*conv(cl(int(A)))*	$\left[\begin{array}{l} \begin{cases} x_1 \geq -7 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 < 36 \end{cases} \\ \begin{cases} -\frac{202}{11} \leq x_1 \leq -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ (x_1+7)^2 + x_2^2 + \ldots + x_n^2 \leq 36 \\ (x_1+18)^2 + x_2^2 + \ldots + x_n^2 \leq 16 \end{array}\right.$	
14	*conv(int(cl(A)))*	$\left[\begin{array}{l} \begin{cases} x_1 \geq -7 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 < 36 \end{cases} \\ \begin{cases} -\frac{202}{11} \leq x_1 \leq -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 < \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ (x_1+7)^2 + x_2^2 + \ldots + x_n^2 < 36 \\ (x_1+18)^2 + x_2^2 + \ldots + x_n^2 < 16 \\ (x_1+25)^2 + x_2^2 + \ldots + x_n^2 < 4 \\ \begin{cases} -\frac{179}{7} \leq x_1 \leq -\frac{134}{7} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 < \left(\frac{2\sqrt{5}}{15} x_1 + \frac{64\sqrt{5}}{15}\right)^2 \end{cases} \end{array}\right.$	
15	*cl(conv(int(A)))*	$\left[\begin{array}{l} \begin{cases} x_1 \geq -7 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq 36 \end{cases} \\ \begin{cases} -\frac{202}{11} \leq x_1 \leq -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8}\right)^2 \end{cases} \\ (x_1+7)^2 + x_2^2 + \ldots + x_n^2 \leq 36 \\ (x_1+18)^2 + x_2^2 + \ldots + x_n^2 \leq 16 \end{array}\right.$	

16	*conv(cl(int(cl(A))))*	$\left[\begin{array}{l} \begin{cases} x_1 \geq -7 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 < 36 \end{cases} \\ \begin{cases} -\frac{202}{11} \leq x_1 \leq -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8} \right)^2 \end{cases} \\ (x_1 + 7)^2 + x_2^2 + \ldots + x_n^2 \leq 36 \\ (x_1 + 18)^2 + x_2^2 + \ldots + x_n^2 \leq 16 \\ (x_1 + 25)^2 + x_2^2 + \ldots + x_n^2 \leq 4 \\ \begin{cases} -\frac{179}{7} \leq x_1 \leq -\frac{134}{7} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{2\sqrt{5}}{15} x_1 + \frac{64\sqrt{5}}{15} \right)^2 \end{cases} \end{array}\right.$	
17	*cl(conv(int(cl(A))))*	$\left[\begin{array}{l} \begin{cases} x_1 \geq -7 \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq 36 \end{cases} \\ \begin{cases} -\frac{202}{11} \leq x_1 \leq -\frac{83}{11} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{\sqrt{30}}{16} x_1 + \frac{15\sqrt{30}}{8} \right)^2 \end{cases} \\ (x_1 + 7)^2 + x_2^2 + \ldots + x_n^2 \leq 36 \\ (x_1 + 18)^2 + x_2^2 + \ldots + x_n^2 \leq 16 \\ (x_1 + 25)^2 + x_2^2 + \ldots + x_n^2 \leq 4 \\ \begin{cases} -\frac{179}{7} \leq x_1 \leq -\frac{134}{7} \\ x_2^2 + x_3^2 + x_4^2 + \ldots + x_n^2 \leq \left(\frac{2\sqrt{5}}{15} x_1 + \frac{64\sqrt{5}}{15} \right)^2 \end{cases} \end{array}\right.$