Problem8: Points on Curves

Team: *Belarus*

Abstract

In this paper we introduce two different problems with the same title. In the case of Problem1 we proved that for any monotonous function $\forall n \in \mathbb{N} \setminus \{0\}$ there exists required sequence $\{t_n\}$ (Theorem 1.4). Also we noted that for n=2 for all continuous functions there exists required sequence. We researched some classes of functions: periodic, symmetrical, and flowing (definitions are given at page 4). It is proved that if $f(x)$ is flowing then for any even $n \in \mathbb{N} \setminus \{0\}$ there exists required sequence $\{t_n\}$ (Theorem 1.7 I). Also if $f(x)$ is symmetrical and flowing then $\forall n \in \mathbb{N} \setminus \{0\}$ there exists required sequence $\{t_n\}$ (Theorem 1.7 I and 1.9). We researched some particular cases with symmetrical and periodic functions (Theorem 1.5-1.9). The cases a) and b) are particularly researched in Corollary 1.1-1.3. We have also generalized Problem1 to the case of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ (Problem 2) and proved analogue of Theorem1.4 about monotonous function (Theorem 2.3). In Theorem 2.1we proved that if $n = 2$ for all continuous functions there exists required sequence. Analogues of theorems about flowing, periodic and symmetrical functions are given in Theorems 2.3–2.5.

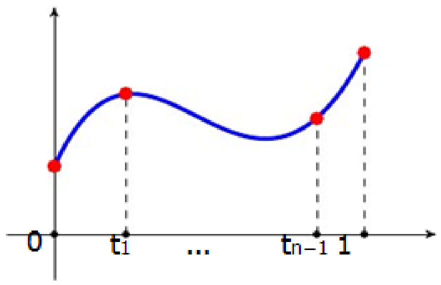
In Problem 3 we researched the case $\alpha \geq \frac{1}{2}$. In Theorem 3.2 it is shown that if $\alpha \geq \frac{1}{2}$ then $\forall C \geq \frac{1}{2}$ there exists C-fitting function. Next, in Theorem3.4, we show that if $\alpha > 1$ then $\forall C \geq \frac{\alpha^\alpha (\alpha - 1)^{\alpha-1}}{(2\alpha - 1)^{2\alpha-1}}$ there exists C-fitting function. Note that $\lim_{\alpha\to\infty} \frac{\alpha^\alpha (\alpha - 1)^{\alpha-1}}{(2\alpha - 1)^{2\alpha-1}} = 0$. A particular case of lines is researched in Theorem 3.5, Corollary 3.1 and Remark3.2. A particular case of Problem3 is Problem4, where we researched C-balancing functions. Finally, in Theorem4.1 we prove that $\forall C \geq \frac{\alpha^\alpha (\alpha - 1)^{\alpha-1}}{2^\alpha (2\alpha - 1)^{2\alpha-1}}$ there exists C-balancing function.

Points on Curves

§1.POINTS AT THE PLANE

Problem1. Let f : [0, 1] → R be a continuous function. Find all (some) positive integers n such that there exists a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ for which the expression

$$\left|f(t_{i+1})-f(t_i)\right|+\left|t_{i+1}-t_i\right|$$



does not depend on $i = 0, 1, ..., n - 1$.
Consider the case that f is

a) a polygonal chain (that is, piecewise linear);
b) a polynomial of degree k = 2, 3, ... ;
c) a trigonometric function.

Theorem1.1. Suppose n=2 and $[a,b]\subset[0,1]$. Then there exist three real numbers $a=t_0<t_1<t_2=b$ *such that* $\left|f(t_0)-f(t_1)\right|+\left|t_0-t_1\right|=\left|f(t_2)-f(t_1)\right|+\left|t_2-t_1\right|$.

Proof. Consider the function $F(x)=\left|f(x)-f(a)\right|+\left|x-a\right|-\left|f(x)-f(b)\right|-\left|x-b\right|$; let $D(F)=[a,b]$. Then $F(x)=\left|f(x)-f(a)\right|-\left|f(x)-f(b)\right|-2x-a-b$. F(x) is continuous as a difference of continuous functions. Since F*(a)<0; F(b)>0* , we obtain that there exists $t_1=x_0\in[a,b]\,\big|\,F(x_0)=0$. According to Problem1. we obtain that for n=2 there always exists required sequence.

Theorem is proved.

Theorem1.2. $\begin{matrix} F(x)=M=const. \\ \forall(c,d)\subset[a,b] \end{matrix} \Leftrightarrow \left[\begin{matrix} if\ \ f(b)\le f(a)\ \forall x\in[c,d]\ f(x)=-x+C\,and\ \ f(b)\le f(x)\le f(a) \\ if\ \ f(a)\le f(b)\ \forall x\in[c,d]\ f(x)=x+C\,and\ \ f(a)\le f(x)\le f(b) \end{matrix}\right.$

Proof.

I. $\Rightarrow$. Consider the function $g(x)=\left|f(x)-f(a)\right|-\left|f(x)-f(b)\right|$. Let $\Delta x\to 0$. Then $\forall x\in(c,d)$

$$M=F(x)=\left|f(x)-f(a)\right|-\left|f(x)-f(b)\right|+2x-a-b$$

$$M=F(x+\Delta x)=\left|f(x+\Delta x)-f(a)\right|-\left|f(x+\Delta x)-f(b)\right|+2x+2\Delta x-a-b$$

Then we get:

$$0=F(x+\Delta x)-F(x)=\left(\left|f(x)-f(a)\right|-\left|f(x)-f(b)\right|\right)-\left(\left|f(x+\Delta x)-f(a)\right|-\left|f(x+\Delta x)-f(b)\right|\right)+2\Delta x\Rightarrow$$

$\Rightarrow 2\Delta x=g(x+\Delta x)-g(x)\Rightarrow \dfrac{g(x+\Delta x)-g(x)}{\Delta x}=2\Rightarrow g'(x)=\lim\limits_{\Delta x\to 0}\dfrac{g(x+\Delta x)-g(x)}{\Delta x}=2$ .Let us consider two cases.

Case1. $f(b)\le f(a)$. Let there exists $x_0\in(c,d)\,\big|\,f(x_0)<f(b)$. By definition $f(x)$ is continuous. It follows that there exists a $\delta-$neighborhood $U_\delta(x_0)$, $\delta>0$ such that $\forall x\in U_\delta(x_0)\ \ f(x)<f(b)$. Then $\forall x\in U_\delta(x_0)\ \ g(x)=f(a)-f(x)-f(b)+f(x)=f(a)-f(b)\Rightarrow\forall x\in U_\delta(x_0)\ g'(x)=0\ne 2$. This contradiction proves that $\forall x\in(c,d)\ \ f(x_0)\ge f(b)$. Similarly we obtain that $\forall x\in(c,d)\ \ f(x_0)\le f(a)$. Then $\forall x\in(c,d)$ we have $g(x)=f(a)+f(b)-2f(x)$. Consider the function $t(x)=-2x+g(x)$.Then $t'(x)=0\Rightarrow t(x)=C_0, where C_0=const.\Rightarrow f(x)=-x+C.$. Thus, $\forall x\in(c,d)$ we have $f(x)=-x+C, where C=const.$ Since $f(x)$ is continuous, it follows that $\forall x\in[c,d]$ we have $f(x)=-x+C, where\,C=const.$

Case2. $f(a)\le f(b)$ Similarly as in Case1 we obtain $g(x)=-f(a)-f(b)+2f(x).$. But it follows that $\forall x\in[c,d]$ $f(x)=x+C$ $, where\,C=const.$ □

II. $\Leftarrow$. Suppose $f(b)\le f(a)\ \forall x\in[c,d]\ f(x)=-x+C\, and\ \ f(b)\le f(x)\le f(a)$. The second case is similar. From $f(b)\le f(x)\le f(a)$ it follows that:

$F(x)= f(a)+ f(b)-2f(x)-2x-a-b= f(a)+ f(b)+2x-2C-2x-a-b= f(a)+ f(b)-2C-a-b.$ □

Theorem is proved.

Now we shall give the following definitions.

Consider a segment $[a,b]\subset[0,1]$. A sequence $a\ =t_0<t_1<...<t_{n-1}<t_n=q\le b$ is called ***q_n-rightfitting*** for [a,b] if $|f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|=C(q)=const\ \forall i=0,1,...,n-1$. Consider the function $dr_n^{(a)}(q)=$ C(q) if there exists such sequence, where $q\in(a,b]$; $dr_n^{(a)}(a)=0$. A sequence $a\le q\ =t_0<t_1<...<t_{n-1}<t_n=b$ is called ***q_n-leftfitting*** for [a,b] if $|f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|=C(q)=const\ \forall i=0,1,...,n-1$. Consider the function $dl_n^{(b)}(q)=C(q)$ if there exists such sequence, where $q\in[a,b)$; $dl_n^{(b)}(b)=0$.

Of course, if there exist two or more required sequences, maybe there exist two or more functions $dl_n^{(b)}$ (or $dr_n^{(a)}$).

Theorem1.3. Consider a segment $[a,b]\subset[0,1]$. Let f(x) be a function such that:

1) There exists a positive integer $k\,|\,\forall\, q\in[a,b]\ \exists\ q_k-rightfitting\ sequence\ for[a,b]\ and\ dr_k^{(a)}(x) is\ continuous\ when\ x\in[a,b].$
2) There exists a positive integer $p\,|\,\forall\, q\in[a,b]\ \exists\ q_p-leftfitting\ sequence\ for\ [a,b]\ and\ dl_p^{(b)}(x) is\ continuous\ when\ x\in[a,b].$

Then $\exists$ sequence of real numbers $a\ =t_0<t_1<...<t_{k+p-1}<t_{k+p}=b\ \ |\ \ |f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|$=const. $\forall i\ =0,1,...,n-1$.

Proof. Consider the function $G(x)=dr_k^{(a)}(x)-dl_p^{(b)}(x)$. G(x) is continuous as a difference of two continuous functions. Since G*(a)>0; G(b)<0* , we obtain $\exists\ x_0\in[a;b]\ \big|\ G(x_0)=0$. Then there exists a sequence of real numbers $a\ =t_0<t_1<...<t_{k+p-1}<t_{k+p}=b$ such that the expression $|f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|$ does not depend on $i=0,1,...,n-1$.

Theorem is proved.

Theorem1.4. Suppose f(x) is monotonous; then $\forall[a,b]\subset[0,1]\ \forall n\in\mathbb{N}\setminus\{0\}\ \exists!\ sequence\, of\ real\ numbers\ \{t_n\}\big|$ $a=t_0<t_1<...<t_{n-1}<t_n=b\ \ and\ \ |f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|=C\ \forall i=0,1,...,n-1$.

Proof. Let us prove that $\forall[a,b]\subset[0,1]\ \ \forall n\in\mathbb{N}\setminus\{0\}\ \exists!\ sequence\, of\ real\ numbers\ \{t_n\}\big|\ \ a=t_0<t_1<...<t_{n-1}<t_n=b$ $|f(t_{i+1})- f(t_i)|+|t_{i+1}-t_i|=C\ \forall i=0,1,...,n-1$; next let us show that $dr_n^{(a)}(x)=\dfrac{|f(x)- f(a)|+|x-a|}{n}$ and $dl_n^{(b)}(x)=\dfrac{|f(b)- f(x)|+|b-x|}{n}$. The proof is by induction on n.

For n=1, there is nothing to prove: $dr_1^{(a)}(x)=|f(x)- f(a)|+|x-a|\wedge dl_1^{(b)}(x)=|f(b)- f(x)|+|b-x|$ are continuous; $dr_1^{(a)}(a)=0$; $dl_1^{(b)}(b)=0$; obviously there exists a unique required sequence. For n=2 from Theorem 1.1 it follows that there exists required sequence; since f(x) is monotonous we obtain that $dr_2^{(a)}(x)=\dfrac{|f(x)- f(a)|+|x-a|}{2}$; $dr_2^{(a)}(a)=0$; $dl_2^{(b)}(x)=\dfrac{|f(b)- f(x)|+|b-x|}{2}$ $dl_2^{(b)}(x)=0$. Since $dr_2^{(a)}(b)=\dfrac{|f(b)- f(a)|+|b-a|}{2}$, we obtain that $\exists!\ t_1\big|dr_2^{(a)}(b)=|f(t_1)- f(a)|+|t_1-a|$. Then there exists a unique required sequence {t2}.

Assume that $\forall[a,b]\ \ \forall k=1,...,n-1\ \exists!\ such\ sequence\ \{t_k\}$ and $dr_k^{(a)}(x)=\dfrac{|f(x)- f(a)|+|x-a|}{k}$;

Belarus

$dl_k^{(b)}(x)=\dfrac{|f(b)-f(x)|+|b-x|}{k}$. Let us prove this for k=n. Since statement is true for k=1 and k=n-1, from Theorem1.3 it follows that $\forall[a,b]$ there exists required sequence {t$_n$}. Since f(x) is monotonous we have $dr_n^{(a)}(x)=\dfrac{|f(x)-f(a)|+|x-a|}{n}\wedge dl_n^{(b)}(x)=\dfrac{|f(b)-f(x)|+|b-x|}{n}$; then $\exists!\ t_1\,\big|dr_2^a(b)=|f(t_1)-f(a)|+|t_1-a|$; similarly $\exists!t_2,...,t_{n-1}\Rightarrow\exists!\{t_n\}$. Thus, induction is complete. According to the problem1.1 we may take a=0 and b=1.

Theorem is proved.

Theorem1.5.

I. Let $f(x)$ be a periodic function $f(x)=f(x+T)$, where $x\in[a,b]\subset[0,1]\,and\,b=a+kT$, $k\in\mathbb{N}\backslash\{0\}$. *Then for* $n=2k$ $\exists$ *sequence* $a=t_0<t_1<...<t_n=b\big|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

II. Let $f(x)$ be a periodic function $f(x)=f(x+T)$, where $x\in[a,b]\subset[0,1]\,and\,b=a+kT+\dfrac{T}{2}$, $k\in\mathbb{N}\backslash\{0\}$. *Then for* $n=2k+1$ $\exists$ *sequence* $a=t_0<t_1<...<t_n=b\big|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

Proof.

I. Let $t_0=a,\ t_1=a+\dfrac{T}{2},t_2=a+T,...,t_i=a+\dfrac{Ti}{2},...,t_{2k}=b.$ Then we have $|t_{i+1}-t_i|=\dfrac{T}{2}$ and $|f(t_{i+1})-f(t_i)|=\left|f\left(a+\dfrac{Ti}{2}+\dfrac{T}{2}\right)-f\left(a+\dfrac{Ti}{2}\right)\right|=\left|f(a)-f\left(a+\dfrac{T}{2}\right)\right|$. Then $|f(t_{i+1})-f(t_i)|+|t_{i+1}-t_i|=const$.

So, we find required sequence. This proves I.

II. Let $t_0=a,\ t_1=a+\dfrac{T}{2},t_2=a+T,...,t_i=a+\dfrac{Ti}{2},...,t_{2k+1}=b.$ Then $|f(t_{i+1})-f(t_i)|+|t_{i+1}-t_i|=\dfrac{T}{2}+\left|f(a)-f\left(a+\dfrac{T}{2}\right)\right|$. So, we find required sequence. This proves II.

Theorem is proved.

Now we shall give the following definitions. A function *f(x)* is called ***symmetrical on [a,b]*** $\subset[0,1]$ if $f\left(\dfrac{a+b}{2}+x\right)=f\left(\dfrac{a+b}{2}-x\right)\forall x\in\left[\dfrac{a-b}{2},\dfrac{b-a}{2}\right]$. A function *f(x)* is called ***flowing on [a,b]*** $\subset[0,1]$ if it is monotonous when $x\in\left[a,\dfrac{a+b}{2}\right]$ and $x\in\left[\dfrac{a+b}{2},b\right]$.

Theorem1.6.

I. Let $f(x)$ be symmetrical on $[a,b]\subset[0,1]$. *Then* $\exists$ *sequence* $a=t_0<t_1<...<t_4=b$ $\big|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \ \forall i=0,1...,3.$

II. Let $f(x)$ be a periodic function $f(x)=f(x+T)\ \forall\ x\in[a,b]\subset[0,1]$ *and* $b=a+kT,\ k\in\mathbb{N}\backslash\{0\}$. Suppose $f(x)$ is symmetrical on $[a,a+T]$. *Then for* $n=4k$ $\exists$ *sequence* $a=t_0<t_1<...<t_n=b$ $\big|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

III. Let $f(x)$ be a periodic function $f(x)=f(x+T)\ \forall\ x\in[a,b]\subset[0,1]$, $b=a+kT+\dfrac{T}{2},\ k\in\mathbb{N}\backslash\{0\}$. Suppose $f(x)$ is symmetrical on $[a,a+T]$. *Then for* $n=4k+2$ $\exists$ *sequence* $a=t_0<t_1<...<t_n=b$ $\big|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

Proof.

I. From Theorem1 it follows that when $x\in\left[a,\dfrac{a+b}{2}\right]$ $\exists x_0\big|\ |f(x_0)-f(a)|+|x_0-a|=$

$=\left|f\left(x_0\right)-f\left(\frac{a+b}{2}\right)\right|+\left|x_0-\frac{a+b}{2}\right|$. Let $t_0=a,\ t_1=x_0,\ t_2=\frac{a+b}{2}$. Since $f(x)$ is symmetrical on $[a,a+T]$, we obtain *for* $y_0=b-x_0$ that $\left|f\left(y_0\right)-f\left(\frac{a+b}{2}\right)\right|+\left|y_0-\frac{a+b}{2}\right|=\left|f\left(y_0\right)-f\left(b\right)\right|+\left|y_0-b\right|=\left|f\left(x_0\right)-f\left(\frac{a+b}{2}\right)\right|+$ $+\left|x_0-\frac{a+b}{2}\right|$. Let $t_3=y_0, t_4=b.$ So, we find required sequence. This proves I.

II. From I it follows that $\exists a=t_0<t_1<t_2<t_3<t_4=a+T\ \left|\ \forall i=0,1,2,3\quad \left|f(t_{i+1})-f(t_i)\right|+\left|t_{i+1}-t_i\right|=const.\right.$ (Because $f(x)$ is symmetrical on $[a,a+T]$). Let $t_5=t_1+T, t_6=t_2+T,...,t_i=t_{i-4}+T,...,\ t_{4k}=a+kT=b.$ Then $\left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \forall i=0,1...,n-1.$ So, we find required sequence. This proves II.

III. From II it follows that $\exists\ sequence\quad a=t_0<t_1<t_2<...<t_{4k-1}<t_{4k}=a+kT$ $\mid\ \left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \ \forall i=0,1...,n-1.$ Using notations of b) let $t_{4k+1}=t_{4k-3}+T$ and $t_{4k+2}=t_{4k-2}+T=b.$ So, we find required sequence. This proves III.

Theorem is proved.

Theorem 1.7.

I. Let $f(x)$ be a flowing function when $x\in[a,b],\ f(a)=f(b)$. Then $\forall n=2k, k\in\mathbb{N}\setminus\{0\}\ \exists\ sequence$ $a=t_0<t_1<...<t_{n-1}<t_n=b\quad\mid\ \left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \forall i=0,1...,n-1.$

II. Let $f(x)$ be a periodic function $f(x)=f(x+T)\ \ \forall\ x\in[a,b]\subset[0,1]$ *and* $b=a+kT,\ k\in\mathbb{N}\setminus\{0\}$. Suppose $f(x)$ is a flowing function on $[a,a+T]$. Then $\forall n=2kp, p\in\mathbb{N}\setminus\{0\}\ \exists\ sequence$ $a=t_0<t_1<...<t_{n-1}<t_n=b\quad\mid\ \left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \forall i=0,1...,n-1.$

III. Let $f(x)$ be a periodic function $f(x)=f(x+T)\ \forall\ x\in[a,b]\subset[0,1]$, $b=a+kT+\frac{T}{2},\ k\in\mathbb{N}\setminus\{0\}$. Suppose $f(x)$ is a flowing function on $[a,a+T]$. Then $\forall n=2kp+p, p\in\mathbb{N}\setminus\{0\}\ \exists\ sequence$ $a=t_0<t_1<...<t_{n-1}<t_n=b\quad\mid\ \left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \forall i=0,1...,n-1.$

Proof.

I. Consider $\forall k\in\mathbb{N}\setminus\{0\}$. From Theorem1.4. it follows that $\exists sequence\{c_k\}: a=c_0<c_1<...<c_k=\frac{a+b}{2}\ \mid$ $\left|c_{i+1}-c_i\right|+\left|f\left(c_{i+1}\right)-f\left(c_i\right)\right|=\frac{\left|f\left(\frac{a+b}{2}\right)-f\left(a\right)\right|+\frac{a+b}{2}-a}{k}\quad \forall i=0,1...,k-1.$ Also from Theorem1.4. it follows that $\exists sequence\{d_k\}: \frac{a+b}{2}=d_0<d_1<...<d_k=b\mid\forall i=0,1...,k-1\left|d_{i+1}-d_i\right|+\left|f\left(d_{i+1}\right)-f\left(d_i\right)\right|=$ $=\frac{\left|f\left(\frac{a+b}{2}\right)-f\left(b\right)\right|+b-\frac{a+b}{2}}{k}$. Let $t_0=a_0,t_1=a_1,...,t_k=a_k,t_{k+1}=b_1,t_{k+2}=b_2,...,t_{2k}=b_k.$ Since $\frac{\left|f\left(\frac{a+b}{2}\right)-f\left(b\right)\right|+b-\frac{a+b}{2}}{k}=\frac{\left|f\left(\frac{a+b}{2}\right)-f\left(a\right)\right|+\frac{a+b}{2}-a}{k}$, it follows that $\{t_k\}$ *is required*. This proves I.

II. Consider $\forall p\in\mathbb{N}\setminus\{0\}$. From I it follows that $\exists\ sequence\ a=t_0<t_1<...<t_{2p-1}<t_{2p}=a+T$ $\mid\ \left|t_{i+1}-t_i\right|+\left|f\left(t_{i+1}\right)-f\left(t_i\right)\right|=const.\ \forall i=0,1...,2p-1.$ Let $t_{2p+1}=t_1+T, t_{2p+2}=t_2+T,...,t_i=t_{i-2p}+T,$

$..., t_{2pk} = t_{2p(k-1)} + T = b.$ We have that $\{t_{2pk}\}$ *is required*. This proves II.

III. From II it follows that $\forall n = 2kp, p \in \mathbb{N} \setminus \{0\}$ $\exists$ *sequence* $a = t_0 < t_1 < ... < t_{n-1} < t_n = a + kT$ $\mid |t_{i+1} - t_i| + |f(t_{i+1}) - f(t_i)| = const.$ $\forall i = 0,1...,n-1.$ Let $t_{2pk+1} = t_{2p(k-1)+1} + T, t_{2pk+2} = t_{2p(k-1)+2} + T,$ $..., t_{2pk+k} = t_{2p(k-1)+k} + T = a + kT + \frac{T}{2} = b$. We have that $\{t_{p(2k+1)}\}$ *is required*. This proves III.

Theorem is proved.

Theorem 1.8.

I. Let $f(x)$ be a monotonous function when $x \in [a,c]$ *and* $[c,b]$. Suppose $\frac{|f(c) - f(a)| + c - a}{|f(b) - f(c)| + b - c} = \frac{p}{q} \in \mathbb{Q},$ $p, q \neq 0$, $\gcd(p,q) = 1$. Then $\forall n = \lambda(p+q), \lambda \in \mathbb{N} \setminus \{0\}$ $\exists$ *sequence* $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ $\mid |t_{i+1} - t_i| + |f(t_{i+1}) - f(t_i)| = const.$ $\forall i = 0,1...,n-1.$

II. Let $f(x)$ be a periodic function $f(x) = f(x+T)$ $\forall x \in [0,1] \mid x + T \in [0,1]$. Consider a segment $[a,b] \subset [0,1]$, where $kT < b < kT + \frac{T}{2}$, $k \in \mathbb{N} \setminus \{0\}$. Let $n = b - kT$, and let $f(x)$ be flowing when $x \in [a, a+T]$. Suppose $\frac{|f(a) - f(a+n)| + n}{\left|f\left(\frac{T}{2} + a\right) - f(a+n)\right| + \frac{T}{2} - n} = \frac{p}{q} \in \mathbb{Q},$ $p, q \neq 0, \gcd(p,q) = 1.$ Then $\forall n = \lambda(2k(p+q) + p),$ $\lambda \in \mathbb{N} \setminus \{0\}$ $\exists$ *sequence* $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ $\mid |t_{i+1} - t_i| + |f(t_{i+1}) - f(t_i)| = const.$ $\forall i = 0,1...,n-1.$

III. Let $f(x)$ be a periodic function $f(x) = f(x+T)$ $\forall x \in [0,1] \mid x + T \in [0,1]$. Consider a segment $[a,b] \subset [0,1]$, where $kT + \frac{T}{2} < b < (k+1)T$, $k \in \mathbb{N} \setminus \{0\}$. Let $n = b - kT - \frac{T}{2}$, and let $f(x)$ be flowing when $x \in [a, a+T]$.

Suppose $\frac{\left|f\left(\frac{T}{2} + a\right) - f(a + n + \frac{T}{2})\right| + n}{\left|f(a) - f\left(a + n + \frac{T}{2}\right)\right| + \frac{T}{2} - n} = \frac{p}{q} \in \mathbb{Q},$ $p, q \neq 0, \gcd(p,q) = 1.$ Then $\forall n = \lambda((2k+1)(p+q) + p),$ $\lambda \in \mathbb{N} \setminus \{0\}$ $\exists$ *sequence* $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$ $\mid |t_{i+1} - t_i| + |f(t_{i+1}) - f(t_i)| = const.$ $\forall i = 0,1...,n-1.$

Proof.

I. Since $f(x)$ is monotonous when $x \in [a,c]$, from Theorem1.4. it follows that for $n = \lambda p$, $\lambda \in \mathbb{N} \setminus \{0\}$ $\exists$ *sequence* $a = t_0 < t_1 < ... < t_{n-1} < t_n = c$ $\mid |t_{i+1} - t_i| + |f(t_{i+1}) - f(t_i)| = \frac{|f(a) - f(c)| + c - a}{\lambda p}$ $\forall i = 0,1...,n-1.$

Since $f(x)$ is monotonous when $x \in [c,b]$, from Theorem1.4. it follows that for $n = \lambda q$ $\exists$ *sequence* $c = a_0 < a_1 < ... < a_{n-1} < a_n = b$ $\mid |a_{i+1} - a_i| + |f(a_{i+1}) - f(a_i)| = \frac{|f(b) - f(c)| + b - c}{\lambda q}$ $\forall i = 0,1...,n-1.$ Since $\frac{|f(c) - f(a)| + c - a}{|f(b) - f(c)| + b - c} = \frac{p}{q} \in \mathbb{Q}$, we have $\frac{|f(a) - f(c)| + c - a}{\lambda p} = \frac{|f(b) - f(c)| + b - c}{\lambda q}$. Let $t_{\lambda p+1} = a_1, t_{\lambda p+2} = a_2,$ $..., t_{\lambda p + \lambda q} = a_{\lambda q}.$ It follows that $\{t_{\lambda p + \lambda q}\}$ *is required*. This proves I.

II. We have that $f(x)$ is monotonous when $x \in [a, a+n]$ *and* $\left[a+n, a+\frac{T}{2}\right]$. From Theorem1.4. it follows that

$\exists sequence \left\{b_{\lambda p}\right\}: a=b_0<b_1<\ldots<b_{\lambda p}=a+n \ |\forall i=0,\ldots,\lambda p-1 \ \left|f(b_{i+1})-f(b_i)\right|+b_{i+1}-b_i=\frac{|f(a)-f(a+n)|+n}{\lambda p}$.

Similarly from Theorem1.4. it follows that $\exists sequence \left\{a_{\lambda q}\right\}: a+n=a_0<a_1<\ldots<a_{\lambda p}=a+\frac{T}{2}$ $|\forall i=0,\ldots,\lambda q-1 \ \left|f(a_{i+1})-f(a_i)\right|+a_{i+1}-a_i=\frac{\left|f(a+n)-f(a+\frac{T}{2})\right|+\frac{T}{2}-n}{\lambda q}$. Since $\frac{|f(a)-f(a+n)|+n}{\left|f\left(\frac{T}{2}+a\right)-f\left(a+n\right)\right|+\frac{T}{2}-n}=\frac{p}{q}$, we have $\frac{|f(a)-f(a+n)|+n}{\lambda p}=\frac{\left|f(a+n)-f(a+\frac{T}{2})\right|+\frac{T}{2}-n}{\lambda q}$. Let $t_0=b_0, t_1=b_1,\ldots,t_{\lambda p}=b_{\lambda p}, t_{\lambda p+1}=a_1, t_{\lambda p+2}=a_2,\ldots,t_{\lambda p+\lambda q}=a_{\lambda q}=a+\frac{T}{2}$. From Theorem1.4. it follows that

$$\exists sequence\left\{c_{\lambda p+\lambda q}\right\}: a+\frac{T}{2}=c_0<c_1<\ldots<c_{\lambda p+\lambda q}=a+T \ | \ \left|f(c_{i+1})-f(c_i)\right|+\left|c_{i+1}-c_i\right|=\frac{\left|f(a)-f\left(a+\frac{T}{2}\right)\right|+\frac{T}{2}}{\lambda p+\lambda q}=$$

$$=\frac{\left|f(a+n)-f\left(a+\frac{T}{2}\right)\right|+\left|f(a)-f(a+n)\right|+n+\frac{T}{2}-n}{\lambda(p+q)}=\frac{|f(a)-f(a+n)|+n}{\lambda p}=\frac{\left|f(a+n)-f(a+\frac{T}{2})\right|+\frac{T}{2}-n}{\lambda q}.$$

Let $t_{\lambda p+\lambda q+1}=c_1, t_{\lambda p+\lambda q+2}=c_2,\ldots,t_{2(\lambda p+\lambda q)}=c_{\lambda p+\lambda q}=a+T$. Now let us take $t_{2\lambda p+2\lambda q+1}=t_1+T$, $t_{2\lambda p+2\lambda q+2}=t_2+T$, $\ldots, t_i=t_{i-2\lambda p-2\lambda q}+T,\ldots,$ $t_{k(2\lambda p+2\lambda q)+\lambda p}=t_{\lambda p}+kT=a+n+kT=b.$ It follows that $\left\{t_{2k\lambda(p+q)+\lambda p}\right\} is\ required$. This proves II.

III. Similarly as in II we obtain $\exists sequence \left\{a_{\lambda p}\right\}: a_0=a+\frac{T}{2}<a_1<\ldots<a_{\lambda p}=a+n+\frac{T}{2}|$ $\forall i=0,\ldots,\lambda p-1 \ \left|f(a_{i+1})-f(a_i)\right|+a_{i+1}-a_i=\frac{\left|f\left(a+\frac{T}{2}+n\right)-f\left(a+\frac{T}{2}\right)\right|+n}{\lambda p}$ and $\exists sequence \left\{b_{\lambda q}\right\}:$ $a+\frac{T}{2}+n=b_0<b_1<\ldots<b_{\lambda q}=a+T \ |\forall i=0,\ldots,\lambda q-1 \ \left|f(b_{i+1})-f(b_i)\right|+b_{i+1}-b_i=\frac{\left|f(a)-f\left(a+n+\frac{T}{2}\right)\right|+\frac{T}{2}-n}{\lambda q}$.

Next similarly as in II we have $\exists sequence\left\{c_{\lambda p+\lambda q}\right\}: a=c_0<c_1<\ldots<c_{\lambda p+\lambda q}=a+\frac{T}{2}$ $|\forall i=0,\ldots,\lambda p+\lambda q-1 \left|f(c_{i+1})-f(c_i)\right|+\left|c_{i+1}-c_i\right|=\frac{\left|f(a)-f\left(a+\frac{T}{2}\right)\right|+\frac{T}{2}}{\lambda p+\lambda q}$. We get analogously as in II that

$$\frac{\left|f(a)-f\left(a+\frac{T}{2}\right)\right|+\frac{T}{2}}{\lambda p+\lambda q}=\frac{\left|f\left(a+\frac{T}{2}+n\right)-f\left(a+\frac{T}{2}\right)\right|+n}{\lambda p}=\frac{\left|f(a)-f\left(a+n+\frac{T}{2}\right)\right|+\frac{T}{2}-n}{\lambda q}.$$

Let $t_0=c_0, t_1=c_1,\ldots,t_{\lambda p+\lambda q}=c_{\lambda p+\lambda q}, t_{\lambda p+\lambda q+1}=a_1, t_{\lambda p+2+\lambda q}=a_2,\ldots,t_{2\lambda p+\lambda q}=a_{\lambda p}, t_{2\lambda p+\lambda q+1}=b_1,\ldots,t_{2\lambda p+2\lambda q}=b_{\lambda q}$. Next let us form sequence recurrently: $t_i=T+t_{i-2\lambda p-2\lambda q}, up\ to\ t_{(2k+1)(\lambda p+\lambda q)+\lambda p}=a+kT+\frac{T}{2}+n=b.$ It follows that

$\{t_{\lambda(2k+1)(p+q)+\lambda p}\}$ *is required*. This proves III.

Theorem is proved.

Corollary1.1. Consider any continuous function $f(x)|D(f)=[a,b]$. Let $f(x)$ be monotonous when $x\in[a=x_0,x_1]$, $x\in[x_1,x_2],...,x\in[x_{n-1},x_n=b]$. Suppose $\forall i=0,...,n-2$ *we have* $0\neq\frac{|f(x_i)-f(x_{i+1})|+x_{i+1}-x_i}{|f(x_{i+2})-f(x_{i+1})|+x_{i+2}-x_{i+1}}\in\mathbb{Q}$. Then $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}\ |\ \forall s=\lambda(q_1+q_2+..+q_n)\ \lambda\in\mathbb{N}\setminus\{0\}\exists sequence\{t_s\}:t_0=a<t_1<...<t_s=b\quad|\ \forall i=0,1..,s-1$ $|f(t_{i+1})-f(t_i)|+|t_{i+1}-t_i|=const.$

Proof. From $\frac{|f(x_i)-f(x_{i+1})|+x_{i+1}-x_i}{|f(x_{i+2})-f(x_{i+1})|+x_{i+2}-x_{i+1}}\in\mathbb{Q}$ it follows that $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}$, $\gcd(q_1,q_2,...,q_n)=1$, such that $(|f(x_0)-f(x_1)|+x_1-x_0):(|f(x_2)-f(x_1)|+x_2-x_1):...:(|f(x_n)-f(x_{n-1})|+x_n-x_{n-1})=q_1:q_2:...:q_n$. From Theorem1.4. it follows that $\forall[x_i,x_{i+1}]$, $where\ i=0,...,n-1\ \exists\ sequence\{d^{(i)}_{\lambda q_{i+1}}\}:x_i=d_0<d_1<...<d_{\lambda q_{i+1}}=x_{i+1}\ |$ $|f(d_{i+1})-f(d_i)|+|d_{i+1}-d_i|=\frac{|f(x_i)-f(x_{i+1})|+x_{i+1}-x_i}{\lambda q_{i+1}}$. But we have that $\forall i=0,..,n-1$ $\frac{|f(x_i)-f(x_{i+1})|+x_{i+1}-x_i}{\lambda q_{i+1}}=const.$ Let $t_0=d^{(0)}_0,t_1=d^{(0)}_1,...,t_{\lambda q_1}=d^{(0)}_{\lambda q_1}$, $t_{\lambda q_1+1}=d^{(1)}_1,t_{\lambda q_1+2}=d^{(1)}_2$, ..., $t_{\lambda q_1+\lambda q_2}=d^{(1)}_{\lambda q_2}$, $t_{\lambda q_1+\lambda q_2+1}=d^{(2)}_1$,..., $t_{\lambda(q_1+q_2+...+q_n)}=d^{(n-1)}_{\lambda q_n}$. This sequence is required.

Corollary is proved.

Corollary1.2. Let us consider the case if $f(x)$ is a polygonal chain when $x\in[a,b]$ (piecewise linear) of lines $l_1:y=k_1x+b_1;l_2:y=k_2x+b_2;...;l_n:y=k_nx+b_n$, *where lines numerated from the left to the right*. Suppose $a,b,k_i,b_i\in\mathbb{Q}\ \forall i=1,...,n$. Then $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}\quad|\ \forall s=\lambda(q_1+q_2+..+q_n)\ \lambda\in\mathbb{N}\setminus\{0\}\quad\exists sequence\{t_s\}$: $t_0=a<t_1<...<t_s=b\quad|\ \forall i=0,1..,s-1\quad|f(t_{i+1})-f(t_i)|+|t_{i+1}-t_i|=const.$

Proof. Note that if $l_i\cap l_j=M(x,f(x))\Rightarrow x,f(x)\in\mathbb{Q}$. Let $m_0=a,m_n=b.$ Suppose $l_i\cap l_{i+1}=Z_i(x_i,y_i)$ $\Rightarrow let\ m_i=x_i$. Then we have that $f(x)$ is monotonous if $x\in[m_0,m_1],...,[m_{n-1},m_n]$. Also it follows that $\forall i=0,1,..,n-2\quad\frac{|f(m_i)-f(m_{i+1})|+m_{i+1}-m_i}{|f(m_{i+2})-f(m_{i+1})|+m_{i+2}-m_{i+1}}\in\mathbb{Q}$. From Corollary1 it follows that $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}$ $|\ \forall s=\lambda(q_1+q_2+..+q_n)\ \lambda\in\mathbb{N}\setminus\{0\}\ \exists\ required\ sequence\{t_s\}$. This completes the proof.

Corollary is proved.

Example1.1. Let $l_1:y=x;l_2:y=3-3x$, where $f(x)$ is defined on $[0,1]$. Then $\frac{\frac{3}{4}+\left|3-\frac{9}{4}\right|}{\frac{1}{4}+\left|3-\frac{9}{4}\right|}=\frac{3}{2}\Rightarrow\forall n=5k\ \exists$ required sequence.

Corollary1.3. Let us consider the case if $f(x)$ is a polynomial when $x\in[a,b]$, where $a,b\in\mathbb{Q}$. Suppose $f(x)\in\mathbb{Q}[x]$, and let all k roots of the equation $f'(x)=0$ be $\in\mathbb{Q}$. Then $\exists q_1,q_2,...,q_k\in\mathbb{N}\setminus\{0\}$ $|\ \forall s=\lambda(q_1+q_2+..+q_k)\ \lambda\in\mathbb{N}\setminus\{0\}\quad\exists sequence\{t_s\}$: $t_0=a<t_1<...<t_s=b\quad|\ \forall i=0,1..,s-1\quad|f(t_{i+1})-f(t_i)|+$ $+|t_{i+1}-t_i|=const.$

Proof. Let $m_1<m_2<...<m_{k-1}$ be roots of the equation $f'(x)=0$, take $m_0=a$ *and* $m_k=b$. we have that $f(x)$ is monotonous if $x\in[m_0,m_1],...,[m_{n-1},m_n]$. Also from $f(x)\in\mathbb{Q}[x]$ it follows that $\forall i=0,1,..,n-2\quad\frac{|f(m_i)-f(m_{i+1})|+m_{i+1}-m_i}{|f(m_{i+2})-f(m_{i+1})|+m_{i+2}-m_{i+1}}\in\mathbb{Q}$. From Corollary1 it follows that $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}$

$|\forall s=\lambda(q_1+q_2+..+q_n)\ \lambda\in\mathbb{N}\setminus\{0\}\ \exists required\ sequence\{t_s\}$. This completes the proof.

Corollary is proved.

Example1.2. Consider the function $f(x)=\left(x-\frac{1}{3}\right)^2$. Then we have $\frac{\left|\frac{1}{3}-\frac{1}{9}\right|+\frac{1}{3}}{\left|\frac{4}{9}-\frac{1}{3}\right|+\frac{2}{3}}=\frac{5}{9}$. From Corollary3 it follows that $\forall n=14k\ \exists required\ \ sequence.$

Theorem1.9. Let $f(x)$ be symmetrical and flowing on $[a,b]\subset[0,1]$. Then $\forall n=2k+1, k\in\mathbb{N}\setminus\{0\}\exists sequence$ $a=t_0<t_1<...<t_{n-1}<t_n=b\quad|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

Proof. Let us consider the function $g(x)=\frac{|f(b)-f(x)|+b-x}{k}-2x+a+b$. We have $g(b)=a-b<0$, $g\left(\frac{a+b}{2}\right)=\frac{\left|f(b)-f\left(\frac{a+b}{2}\right)\right|+\frac{b-a}{2}}{k}>0.$ Note that $g(x)$ is continuous as a sum/difference of continuous functions. It follows that $\exists x_0\in\left(\frac{a+b}{2},b\right)\Big|\ g(x_0)=0$. From Theorem 1.4. it follows that $\exists sequence$ $\{m_k\}:x_0=m_0<m_1<...<m_k=b|\ |\forall i=0,1...,k-1|t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=\frac{|f(b)-f(x_0)|+b-x_0}{k}.$ Let us take $t_0=a+b-m_k=a,\ t_1=a+b-m_{k-1},...,t_k=a+b-m_0=a+b-x_0,t_{k+1}=m_0=x_0,t_{k+2}=m_1,...,t_{2k+1}=m_k=b.$ From $f\left(\frac{a+b}{2}-x\right)=f\left(\frac{a+b}{2}-x\right)\ \forall x\in[a,b]$ it follows that $f(a+b-m_i)=f(m_i)$. Thus, we have $|t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=\frac{|f(b)-f(x_0)|+b-x_0}{k}=2x_0-a-b=const.\ \forall i=0,1...,n-1.$

Theorem is proved.

Corollary1.4. Let $f(x)$ be a periodic function $f(x)=f(x+T)\forall x\in[a,b]\subset[0,1],b=a+kT,k\in\mathbb{N}\setminus\{0\}$. Suppose $f(x)$ is flowing and symmetrical on $[a,a+T]$. Then $\forall p\in\mathbb{N}\setminus\{0\}\ for\ n=k(2p+1)\ \exists sequence$ $a=t_0<t_1<...<t_{n-1}<t_n=b\quad|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

Proof. Consider $\forall p\in\mathbb{N}\setminus\{0\}$. From Theorem1.9. it follows that $\exists sequence\ \ a=t_0<t_1<...<t_{2p}<t_{2p+1}=a+T$ $|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,2p.$ Let $t_{2p+2}=t_1+T,t_{2p+3}=t_2+T,...,t_i=t_{i-2p-1}+T,...,t_{k(2p+1)}=b.$ We have that $\{t_n\}$ is required.

Corollary is proved.

Example1.3. From Theorem 1.7. and Theorem 1.9. it follows that $\forall n\in\mathbb{N}\setminus\{0\}\ for\ function\ y=\left(x-\frac{1}{2}\right)^2, x\in[0,1]$ $\exists sequence\ \ a=t_0<t_1<...<t_{n-1}<t_n=b\quad|\ |t_{i+1}-t_i|+|f(t_{i+1})-f(t_i)|=const.\ \forall i=0,1...,n-1.$

§2.POINTS IN P-DIMENSIONAL EUCLIDEAN SPACE

Problem2.(a generalization of problem 1.) Let $f:S\to\mathbb{R}$ be a continuous function, where S is a segment in Euclidean space E^p such that:

$$X=(x^{(1)},x^{(2)},...,x^{(p)})\in S\Leftrightarrow\begin{cases}0\le x^{(1)}\le 1\\0\le x^{(2)}\le 1\\\dots\\0\le x^{(p)}\le 1\end{cases}$$

Let us find all(some) positive integers n such that there exists a sequence $\{T_i\}_{i=0}^n\equiv\left\{\left(t_i^{(1)},t_i^{(2)},...,t_i^{(p)}\right)\right\}_{i=0}^n$, where

Belarus

$$0 = t_0^{(1)} < t_1^{(1)} < ... < t_n^{(1)} = 1 \qquad \forall i = 0,1,..,n-1$$
$$0 = t_0^{(2)} < t_1^{(2)} < ... < t_n^{(2)} = 1 \quad and \quad const. = \left| f(T_{i+1}) - f(T_i) \right| + \sum_{j=1}^{p} \left| t_{i+1}^{(j)} - t_i^{(j)} \right|$$
$$...$$
$$0 = t_0^{(p)} < t_1^{(p)} < ... < t_n^{(p)} = 1$$

Theorem2.1. Let $A = \left(a^{(1)}, a^{(2)}, ..., a^{(p)}\right)$ and $B = \left(b^{(1)}, b^{(2)}, ..., b^{(p)}\right)$ be points in Euclidean space E^p such that there exists a segment S_1, where

$$X = (x^{(1)}, x^{(2)}, ..., x^{(p)}) \in S_1 \Leftrightarrow \begin{cases} a^{(1)} \le x^{(1)} \le b^{(1)} \\ a^{(2)} \le x^{(2)} \le b^{(2)} \\ ... \\ a^{(p)} \le x^{(p)} \le b^{(p)} \end{cases}$$

Suppose $S_1 \subset S$. Then there exists a sequence $\{T_i\}_{i=0}^{2}$ such that

$$a^{(1)} = t_0^{(1)} < t_1^{(1)} < t_2^{(1)} = b^{(1)} \qquad \forall i = 0,1$$
$$a^{(2)} = t_0^{(2)} < t_1^{(2)} < t_2^{(2)} = b^{(2)} \quad and \quad const. = \left| f(T_{i+1}) - f(T_i) \right| + \sum_{j=1}^{p} \left| t_{i+1}^{(j)} - t_i^{(j)} \right|$$
$$...$$
$$a^{(p)} = t_0^{(p)} < t_1^{(p)} < t_2^{(p)} = b^{(p)}$$

Proof. We have $T_0 = A; T_2 = B$. Consider a segment $[AB]$ (a part of line AB connecting A and B). Obviously $[AB]$ is a connect set. Consider the function $G : [AB] \to \mathbb{R}$, where $\forall X = (x^{(1)}, x^{(2)}, ..., x^{(p)}) \in [AB]$ we have:

$$G(X) = \left| f(X) - f(T_0) \right| + \sum_{j=1}^{p} \left| t_0^{(j)} - x^{(j)} \right| - \left| f(X) - f(T_2) \right| - \sum_{j=1}^{p} \left| t_2^{(j)} - x^{(j)} \right|$$

$G(X)$ is continuous as a sum/difference of continuous functions. We have that $G(X)$ is continuous and $D(G)$ is connected. From $G(T_0) < 0$ and $G(T_2) > 0$ it follows that there exists $X_0 \in [AB] \left| G(X_0) = 0. \right.$ So, we may take $T_1 = X_0.$ Sequence $\{T_i\}_{i=0}^{2}$ is required.

According to problem1.2. we may take $A = (0,0,...,0)$ *and* $B = (1,1,...,1)$. We obtain that for n=2 there always exists required sequence.

Theorem is proved.

Now we shall give the following definitions. Let $A = \left(a^{(1)}, a^{(2)}, ..., a^{(p)}\right)$ and $B = \left(b^{(1)}, b^{(2)}, ..., b^{(p)}\right)$ be two points in Euclidean space E^p such that $0 \le a^{(i)} \le b^{(i)} \le 1, where\, i = 1,2,..., p.$

Definition1. Consider a segment $K \subset S$, such that

$$X = (x^{(1)}, x^{(2)}, ..., x^{(p)}) \in K \Leftrightarrow \begin{cases} a^{(1)} \le x^{(1)} \le b^{(1)} \\ a^{(2)} \le x^{(2)} \le b^{(2)} \\ ... \\ a^{(p)} \le x^{(p)} \le b^{(p)} \end{cases}$$

A sequence $\{T_i\}_{i=0}^{n} \equiv \left\{ \left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right) \right\}_{i=0}^{n} \Big|$

$$a^{(1)} = t_0^{(1)} < t_1^{(1)} < ... < t_n^{(1)} = b^{(1)} \qquad \forall i = 0,1,...,n-1$$
$$a^{(2)} = t_0^{(2)} < t_1^{(2)} < ... < t_n^{(2)} = b^{(2)} \quad and \quad \left| f(T_{i+1}) - f(T_i) \right| + \sum_{j=1}^{p} \left| t_{i+1}^{(j)} - t_i^{(j)} \right| = const.$$
$$...$$
$$a^{(p)} = t_0^{(p)} < t_1^{(p)} < ... < t_n^{(p)} = b^{(p)}$$

is called ***n-fitting for segment K***.

Definition2. Suppose $Q(q^{(1)}, q^{(2)}, ..., q^{(p)})\, is\, a\, point\, in\, E^p, Q \in [AB]$.

Belarus

A sequence $\{T_i\}_{i=0}^{n} \equiv \left\{\left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right)\right\}_{i=0}^{n}$, such that

$$\begin{array}{l} a^{(1)} = t_0^{(1)} < t_1^{(1)} < ... < t_n^{(1)} = q^{(1)} \\ a^{(2)} = t_0^{(2)} < t_1^{(2)} < ... < t_n^{(2)} = q^{(2)} \\ ... \\ a^{(p)} = t_0^{(p)} < t_1^{(p)} < ... < t_n^{(p)} = q^{(p)} \end{array} \quad and \quad \begin{array}{l} \forall i = 0,1,...,n-1 \\ C(Q) = \left| f(T_{i+1}) - f(T_i) \right| + \sum_{j=1}^{p} \left| t_{i+1}^{(j)} - t_i^{(j)} \right| = const. \end{array}$$

is called ***Qn-rightfitting for segment K.*** Consider the function $dr_n^{(A)}(X) : [AB] \to \mathbb{R} \mid \ dr_n^{(A)}(Q) = C(Q)$ if there exists such sequence $\forall Q \in [AB] \setminus \{A\}$, let $dr_n^{(A)}(A) = 0$. Of course, if there exist two or more required sequences, maybe there exist two or more functions $dr_n^{(A)}(X)$.

Definition3. A sequence $\{T_i\}_{i=0}^{n} \equiv \left\{\left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right)\right\}_{i=0}^{n}$, such that

$$\begin{array}{l} q^{(1)} = t_0^{(1)} < t_1^{(1)} < ... < t_n^{(1)} = b^{(1)} \\ q^{(2)} = t_0^{(2)} < t_1^{(2)} < ... < t_n^{(2)} = b^{(2)} \\ ... \\ q^{(p)} = t_0^{(p)} < t_1^{(p)} < ... < t_n^{(p)} = b^{(p)} \end{array} \quad and \quad \begin{array}{l} \forall i = 0,1,...,n-1 \\ C(Q) = \left| f(T_{i+1}) - f(T_i) \right| + \sum_{j=1}^{p} \left| t_{i+1}^{(j)} - t_i^{(j)} \right| = const. \end{array}$$

is called ***Qn-leftfitting for segment K.*** Consider the function $dl_n^{(B)}(X) : [AB] \to \mathbb{R} \mid \ dl_n^{(B)}(Q) = C(Q)$ if there exists such sequence $\forall Q \in [AB] \setminus \{B\}$, let $dl_n^{(B)}(B) = 0$. Of course, if there exist two or more required sequences, maybe there exist two or more functions $dl_n^{(B)}(X)$.

Theorem2.2. Let $f(x^{(1)}, x^{(2)}, ..., x^{(p)})$ be a function such that:

1) $\exists$ positive integer $m \left| \forall Q(q^{(1)}, q^{(2)}, ..., q^{(p)}) \in [AB] \exists\ Q_m - rightfitting\ sequence \right.$ and $dr_m^{(A)}(X) : [AB] \to \mathbb{R}$ is continuous;
2) $\exists$ positive integer $v \left| \forall Q(q^{(1)}, q^{(2)}, ..., q^{(p)}) \in [AB] \exists\ Q_v - leftfitting\ sequence \right.$ and $dl_n^{(B)}(X) : [AB] \to \mathbb{R}$ is continuous;

Then there exists an $(m+v)-\ fitting$ sequence for segment K $\{T_i\}_{i=0}^{m+v} \equiv \left\{\left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right)\right\}_{i=0}^{m+v}$.

Proof. Consider the function $G : [AB] \to \mathbb{R} \mid \ \forall X \in [AB] \ \ G(X) = dr_m^{(A)}(X) - dl_v^{(B)}(X)$. Note that $G(X)$ is continuous as the difference of two continuous functions. We obtain that $D(G)$ is a connected set. From $G(A) > 0\ and\ G(B) < 0$ it follows that $\exists\ X_0 \in K \left| G(X_0) = 0. \right.$ Let $X_0 = (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(p)})$. Consider two segments $Y_1\ and\ Y_2$:

$$X = (x^{(1)}, x^{(2)}, ..., x^{(p)}) \in Y_1 \Leftrightarrow \begin{cases} a^{(1)} \le x^{(1)} \le x_0^{(1)} \\ a^{(2)} \le x^{(2)} \le x_0^{(2)} \\ ... \\ a^{(p)} \le x^{(p)} \le x_0^{(p)} \end{cases} \quad X = (x^{(1)}, x^{(2)}, ..., x^{(p)}) \in Y_2 \Leftrightarrow \begin{cases} x_0^{(1)} \le x^{(1)} \le b^{(1)} \\ x_0^{(2)} \le x^{(2)} \le b^{(2)} \\ ... \\ x_0^{(p)} \le x^{(p)} \le b^{(p)} \end{cases}$$

$$Y_1 \cap Y_2 = X_0 \text{ ; and } dr_m^{(A)}(X_0) = dl_v^{(B)}(X_0).$$

Consider $(X_0)_m - rightfitting\ sequence \{R_i\}_{i=0}^{m} \subset Y_1$ and $(X_0)_m - leftfitting\ sequence \{S_i\}_{i=0}^{v} \subset Y_2$. Let $T_0 = R_0$, $T_1 = R_1, ..., T_m = R_m = X_0 = S_0, T_{m+1} = S_1, ..., T_{m+v} = S_v$. From $dr_m^{(A)}(X_0) = dl_v^{(B)}(X_0)$ it follows that the sequence

$\{T_i\}_{i=0}^{m+v} \equiv \left\{\left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right)\right\}_{i=0}^{n}$ is required.

Theorem is proved.

Now we shall give the following the following definitions. A function $f(X): S \to \mathbb{R}$ is called ***periodic on segment K*** if $\forall X(x^{(1)},...,x^{(p)}) \in [AB]$ $f(X) = f(x^{(1)} + T(b^{(1)} - a^{(1)}),...,x^{(p)} + T(b^{(p)} - a^{(p)}))$. The number $T \in \mathbb{R}^+$ is called the period of $f(X)$. A function $f(X)$ is called ***symmetrical on segment K*** if $\forall X(x^{(1)},...,x^{(p)}) \in [AB]$ $f(X) = f\left(a^{(1)} + b^{(1)} - x^{(1)}, a^{(2)} + b^{(2)} - x^{(2)},...,a^{(p)} + b^{(p)} - x^{(p)}\right)$. Let M be the midpoint of $[AB]$. A function $f(X): S \subset \mathbb{R}^p \to \mathbb{R}$ is called ***monotonous on*** $X \subset S$ if $\forall X_1, X_2 \in X \begin{cases} x_1^{(1)} \geq x_2^{(1)} \\ ... \\ x_1^{(p)} \geq x_2^{(p)} \end{cases}$ or $\forall X_1, X_2 \in X \begin{cases} x_1^{(1)} \leq x_2^{(1)} \\ ... \\ x_1^{(p)} \leq x_2^{(p)} \end{cases}$. A function $f(X)$ is called ***flowing on segment K*** if $f(X)$ is monotonous on $[AM]$ and on $[MB]$. Here and below K is the segment mentioned in definitions, $[AB]$ is line segment of line AB. For any periodic on segment K function with period T let us consider the segment P such that

$$X = (x^{(1)}, x^{(2)},...,x^{(p)}) \in P \Leftrightarrow \begin{cases} a^{(1)} \leq x^{(1)} \leq a^{(1)} + T\left(b^{(1)} - a^{(1)}\right) \\ a^{(2)} \leq x^{(2)} \leq a^{(2)} + T\left(b^{(2)} - a^{(2)}\right) \\ ... \\ a^{(p)} \leq x^{(p)} \leq a^{(p)} + T\left(b^{(p)} - a^{(p)}\right) \end{cases}$$

Theorem 2.3. Suppose $f(X)$ is monotonous on $[AB]$. Then $\forall n \in \mathbb{N} \setminus \{0\}\ \exists\ n-\mathit{fitting}$ sequence for segment K.

Proof. Consider $\forall$ segment $[CD] \subset [AB]$. Let us show that $\forall\ [CD] \subset [AB]\ \forall n \in \mathbb{N} \setminus \{0\}\ \exists$ *sequence*

$$\{T_i\}_{i=0}^{n} \equiv \left\{\left(t_i^{(1)}, t_i^{(2)}, ..., t_i^{(p)}\right)\right\}_{i=0}^{n} \subset [CD] \Big|$$

$$\begin{matrix} c^{(1)} = t_0^{(1)} < t_1^{(1)} < ... < t_n^{(1)} = d^{(1)} \\ c^{(2)} = t_0^{(2)} < t_1^{(2)} < ... < t_n^{(2)} = d^{(2)} \\ ... \\ c^{(p)} = t_0^{(p)} < t_1^{(p)} < ... < t_n^{(p)} = d^{(p)} \end{matrix} \quad \mathit{and} \quad \begin{matrix} \forall i = 0,1,...,n-1 \\ \left|f(T_{i+1}) - f(T_i)\right| + \sum_{j=1}^{p}\left|t_{i+1}^{(j)} - t_i^{(j)}\right| = \mathit{const}. \end{matrix}$$

Also let us prove that $dr_n^{(C)}(X) = \dfrac{\left|f(X) - f(C)\right| + \sum_{j=1}^{p}\left|c^{(j)} - x^{(j)}\right|}{n}$ and $dl_n^{(D)}(X) = \dfrac{\left|f(X) - f(D)\right| + \sum_{j=1}^{p}\left|d^{(j)} - x^{(j)}\right|}{n}$.

The proof is by induction over n.

For n=1, there is nothing to prove; and we have $dr_1^{(C)}(X) = \left|f(X) - f(C)\right| + \sum_{j=1}^{p}\left|c^{(j)} - x^{(j)}\right|$, $dl_1^{(D)}(X) = \left|f(X) - f(D)\right| + \sum_{j=1}^{p}\left|d^{(j)} - x^{(j)}\right|$. Next, $dr_1^{(C)}(X)$ and $dl_1^{(D)}(X)$ are continuous, and $dr_1^{(C)}(C) = dl_1^{(D)}(D) = 0$.

For n=2, from Theorem2.1. it follows that $\exists$ *required sequence* $\{T_i\}_{i=0}^{2} \subset [CD]$, and we have that $f(X)$ is monotonous. It follows that $dr_2^{(C)}(X) = \dfrac{\left|f(X) - f(C)\right| + \sum_{j=1}^{p}\left|c^{(j)} - x^{(j)}\right|}{2}$ and $dl_2^{(D)}(X) = \dfrac{\left|f(X) - f(D)\right| + \sum_{j=1}^{p}\left|d^{(j)} - x^{(j)}\right|}{2}$.

Note that $dr_2^{(C)}(X)$ and $dl_2^{(D)}(X)$ are continuous, and $dr_2^{(C)}(C) = dl_2^{(D)}(D) = 0$.

Assume that $\forall\ [CD] \subset [AB]\ \forall k = 1,...,n-1\ \exists$ *required* sequence $\{T_i\}_{i=0}^{k} \subset [CD]$ and

Belarus

$$dr_k^{(C)}(X)=\frac{|f(X)-f(C)|+\sum_{j=1}^{p}\left|c^{(j)}-x^{(j)}\right|}{k}\quad dl_k^{(D)}(X)=\frac{|f(X)-f(D)|+\sum_{j=1}^{p}\left|d^{(j)}-x^{(j)}\right|}{k}.$$

Let us prove this for $k=n$. Since statement is true for $k=1\,and\,k=n-1$, from Theorem2.2. it follows that $\forall[CD], C\neq D, \exists$ required sequence $\{T_i\}_{i=0}^{n}\subset[CD]$ (Because $X_0\in[CD]$,$\{T_i\}_{i=0}^{1}\subset[CD]$,and $\{T_i\}_{i=0}^{n-1}\subset[CD]$).

Since $f(X)$ is monotonous, we have $dr_n^{(C)}(X)=\dfrac{|f(X)-f(C)|+\sum_{j=1}^{p}\left|c^{(j)}-x^{(j)}\right|}{n}$ and $dl_n^{(D)}(X)=\dfrac{|f(X)-f(D)|+\sum_{j=1}^{p}\left|d^{(j)}-x^{(j)}\right|}{n}$.

Note that $dr_n^{(C)}(X)$ and $dl_n^{(D)}(X)$ are continuous, and $dr_n^{(C)}(C)=dl_n^{(D)}(D)=0$ .Thus, induction is complete and we may take $C=A$ and $D=B$.

Theorem is proved.

Theorem 2.4.

I. Let $f(X)$ be periodic on K, $f(X)=f(x^{(1)}+T(b^{(1)}-a^{(1)}),...,x^{(p)}+T(b^{(p)}-a^{(p)}))$. Suppose $kT=1, k\in\mathbb{N}\setminus\{0\}$. Then $for\,n=2k\ \exists n-\ fitting\ sequence\ for\,segment\,K.$

II. Let $f(X)$ be periodic on K, $f(X)=f(x^{(1)}+T(b^{(1)}-a^{(1)}),...,x^{(p)}+T(b^{(p)}-a^{(p)}))$. Suppose $kT+\dfrac{T}{2}=1, k\in\mathbb{N}\setminus\{0\}$. Then $for\,n=2k+1\ \exists n-\ fitting\ sequence\ for\,segment\,K.$

Proof.

I. Let us take $T_0=A, T_1=\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{T}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right),...,$
$T_i=\left(a^{(1)}+\frac{Ti}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{Ti}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{Ti}{2}\left(b^{(p)}-a^{(p)}\right)\right),...,T_{2k}=B.$ Then we have $\left|f(T_{i+1})-f(T_i)\right|+$
$+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=\left|f(A)-f\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{T}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right)\right|+\sum_{j=1}^{p}\left|\frac{T}{2}\left(b^{(j)}-a^{(j)}\right)\right|=$
$=const$. It follows that sequence $\{T_i\}_{i=0}^{2k}$ is required. This proves I.

II. Let us take $T_0=A, T_1=\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{T}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right),...,$
$T_i=\left(a^{(1)}+\frac{Ti}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{Ti}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{Ti}{2}\left(b^{(p)}-a^{(p)}\right)\right),...,T_{2k+1}=B.$ Then we have $\left|f(T_{i+1})-f(T_i)\right|+$
$+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=\left|f(A)-f\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right), a^{(2)}+\frac{T}{2}\left(b^{(2)}-a^{(2)}\right),...,a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right)\right|+\sum_{j=1}^{p}\left|\frac{T}{2}\left(b^{(j)}-a^{(j)}\right)\right|=$
$=const$. It follows that sequence $\{T_i\}_{i=0}^{2k+1}$ is required. This proves II.

Theorem is proved.

Theorem 2.5.

I. Let $f(X)$ be symmetrical on segment $K\subset S$. Then $\exists\ 4-\ fitting$ sequence $\{T_i\}_{i=0}^{4}$ for segment K.

II. Let $f(X)$ be periodic on K with period T. Suppose $kT=1, k\in\mathbb{N}\setminus\{0\}$. Suppose $f(X)$ is symmetrical on the segment P. Then for $n=4k\,\exists n-\ fitting\ sequence\ \{T_i\}_{i=0}^{n}$ for segment K.

III. Let $f(X)$ be periodic on K with period T. Suppose $kT+\dfrac{T}{2}=1, k\in\mathbb{N}\setminus\{0\}$ and $f(X)$ is symmetrical on the segment P. Then for $n=4k+2\ \exists n-\ fitting\ sequence\{T_i\}_{i=0}^{n}$ for segment K.

Proof.

I. Let M be the midpoint of $[AB]$.From Theorem2.1. it follows that $\exists X_0\in[AM]\,|$

$\left|F(X_0)-F(A)\right|+\sum_{j=1}^{p}\left|x_0^{(j)}-a^{(j)}\right|=\left|F(X_0)-F(M)\right|+\sum_{j=1}^{p}\left|x_0^{(j)}-m^{(j)}\right|$. Let us take $T_0=A, T_1=X_0, T_2=M$. Let $Y_0\in[MB], |Y_0M|=|X_0M|$. Then $Y_0\left(a^{(1)}+b^{(1)}-x^{(1)}, a^{(2)}+b^{(2)}-x^{(2)},...,a^{(p)}+b^{(p)}-x^{(p)}\right)$. Take $T_3=Y_0, T_4=B.$ Since $f(X)$ is symmetrical, we obtain that:

$$\left|F(Y_0)-F(B)\right|+\sum_{j=1}^{p}\left|y_0^{(j)}-b^{(j)}\right|=\left|F(X_0)-F(A)\right|+\sum_{j=1}^{p}\left|x_0^{(j)}-a^{(j)}\right|=\left|F(X_0)-F(M)\right|+\sum_{j=1}^{p}\left|x_0^{(j)}-m^{(j)}\right|=\left|F(Y_0)-F(M)\right|+\sum_{j=1}^{p}\left|y_0^{(j)}-m^{(j)}\right|$$

Thus, it follows that $\{T_i\}_{i=0}^{4}$ is required. This proves I.

II. From I it follows that $\exists\, 4-$ *fitting sequence* $\{T_i\}_{i=0}^{4}$ for segment P. Let us take $T_5=\left((t_0^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_0^{(p)}+T(b_0-a_0))\right),...,T_i=\left((t_{i-4}^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_{i-4}^{(p)}+T(b_0-a_0))\right),...,T_{4k}=B.$

Then $\left|f(T_{i+1})-f(T_i)\right|+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=const.$ It follows that $\{T_i\}_{i=0}^{4k}$ is required. This proves II.

III. From I it follows that $\exists\, 4-$ *fitting sequence* $\{T_i\}_{i=0}^{4}$ for segment P. Let us take $T_5=\left((t_0^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_0^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,T_i=\left((t_{i-4}^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_{i-4}^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,T_{4k+2}=B.$

Then $\left|f(T_{i+1})-f(T_i)\right|+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=const.$ It follows that $\{T_i\}_{i=0}^{4k}$ is required. This proves III.

Theorem is proved.

Theorem 2.6.

I. Let $f(X)$ be flowing on K; suppose $F(A)=F(B)$. Then $\forall k\in\mathbb{N}\setminus\{0\}$ for $n=2k\,\exists\, n-$ *fitting* sequence for segment K.

II. Let $f(X)$ be periodic on segment K with period T, where $kT=1, k\in\mathbb{N}\setminus\{0\}$. Let $f(X)$ be flowing on P. Then $\forall p\in\mathbb{N}\setminus\{0\}$ *for* $n=2kp\,\exists n-$ *fitting* sequence for segment K.

III. Let $f(X)$ be periodic on segment K with period T, where $kT+\frac{T}{2}=1, k\in\mathbb{N}\setminus\{0\}$. Let $f(X)$ be flowing on P. Then $\forall p\in\mathbb{N}\setminus\{0\}$ *for* $n=p(2k+1)$ $\exists n-$ *fitting* sequence for segment K.

Proof. Let $M\left(\frac{a^{(1)}+b^{(1)}}{2},...,\frac{a^{(p)}+b^{(p)}}{2}\right)\equiv M\left(m^{(1)},...,m^{(p)}\right)$ be the midpoint of $[AB]$.

I. Consider $\forall k\in\mathbb{N}\setminus\{0\}$. Consider two segments Z_1 *and* Z_2 |

$$X=(x^{(1)},x^{(2)},...,x^{(p)})\in Z_1\Leftrightarrow\begin{cases}a^{(1)}\le x^{(1)}\le m^{(1)}\\ a^{(2)}\le x^{(2)}\le m^{(2)}\\ \quad ...\\ a^{(p)}\le x^{(p)}\le m^{(p)}\end{cases}\quad X=(x^{(1)},x^{(2)},...,x^{(p)})\in Z_2\Leftrightarrow\begin{cases}m^{(1)}\le x^{(1)}\le b^{(1)}\\ m^{(2)}\le x^{(2)}\le b^{(2)}\\ \quad ...\\ m^{(p)}\le x^{(p)}\le b^{(p)}\end{cases}$$

From Theorem2.3. it follows that $\exists k-$ *fitting sequence* $\{D_i\}_{i=0}^{n}$ for segment Z_1 and $\exists k-$ *fitting sequence* $\{R_i\}_{i=0}^{n}$ for segment Z_2. Let us take $T_0=D_0=A, T_1=D_1,...,T_k=D_k=R_0, T_{k+1}=R_1,...,T_{2k}=R_k=B$. It follows that sequence $\{T_i\}_{i=0}^{2k}$ is required. Really, we have that:

$$\left|f(T_{i+1})-f(T_i)\right|+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=\frac{\left|f(A)-f(M)\right|+\sum_{j=1}^{p}\left|m^{(j)}-a^{(j)}\right|}{k}=\frac{\left|f(B)-f(M)\right|+\sum_{j=1}^{p}\left|m^{(j)}-b^{(j)}\right|}{k}$$

. This proves I.

II. Consider $\forall p\in\mathbb{N}\setminus\{0\}$. From I it follows that $\exists\, 2p-$ *fitting* sequence $\{T_i\}_{i=0}^{2p}$ for segment P. Let us take

$T_{2p+1}=\left((t_1^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_1^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,T_i=\left((t_{i-2p}^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_{i-2p}^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,$
$T_{2kp}=B.$ It follows that $\{T_i\}_{i=0}^{2kp}$ is required. This proves II.

III. Consider $\forall p\in\mathbb{N}\setminus\{0\}$. From I it follows that $\exists\ 2p-\mathit{fitting}$ sequence $\{T_i\}_{i=0}^{2p}$ for segment P. Let us take $T_{2p+1}=\left((t_1^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_1^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,T_i=\left((t_{i-2p}^{(1)}+T(b^{(1)}-a^{(1)})),...,(t_{i-2p}^{(p)}+T(b^{(p)}-a^{(p)}))\right),...,$
$T_{2kp+p}=B.$ It follows that $\{T_i\}_{i=0}^{(2k+1)p}$ is required. This proves III.

Theorem is proved.

Theorem 2.7.

I. Let $C_1,C_2,...,C_{n-1}$ be distinct points of $[AB]\setminus\{A,B\},|AC_1|<|AC_2|<...<|AC_{n-1}|<|AB|$. Take $C_0=A,C_n=B.$ Suppose $f(X)$ is monotonous on each of line segments $[C_iC_{i+1}],\mathit{where}\,i=0,...,n-1.$ Let

$$\forall i=0,...,n-2\ \ \frac{|f(C_i)-f(C_{i+1})|+\sum_{j=1}^{p}\left|c_i^{(j)}-c_{i+1}^{(j)}\right|}{|f(C_{i+2})-f(C_{i+1})|+\sum_{j=1}^{p}\left|c_{i+2}^{(j)}-c_{i+1}^{(j)}\right|}\in\mathbb{Q}.\qquad\text{Then}\qquad \exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}\,|\,\forall\lambda\in\mathbb{N}\setminus\{0\}$$

$\mathit{for}\,s=\lambda\left(q_1+q_2+...+q_n\right)\ \exists s-\mathit{fitting\ sequence\ for\ segment}\,K.$

II. Let $f(X)\,\mathit{be\,a\ periodic\ function\ with\ period}\,T$, where $kT<1<kT+\frac{T}{2},k\in\mathbb{N}\setminus\{0\}$. Suppose $n=1-kT$, and let $f(X)$ be flowing on segment P. Suppose

$$\frac{\left|f(A)-f\left(\left(a^{(1)}+n\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+n\left(b^{(p)}-a^{(p)}\right)\right)\right)\right|+n\sum_{j=1}^{p}\left|b^{(j)}-a^{(j)}\right|}{\left|f\left(\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right)\right)-f\left(\left(a^{(1)}+n\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+n\left(b^{(p)}-a^{(p)}\right)\right)\right)\right|+\left(\frac{T}{2}-n\right)\sum_{j=1}^{p}\left|b^{(j)}-a^{(j)}\right|}=\frac{r}{q}\in\mathbb{Q}\setminus\{0\}$$

Then $\forall\lambda\in\mathbb{N}\setminus\{0\}$ $\mathit{for}\,m=\lambda(2k(r+q)+r)\exists m-\mathit{fitting}$ sequence for segment K.

III. Let $f(X)\,\mathit{be\,a\ periodic\ function\ with\ period}\,T$, where $kT+\frac{T}{2}<1<(k+1)T,k\in\mathbb{N}\setminus\{0\}$. Suppose $n=(k+1)T-1$, and let $f(X)$ be flowing on segment P. Suppose

$$\frac{\left|f\left(\left(a^{(1)}+\frac{T}{2}\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+\frac{T}{2}\left(b^{(p)}-a^{(p)}\right)\right)\right)-f\left(\left(a^{(1)}+\left(n+\frac{T}{2}\right)\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+\left(n+\frac{T}{2}\right)\left(b^{(p)}-a^{(p)}\right)\right)\right)\right|+n\sum_{j=1}^{p}\left|b^{(j)}-a^{(j)}\right|}{\left|f(A)-f\left(\left(a^{(1)}+\left(n+\frac{T}{2}\right)\left(b^{(1)}-a^{(1)}\right)\right),...,\left(a^{(p)}+\left(n+\frac{T}{2}\right)\left(b^{(p)}-a^{(p)}\right)\right)\right)\right|+\left(\frac{T}{2}-n\right)\sum_{j=1}^{p}\left|b^{(j)}-a^{(j)}\right|}=\frac{r}{q}\in\mathbb{Q}\setminus\{0\}$$

Then $\forall\lambda\in\mathbb{N}\setminus\{0\}$ $\mathit{for}\,m=\lambda((2k+1)(r+q)+r)\exists m-\mathit{fitting}$ sequence for segment K.

Proof.

I. From $\dfrac{|f(C_i)-f(C_{i+1})|+\sum_{j=1}^{p}\left|c_i^{(j)}-c_{i+1}^{(j)}\right|}{|f(C_{i+2})-f(C_{i+1})|+\sum_{j=1}^{p}\left|c_{i+2}^{(j)}-c_{i+1}^{(j)}\right|}\in\mathbb{Q}$ it follows that $\exists q_1,q_2,...,q_n\in\mathbb{N}\setminus\{0\}$ such that

$$\left(|f(C_0)-f(C_1)|+\sum_{j=1}^{p}\left|c_0^{(j)}-c_1^{(j)}\right|\right):\left(|f(C_1)-f(C_2)|+\sum_{j=1}^{p}\left|c_2^{(j)}-c_1^{(j)}\right|\right):...:\left(|f(C_{n-1})-f(C_n)|+\sum_{j=1}^{p}\left|c_{n-1}^{(j)}-c_n^{(j)}\right|\right)=q_1:q_2:...:q_n$$

Consider $\forall\lambda\in\mathbb{N}\setminus\{0\}$. For all $i=1,...,n$ consider the segment H_i:

$$X=(x^{(1)},x^{(2)},...,x^{(p)})\in H_i\Leftrightarrow\begin{cases}c_{i-1}^{(1)}\le x^{(1)}\le c_i^{(1)}\\ c_{i-1}^{(2)}\le x^{(2)}\le c_i^{(2)}\\ ...\\ c_{i-1}^{(p)}\le x^{(p)}\le c_i^{(p)}\end{cases}$$

From Theorem2.3. it follows that $\exists\lambda q_j - fitting$ sequence $\left\{ {}^{(j)}R_i \right\}_{i=0}^{\lambda q_j}$ for each segment $U_j \;\forall\; j=1,..,n$ such that

$$\left| F\left({}^{(j)}R_i\right) - F\left({}^{(j)}R_{i+1}\right)\right| + \sum_{s=1}^{p}\left| {}^{(j)}r_{i+1}^{(s)} - {}^{(j)}r_i^{(s)} \right| = \frac{\left| f(C_{j-1}) - f(C_j)\right| + \sum_{s=1}^{p}\left| c_{j-1}^{(s)} - c_j^{(s)}\right|}{\lambda q_j}.$$

But we have that $\forall i=0,..,n-1 \;\; \frac{\left| f(C_i) - f(C_{i+1})\right| + \sum_{s=1}^{p}\left| c_i^{(s)} - c_{i+1}^{(s)}\right|}{\lambda q_{i+1}} = const.$ Then let us take $T_0 = {}^{(1)}R_0, T_1 = {}^{(1)}R_1, ..., T_{\lambda q_1} = {}^{(1)}R_{\lambda q_1} = {}^{(2)}R_0, T_{\lambda q_1+1} = {}^{(2)}R_1, ..., T_{\lambda q_1+\lambda q_2} = {}^{(2)}R_{\lambda q_2}, .., T_{\lambda q_1+\lambda q_2+..+\lambda q_n} = {}^{(n)}R_{\lambda q_n} = B.$ It follows that sequence $\left\{T_i\right\}_{i=0}^{\lambda(q_1+q_2+...+q_n)}$ is required. This proves I.

II. Let $A_0 = A, A_1 = \left(\left(a^{(1)} + \frac{T}{2}\left(b^{(1)} - a^{(1)}\right)\right), ..., \left(a^{(p)} + \frac{T}{2}\left(b^{(p)} - a^{(p)}\right)\right)\right),\; \overrightarrow{AA_2} = 2\overrightarrow{AA_1}, \overrightarrow{AA_3} = 3\overrightarrow{AA_1}, ..., \overrightarrow{AA_{2k}} = 2k\overrightarrow{AA_1}, A_{2k+1} = B.$ We have that $\frac{\left| f(A) - f\left(\left(a^{(1)} + n\left(b^{(1)} - a^{(1)}\right)\right), ..., \left(a^{(p)} + n\left(b^{(p)} - a^{(p)}\right)\right)\right)\right| + n\sum_{j=1}^{p}\left| b^{(j)} - a^{(j)}\right|}{\left| f(A) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| b^{(j)} - a^{(j)}\right|} = \frac{r}{r+q}$. It follows that

$$\left(\left| f(A_0) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) : \left(\left| f(A_2) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) : ... : \left(\left| f(A_{2k+1}) - f(A_{2k})\right| + n\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) = (r+q) : (r+q) : ... : (r+q) : r$$

We get by definition that $f(X)$ is monotonous on each line segment $\left[A_i A_{i+1}\right]$. From I it follows that for $m = \lambda(2k(r+q)+r)\;\; \exists m - fitting$ sequence for segment K.

II. Let $A_0 = A, A_1 = \left(\left(a^{(1)} + \frac{T}{2}\left(b^{(1)} - a^{(1)}\right)\right), ..., \left(a^{(p)} + \frac{T}{2}\left(b^{(p)} - a^{(p)}\right)\right)\right),\;\; \overrightarrow{AA_2} = 2\overrightarrow{AA_1}, \overrightarrow{AA_3} = 3\overrightarrow{AA_1}, ..., \overrightarrow{AA_{2k+1}} = 2k\overrightarrow{AA_1}, A_{2k+2} = B.$

Next, from definition we obtain that:

$$\frac{\left| f\left(\left(a^{(1)} + \frac{T}{2}\left(b^{(1)} - a^{(1)}\right)\right), ..., \left(a^{(p)} + \frac{T}{2}\left(b^{(p)} - a^{(p)}\right)\right)\right) - f\left(\left(a^{(1)} + \left(n + \frac{T}{2}\right)\left(b^{(1)} - a^{(1)}\right)\right), ..., \left(a^{(p)} + \left(n + \frac{T}{2}\right)\left(b^{(p)} - a^{(p)}\right)\right)\right)\right| + n\sum_{j=1}^{p}\left| b^{(j)} - a^{(j)}\right|}{\left| f(A) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| b^{(j)} - a^{(j)}\right|} = \frac{r}{r+q}.$$

It follows that:

$$\left(\left| f(A_0) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) : \left(\left| f(A_2) - f(A_1)\right| + \frac{T}{2}\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) : ... : \left(\left| f(A_{2k+1}) - f(A_{2k+2})\right| + n\sum_{j=1}^{p}\left| a^{(j)} - b^{(j)}\right|\right) = (r+q) : (r+q) : ... : (r+q) : r$$

We get by definition that $f(X)$ is monotonous on each line segment $\left[A_i A_{i+1}\right]$. From I it follows that for $m = \lambda((2k+1)(r+q)+r)\;\; \exists m - fitting$ sequence for segment K.

Theorem is proved.

Theorem2.8.Let $f(X)$ be symmetrical and flowing on K. Then $\forall n = 2k+1, k \in \mathbb{N}\setminus\{0\}\, \exists n - fitting$ sequence for segment K.

Proof. Suppose M is the midpoint of $[AB]$. Let us consider the function $g : [MB] \to \mathbb{R}$ such that

$$g(X) = \frac{\left| f(B) - f(X)\right| + \sum_{j=1}^{p}\left| b^{(j)} - x^{(j)}\right|}{k} + \sum_{j=1}^{p}\left(a^{(j)} + b^{(j)} - 2x^{(j)}\right)$$

We have that $g(B)<0, g(M)>0$. Note that $g(X)$ is continuous as a sum\difference of continuous functions. It follows that $\exists X_0 \in [MB]\setminus\{M,B\} \mid g(X_0)=0$. Consider the segment Z, where

$$X=(x^{(1)},x^{(2)},...,x^{(p)})\in Z \Leftrightarrow \begin{cases} x_0^{(1)} \le x^{(1)} \le b^{(1)} \\ x_0^{(2)} \le x^{(2)} \le b^{(2)} \\ \dots \\ x_0^{(p)} \le x^{(p)} \le b^{(p)} \end{cases}$$

From Theorem2.3. it follows that $\exists k-\mathit{fitting\,sequence}\{M_i\}_{i=0}^{k} \subset [X_0B]$ for segment Z. Let us take $T_0=\left(a^{(1)}+b^{(1)}-m_k^{(1)},...,a^{(p)}+b^{(p)}-m_k^{(p)}\right)=A, T_1=\left(a^{(1)}+b^{(1)}-m_{k-1}^{(1)},...,a^{(p)}+b^{(p)}-m_{k-1}^{(p)}\right),...,T_k=$ $=\left(a^{(1)}+b^{(1)}-m_0^{(1)},...,a^{(p)}+b^{(p)}-m_0^{(p)}\right), T_{k+1}=M_0=X_0, T_{k+2}=M_1,...,T_{2k+1}=B.$ By definition we have that $f\left(a^{(1)}+b^{(1)}-x^{(1)},...,a^{(p)}+b^{(p)}-x^{(p)}\right)=f\left(x^{(1)},...,x^{(p)}\right)$. Finally, we have that $\left|f(T_{i+1})-f(T_i)\right|+\sum_{j=1}^{p}\left|t_{i+1}^{(j)}-t_i^{(j)}\right|=$

$$=\frac{\left|f(B)-f(X)\right|+\sum_{j=1}^{p}\left|b^{(j)}-x^{(j)}\right|}{k}=\sum_{j=1}^{p}\left(-a^{(j)}-b^{(j)}+2x_0{}^{(j)}\right).$$ Thus, sequence $\{T_i\}_{i=0}^{2k+1}$ is required.

Theorem is proved.

Corollary2.1. Let $f(X)$ be a periodic function on K with period $\mathrm{T}\left|1=kT, k\in\mathbb{N}\setminus\{0\}\right.$. Suppose $f(X)$ is symmetrical and flowing on P. Then $\forall r\in\mathbb{N}\setminus\{0\}$ $\mathit{for}\ n=k(2r+1)\ \exists n-\mathit{fitting}$ sequence for segment K.

Proof. Consider $\forall r\in\mathbb{N}\setminus\{0\}$. From Theorem 2.8 it follows that $\exists\,(2r+1)-\mathit{fitting}$ sequence for segment P. Let $T_{2r+2}=\left(t^{(1)}+T\left(b^{(1)}-a^{(1)}\right),...,t^{(p)}+T\left(b^{(p)}-a^{(p)}\right)\right),...,T_i=\left(t_{i-2r-1}^{(1)}+T\left(b^{(1)}-a^{(1)}\right),...,t_{i-2r-1}^{(p)}+T\left(b^{(p)}-a^{(p)}\right)\right),...,$ $T_{k(2r+1)}=B.$ We have that $\{T_i\}_{i=0}^{k(2p+1)}$ is required.

Corollary is proved.

§3. $\alpha-\mathit{tight}$ SEQUENCES AND $C-\mathit{fitting}$ FUNCTIONS

Problem3. A sequence $A_0,A_1,\dots,A_n$ of n+1 points of the plane is called $\alpha-\mathit{tight}, \mathit{where}\,\alpha\in\mathbb{R},$ if $A_0=(0;0)$ and the distance between points A_i and A_j satisfies

$$d(A_iA_j)\le\left(\frac{j-i}{n}\right)^{\alpha},\ \mathit{for\ all}\ 0\le i<j\le n.$$

Let $\{A_i=(x_i;y_i)\}_{i=0}^{n}$ be an $\alpha-\mathit{tight}$ sequence of n+1 points. A function $F:\mathbb{R}^2\to\mathbb{R}$ is called $C-\mathit{fitting}$ for the sequence $\{A_i\}_{i=0}^{n}$ if the following inequalities hold

$$\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|\le C\cdot\frac{j-i}{n},\ \mathit{for\ all}\ 0\le i<j\le n.$$

Find all (some) numbers C such that for any positive integer n and any $\alpha-\mathit{tight}$ sequence there exists a $C-\mathit{fitting}$ function for $\{A_i\}_{i=0}^{n}$. Consider the case that

a) $\alpha=1$ b) $\alpha=\frac{2010}{2011}$ c) $\alpha=\frac{1}{2}$

Remark3.1. Note that if $\alpha\ge0$ then $\forall i$ all points $\{A_i\}_{i=0}^{n}$ are situated in the circle $\gamma(A_i;r), \mathit{where}\, r=\max\left\{\left(\frac{i}{n}\right)^{\alpha};\left(\frac{n-i}{n}\right)^{\alpha}\right\}$. It follows that $\{A_i\}_{i=0}^{n}\subset\gamma(O;1)$.

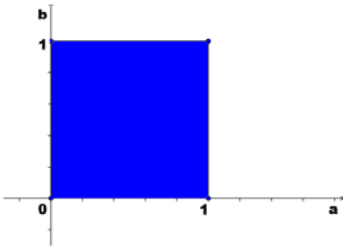
Theorem3.1. (constructing an $\alpha-\mathit{tight}$ sequence). Suppose $d(A_iA_j)\le\left(\frac{j-i}{n}\right)^{\alpha}$ $\mathit{for\ all}\ 0\le i<j\le k,\ 1\le\alpha,$ and

$d(A_kA_{k+1}) \le \left(\frac{1}{n}\right)^{\alpha}$. Then $d(A_iA_j) \le \left(\frac{j-i}{n}\right)^{\alpha}$, *for all* $0 \le i < j \le k+1$.

Proof. Let us prove the following Lemma.

Lemma1. Let $0 \le a \le 1,\ 0 \le b \le 1,\ \ and\ \ \alpha > 1$. Then $a^{\alpha} + b^{\alpha} \le (a+b)^{\alpha}$.

Proof. The case if $\alpha = 1$ is obvious. Without loss of generality it can be assumed that $\alpha > 1$. Consider the function $h(a,b) = a^{\alpha} + b^{\alpha} - (a+b)^{\alpha}$. From $0 \le a \le 1\ and\ \ 0 \le b \le 1$ it follows that such (a,b) form a compact set.



Then there exist min(h) and max(h). Let us find all points that can be extremums.

1) $\begin{cases} h'_a = \alpha a^{\alpha-1} - \alpha(a+b)^{\alpha-1} = 0 \\ h'_b = \alpha b^{\alpha-1} - \alpha(a+b)^{\alpha-1} = 0 \end{cases} \Leftrightarrow a = b = 0 \Rightarrow h(a,b) = 0$;

2) $a = 0 \Rightarrow h(0,b) = 0$;

3) $b = 0 \Rightarrow h(a,0) = 0$;

4) $a = 1 \Rightarrow h(1,b) = g(b) = 1 + b^{\alpha} - (b+1)^{\alpha}$. Consider $g'_b = \alpha b^{\alpha-1} - \alpha(b+1)^{\alpha-1}$. Then $g' = 0 \Leftrightarrow b^{\alpha-1} = (b+1)^{\alpha-1} \Leftrightarrow \varnothing$. We have $g(0) = 0\ and\ g(1) = 2 - 2^{\alpha} < 0$.

5) $b = 1 \Rightarrow h(a,1) = 1 + a\alpha - (a+1)^{\alpha}$. This case is similar to case 4) and gives $h(a,1) \le 0$.

So, we have $max(h) \le 0 \Rightarrow h(a,b) \le 0 \Rightarrow a^{\alpha} + b^{\alpha} \le (a+b)^{\alpha}$.

Lemma1 is proved.

From the triangle inequality it follows that

$$D(A_{k+1}A_i) \le D(A_kA_{k+1}) + D(A_kA_i) \le \left(\frac{1}{n}\right)^{\alpha} + \left(\frac{k-i}{n}\right)^{\alpha} \le [Lemma1] \le \left(\frac{k+1-i}{n}\right)^{\alpha}.$$

Theorem is proved.

Theorem3.2.

$\forall \alpha \in \mathbb{R}; \forall C \ge \sup\left\{\frac{1}{2}; \frac{1}{2} \bullet \left(\frac{1}{n}\right)^{2\alpha-1} \middle| n \in \mathbb{N} \setminus \{0\}\right\}$, *there exists a C − fitting function. Furthermore, if* $\alpha \ge \frac{1}{2}$ $\forall C \ge \frac{1}{2}$ *there exists a C − fitting function.*

Proof. Consider the function $F(A_i) = x_i y_i$. Hereby,

$$\left|F(A_j) - F(A_i) - x_j y_i + x_i y_j\right| = \left|x_j y_j - x_i y_i - x_j y_i + x_i y_j\right| = \left|(x_i - x_j)(y_i - y_j)\right| \le \frac{(x_i - x_j)^2 + (y_i - y_j)^2}{2} = \frac{\left(D(A_iA_j)\right)^2}{2} \le \frac{1}{2}\left(\frac{j-i}{n}\right)^{2\alpha}$$

Then $\forall C \ge \sup\left\{\frac{1}{2} \bullet \left(\frac{j-i}{n}\right)^{2\alpha-1}\right\} = \sup\left\{\frac{1}{2}; \frac{1}{2} \bullet \left(\frac{1}{n}\right)^{2\alpha-1}\right\}$, *there exists a C − fitting function.*

Furthermore, if $\alpha \ge \frac{1}{2}$ $\forall C \ge \frac{1}{2}$ *there exists a C − fitting function.*

(Of course, it might be $sup = \infty$).

Theorem is proved.

Theorem3.3. $\forall C \ge \sup\left\{\frac{i^{\alpha}(j-i)^{\alpha-1}}{n^{2\alpha-1}} \middle| 0 \le i < j \le n\right\}$ *there exists a C − fitting function.*

Belarus

Proof. Let us prove the following Lemma.

Lemma2. $\left|x_i y_j - x_j y_i\right| \le \dfrac{i^\alpha (j-i)^\alpha}{n^{2\alpha}}$.

Proof. Consider a vector product $\left|\overline{A_0A_i} \times \overline{A_0A_j}\right| = 2S_{A_0A_iA_j} = \left|x_i y_j - x_j y_i\right|$. Altitude of $\triangle A_0A_iA_j$ from A_0 $h_{A_0} \le A_0A_i \le \left(\dfrac{i}{n}\right)^\alpha$ and $A_iA_j \le \left(\dfrac{(j-i)}{n}\right)^\alpha$. Then $\left|x_i y_j - x_j y_i\right| \le \dfrac{i^\alpha (j-i)^\alpha}{n^{2\alpha}}$.

Lemma2 is proved.

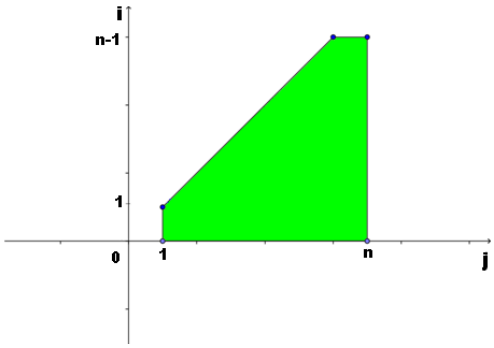
Let $F(A_i) = const$. Then if $C \ge \dfrac{i^\alpha (j-i)^{\alpha-1}}{n^{2\alpha-1}} \forall 0 \le i < j \le n, n \in \mathbb{N} \setminus \{0\}$ there exists a C-fitting function.

Theorem is proved.

Theorem 3.4. $\forall C \ge \dfrac{\alpha^\alpha (\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}$, where $\alpha > 1$, there exists a C-fitting function.

Proof. From Theorem3.3 it follows that $\forall C \ge A = \sup\left\{ \dfrac{i^\alpha (j-i)^{\alpha-1}}{n^{2\alpha-1}} \middle| 0 \le i < j \le n \right\}$ *there exists* *a C − fitting function*. Let us show that $A \le \dfrac{\alpha^\alpha (\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}$.

Case n=1 is obvious: $x_1 \bullet 0 - y_1 \bullet 0 = 0$; so we can take $F(A_i) = const.$, then $\forall C \ge 0$ *there exists a C − fitting function.* Without loss of generality it can be assumed that $n \ge 2$. Consider the function $d(i,j,n) = \dfrac{i^\alpha (j-i)^{\alpha-1}}{n^{2\alpha-1}}$, where $i \in [0, n-1], j \in [1, n], i \le j$. Obviously $d \to \max$ *when* $i^\alpha (j-i)^{\alpha-1} \to \max$, *where* $i \in [0, n-1], j \in [1, n], i \le j$. Consider the function $f(i,j) = i^\alpha (j-i)^{\alpha-1}$, where $i \in [0, n-1], j \in [1, n]$, $i \le j$. From $\alpha > 1$ it follows that f is continuous. Note that such (i,j) form a compact set:



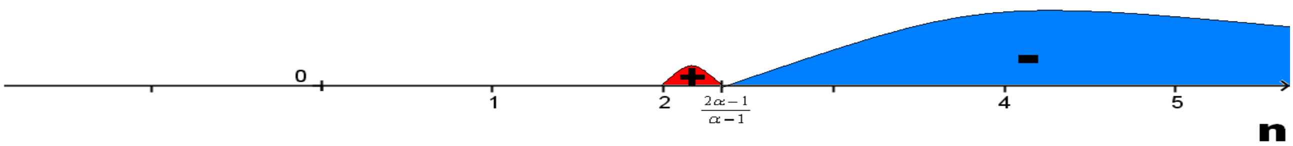
It follows that there exist min(f) and max(f). Let us find points that may be extremums.

1. We have $\begin{array}{l} f'_i = \alpha i^{\alpha-1}(j-i)^{\alpha-1} - (\alpha-1)i^\alpha (j-i)^{\alpha-2} \\ f'_j = (\alpha-1)i^\alpha (j-i)^{\alpha-1} \end{array}$. Then $\begin{cases} f'_i = 0 \text{ or not defined} \\ f'_j = 0 \text{ or not defined} \end{cases} \Rightarrow i = 0 \text{ or } i = j \Rightarrow$ $\Rightarrow f(i,j) = 0$. Now let us consider the edge points.
2. $j = i \Rightarrow f(i,j) = 0$.
3. $i = 0 \Rightarrow f(0,j) = 0$.
4. $j = 1; i \in [0,1]$. Let $q(i) = f(i,1) = i^\alpha (1-i)^{\alpha-1}$. Then $q'(i) = \alpha i^{\alpha-1}(1-i)^{\alpha-1} - (\alpha-1)(1-i)^{\alpha-2} i^\alpha$. Note that $q' = 0 \text{ or not defined} \Leftrightarrow \left[\begin{array}{l} i = 0 \\ i = 1 \\ \alpha(1-i) - (\alpha-1)i = 0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} i = 0 \\ i = 1 \\ i = \dfrac{\alpha}{2\alpha-1} \end{array}\right.$. Finally we obtain $q(0) = 0; q(1) = 0; q\left(\dfrac{\alpha}{2\alpha-1}\right) = \dfrac{\alpha^\alpha (\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}$.

5. $i = n-1;\quad j \in [n-1;n]$. Let $t(j) = f(n-1, j) = (n-1)^{\alpha}(j+1-n)^{\alpha-1}$. Then we have $t'(j) = (\alpha-1)(n-1)^{\alpha}(j+1-n)^{\alpha-2}$. Note that $(t' = 0 \text{ or not defined}) \Leftrightarrow j = n-1$. Finally we have $t(n-1) = 0;\ t(n) = (n-1)^{\alpha}$.
6. $j = n;\quad i \in [0, n-1]$. Consider the function $g(i) = i^{\alpha}(n-i)^{\alpha}$. Then $g'(i) = \alpha i^{\alpha-1}(n-i)^{\alpha-1} - (\alpha-1)i^{\alpha}(n-i)^{\alpha-2}$. Note that $g' = 0 \text{ or not defined} \Leftrightarrow \left[\begin{array}{l} i = 0 \\ i = n \\ i = \dfrac{\alpha n}{2\alpha-1} \end{array}\right.$. Finally we have $g(0) = 0;\ g(n) = 0; g(n-1) = (n-1)^{\alpha}; g\left(\dfrac{\alpha n}{2\alpha-1}\right) = \dfrac{n^{2\alpha-1}\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}$.

Now we have that $A = \sup(d) \le \sup \left\{0; \dfrac{(n-1)^{\alpha}}{n^{2\alpha-1}}; \dfrac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}; \dfrac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{n^{2\alpha-1}(2\alpha-1)^{2\alpha-1}}\right\} = \sup\left\{\dfrac{(n-1)^{\alpha}}{n^{2\alpha-1}}; \dfrac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}\right\}$.

Consider the function $p(n) = \dfrac{(n-1)^{\alpha}}{n^{2\alpha-1}}$, *where* $D(p) = [2;+\infty)$. Then $p'(n) = \dfrac{\alpha(n-1)^{\alpha-1}n^{2\alpha-1} - (2\alpha-1)n^{2\alpha-2}(n-1)^{\alpha}}{n^{4\alpha-2}} = \dfrac{(\alpha-1)(n-1)^{\alpha-1}\left(\dfrac{2\alpha-1}{\alpha-1} - n\right)}{n^{2\alpha}}$. Note that $p' = 0$ *or not defined when* $\left[\begin{array}{l} n = 1 \\ n = \dfrac{2\alpha-1}{\alpha-1} > 2 \end{array}\right.$. It follows that when $n \in \left[2, \dfrac{2\alpha-1}{\alpha-1}\right) \Rightarrow p' > 0$, when $n \in \left(\dfrac{2\alpha-1}{\alpha-1}, +\infty\right] \Rightarrow p' < 0$, and $p' = 0 \text{ if } n = \dfrac{2\alpha-1}{\alpha-1}$.



Then $max(p) = p\left(\dfrac{2\alpha-1}{\alpha-1}\right) = \dfrac{(\alpha-1)^{\alpha-1}\alpha^{\alpha}}{(2\alpha-1)^{2\alpha-1}}$. Thus, $A = \sup(d) \le \dfrac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{(2\alpha-1)^{2\alpha-1}}$.

Theorem is proved.

Theorem3.5. Let all points of $\alpha-tight$ sequence be situated on the lines $l_1 : y = k_1 x, l_2 : y = k_2 x, ..., l_k : y = k_k x$. Then $\forall C \ge \sup\left\{\dfrac{|k_p - k_q|}{\sqrt{(k_q^2+1)(k_p^2+1)}}\left(\dfrac{ij}{n}\right)^{\alpha}\dfrac{n}{j-i}\right\}$, *where* $0 \le i < j \le n,\ 1 \le p < q \le k$, *there exists a* $C-$ *fitting function.*

Proof. Consider l_i *and* l_j. Let us find $|\sin l_i \wedge l_j|$. Consider two points $A(1, k_i)$ and $B(1, k_j)$. Then

$$2S_{AOB} = |\overline{OA} \times \overline{OB}| = |k_i - k_j| = OA \cdot OB |\sin l_i \wedge l_j| = \sqrt{(k_q^2+1)(k_p^2+1)}|\sin l_i \wedge l_j|.$$

Let $F(A_i) = const$. Note that

$$\left|\overline{A_0A_i}\times\overline{A_0A_j}\right|=2S_{A_0A_iA_j}=\left|x_iy_j-x_jy_i\right|=A_0A_i\cdot A_0A_j\sin A_iA_0A_j\le\left(\frac{ij}{n^2}\right)^{\alpha}\sin A_iA_0A_j\le\left(\frac{ij}{n^2}\right)^{\alpha}\max\left(\frac{\left|k_p-k_q\right|}{\sqrt{\left(k_q^2+1\right)\left(k_p^2+1\right)}}\right)$$

Then $\forall C\ge\sup\left\{\frac{\left|k_p-k_q\right|}{\sqrt{\left(k_q^2+1\right)\left(k_p^2+1\right)}}\left(\frac{ij}{n}\right)^{\alpha}\frac{n}{j-i}\right\}, where\ 0\le i<j\le n,\ 1\le p<q\le k,\ there\,exists\ a\,C-fitting$ *function*.

Theorem is proved.

Corollary3.1. Let $\alpha\ge0\,and\left|\sin l_i\wedge l_j\right|\le\frac{1}{tn}\forall i,j=1,..,k.$ Then $\forall C\ge\frac{1}{t}\exists C-fitting\ function.$

Proof. We have

$$\sup\left\{\frac{\left|k_p-k_q\right|}{\sqrt{\left(k_q^2+1\right)\left(k_p^2+1\right)}}\left(\frac{ij}{n}\right)^{\alpha}\frac{n}{j-i}\right\}\le\sup\left\{\frac{(ij)^{\alpha}}{n^{2\alpha}t(j-i)}\right\}\le\frac{1}{t}.$$

Corollary is proved.

Remark3.2. Let us generalize the definition of C-fitting function. A function $F:\mathbb{R}^2\to\mathbb{R}$ is called $C-fitting$ for $\forall$ sequence $\{A_i\}_{i=0}^n$ (not necessary $\alpha-tight$) if the following inequalities hold:

$$\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|\le C\cdot\frac{j-i}{n},\ for\ all\ 0\le i<j\le n.$$

Note that if all points $\{A_i\}_{i=0}^n$ are situated on the same line, $\forall C\ge0$ there always exists a C-fitting function. Really, let $\{A_i\}_{i=0}^n\in l, where\ l\,is\,defined\ y=kx+b.$ Then consider the function $F:\mathbb{R}^2\to\mathbb{R}$ defined by $F(A_i)=bx_i$. Then we have $\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|=\left|bx_j-bx_i-x_j(kx_i+b)+x_i(kx_j+b)\right|=0$, so we obtain that $\forall C\ge0$ there always exists a C-fitting function. If l is defined $x=c$ consider the function $F(A_i)=-cy_i$. Then we obtain $\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|=\left|-cy_j+cy_i-cy_i+cy_j\right|=0$, so we obtain that $\forall C\ge0$ there always exists a C-fitting function.

§4. $C-balancing$ FUNCTIONS

Problem 4. We have also researched the following problem. Now we shall give the following definitions.
A sequence $A_0,A_1,\ldots,A_n$ of n+1 points of the plane is called $\alpha-balanced, where\,\alpha\in\mathbb{R}$, if A_0=(0;0) and the distance between points A_i and A_j $for\ all\ 0\le i<j\le n,\ i+j=n$ satisfies $d(A_iA_j)\le\left(\frac{j-i}{n}\right)^{\alpha}$ and $D(A_0A_i)\le\left(\frac{i}{n}\right)^{\alpha}$ $for\,all\ i=0,1,...,\left[\frac{n}{2}\right]$. It is obvious that $\forall\alpha-tight$ sequence is $\alpha-balanced$. Let $\{A_i=(x_i;y_i)\}_{i=0}^n$ be an $\alpha-balanced$ sequence of n+1 points. A function $F:\mathbb{R}^2\to\mathbb{R}$ is called $C-balancing$ for the sequence $\{A_i\}_{i=0}^n$ if the following inequalities hold

$$\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|\le C\cdot\frac{j-i}{n},\ for\ all\ 0\le i<j\le n, such\,that\,i+j=n.$$

Let us find all (some) numbers C such that for any positive integer n and any $\alpha-balanced$ sequence there exists a $C-balancing$ function for $\{A_i\}_{i=0}^n$.

Note that if $\alpha>0$, we can take $F(A_i)=x_iy_i$ and similar as in Theorem 3.2 obtain $\left|F(A_j)-F(A_i)-x_jy_i+x_iy_j\right|\le\frac{1}{2}\left(\frac{j-i}{n}\right)^{\alpha}$, then $\forall C\ge\frac{1}{2}there\ exists\ a\ C-balancing\ function.$ But more interesting result gives the following Theorem.

Theorem4.1. $\forall C \geq \frac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{2^{\alpha}(2\alpha-1)^{2\alpha-1}}; \alpha > 1,$ *there exists a* $C-$*balancing function.*

Proof. Note that $|x_i y_{n-i} - x_{n-i} y_i| \leq \frac{i^{\alpha}(n-2i)^{\alpha}}{n^{2\alpha}}, where\, i \leq \left[\frac{n}{2}\right]$. Really, let us consider a vector product $\left|\overrightarrow{A_0 A_i} \times \overrightarrow{A_0 A_{n-i}}\right| = 2S_{A_0 A_i A_{n-i}} = |x_i y_{n-i} - x_{n-i} y_i|$. Altitude of $\triangle A_0 A_i A_j$ from A_0 $h_{A_0} \leq A_0 A_i \leq \left(\frac{i}{n}\right)^{\alpha}$ and $A_i A_j \leq \left(\frac{n-2i}{n}\right)^{\alpha}$. Then $|x_i y_{n-i} - x_{n-i} y_i| \leq \frac{i^{\alpha}(n-2i)^{\alpha}}{n^{2\alpha}}, where\, i \leq \left[\frac{n}{2}\right]$. Let $F(A_i) = const$. Then if $C \geq \frac{i^{\alpha}(n-2i)^{\alpha}}{n^{2\alpha}}, \forall i \leq \left[\frac{n}{2}\right]$ there exists a C-balancing function. Consider the function $f(i) = i^{\alpha}(n-2i)^{\alpha-1}$, where $i \in \left[0, \frac{n}{2}\right]$. Let us fix n. Since $\alpha > 1$, we obtain f *is continuous*; and such *i* form a compact set. Let us find max(f). In the edge points *f(i)=0*.We have $f'(i) = \alpha i^{\alpha-1}(n-2i)^{\alpha-1} - 2i^{\alpha}(\alpha-1)(n-2i)^{\alpha-2}$ Then $f' = 0\, or\, not\, defined \Leftrightarrow \left[\begin{array}{l} n = 2i \\ i = 0 \\ i = \frac{n\alpha}{4\alpha-2} \end{array}\right.$ We obtain $\max(f) = f\left(\frac{n\alpha}{4\alpha-2}\right) = \left(\frac{n\alpha}{4\alpha-2}\right)^{\alpha}\left(\frac{n(\alpha-1)}{2\alpha-1}\right)^{\alpha-1}$. Then $i^{\alpha}(n-2i)^{\alpha} \leq \left(\frac{n\alpha}{4\alpha-2}\right)^{\alpha}\left(\frac{n(\alpha-1)}{2\alpha-1}\right)^{\alpha-1} \Rightarrow \frac{i^{\alpha}(n-2i)^{\alpha}}{n^{2\alpha}} \leq \frac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{2^{\alpha}(2\alpha-1)^{2\alpha-1}}\ \forall i \leq \left[\frac{n}{2}\right]$.

Thus, $\forall C \geq \frac{\alpha^{\alpha}(\alpha-1)^{\alpha-1}}{2^{\alpha}(2\alpha-1)^{2\alpha-1}}; \alpha > 1,$ *there exists a* $C-$*balancing function.*

Theorem is proved.