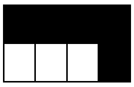
4. A *Bab*y Chess

Team: Belarus

Abstract

The first point of the problem was considered and solved completely for all pieces. It means that we have winning strategies on rectangular boards for pieces that don`t «leave» a trace. The most important result is that we have constructed a method that allows us to investigate boards with a complex structure («stepped» ones; not simply connected ones etc.) and another pieces (unusual pieces with non-standard moves). Particularly we have applied this method to some types of «stepped» boards and pieces with the following moves:



Also the second point for a rook was considered.

We call two squares which are connected by a move of concerned piece *adherent* squares.

The main ideas are the following:

1) Division of the given board into pairs of adherent squares. For this purpose we consider some smaller boards and divide them into pairs of adherent squares. Call these boards *elementary boards* and denote I(k;l) or shortly I_d. If the given board can be divided into adherent squares then the first player wins, because he can always make move on the corresponding adherent square.

2) Reduction of some games to easier ones using equivalent games on certain boards and for certain pieces.

The main results for usual pieces can be expressed in general statements (see theorems 1-3).

Also we have the following results in the second point:

Theorem 4. *(for a rook)* For a rook which leaves «trace» the first player wins on all boards except Π(1;1).

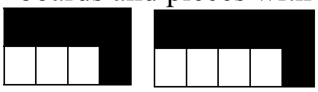
§1 Introduction

In this problem we consider rectangular boards Π(m;n) (m is a number of lines; n is a number of columns). Place a piece on the square with coordinates (a; b) (a is a serial number of line bottom-up; b is a serial number of column from left to right). Assume the upper left square is *black*.

We call the square with coordinates (a; b) *the first* square.

We call two squares which are connected by a move of considered piece *adherent* squares.

The first point of the problem was solved for all pieces. It means that we have winning strategies on rectangular boards for pieces that doesn`t leave a trace. The most important result is that we have constructed a method which allows us to investigate boards with a complex structure («stepped» ones; not simply connected ones etc.) and another pieces (non-classical pieces with non-standard moves). Particularly we apply this method to some types of «stepped» boards and pieces with the following moves:



The main ideas are the following:

1) Division of the given board into pairs of adherent squares. For this purpose we consider some smaller boards and divide them into pairs of adherent squares. Call these boards *elementary boards* and denote I(k;l) or shortly I_d. If the given board can be divided into adherent squares then the first player wins, because he can always make move on the corresponding adherent square.

2) Reduction of some games to easier ones using equivalent games on certain boards and for certain pieces.

Note that if we turn the initial board 90º or 180º clockwise then we obtain analogous board.

§2 Elementary boards for a *knight*

Denote the set of elementary boards as *J*. Denote a board with «excluded» square $\Pi(m;n)_{(i;j)}$ where (i;j) are coordinates of «excluded» square.

Lemma 1. If Π(m;n) can be divided into adherent squares then the first player wins. In particular, if $\Pi(m;n)=\coprod_{d=1}^{s} I_d$ where s is a number of I_d; $I_d \in$ J.

Lemma 2. If $\Pi(m;n)_{(a;b)}$ can be divided into adherent squares then the second player wins. In particular, if $\Pi(m;n)_{(a;b)}=\coprod_{d=1}^{s} I_d$ where s is a number of I_d; $I_d \in$ J.

Here you can see some elementary boards: I(4;2); I(4;3); I(6;3); I(6;5):

1	2
3	4
2`	1`
4`	3`

1	2	3
3`	4	5
2`	1`	6
6`	5`	4`

1	2	3
3`	4	5
2`	1`	6
7	5`	4`
8	6`	9
9`	7`	8`

		7`	6`	5`
		4`	3`	7
		6	5	4
		2`	1`	3
				2
				1

(*Fig. 1*)

Adherent squares are marked by the stroke.

Later we will show that any board $\Pi(m;n)$ where $m \cdot n$ is even and $m \geq 3$; $n \geq 3$ can be divided into pairs of adherent squares using elementary boards and additional research. Analogously any board $\Pi(m;n)$ where $m \cdot n$ is odd with one excluded square can be divided into pairs of adherent squares.

§3 General strategy for a *knight*

First of all consider $\Pi(k;1)$ and $\Pi(k;2)$:

It\`s clear that the second player wins on $\Pi(k;1)$. All moves except the first one are uniquely specified on the $\Pi(k;2)$, and after any move a either decreases or increases by 2. Thus if either (a-1) or (k-a) equals either (4u+2) or (4u+3) then the first player wins, otherwise the second player wins.

Proposition 1. If either m or n is even, then the first player wins.

Proof.

First divide $\Pi(4;l)$ into adherent squares. For this purpose we use $I(4;2)$ and $I(4;3)$. Thus we can divide $\Pi(4k;l)$ using $\Pi(4;l)$ k times. Note that $k \geq 1$, $l \geq 2$.

Now divide $\Pi(4k+2;4l+2)$. For this purpose divide $\Pi(6;6)$ in the top right corner of the initial board using $I(6;3)$. So the initial board is divided into two boards: $\Pi(6;4l-4)$ and $\Pi(4k-4;4l+2)$. There is at least one side that divisible by 4 in these boards, so we can divide them. Note that k, $l \geq 1$.

Divide $\Pi(4k+2;2l+1)$ into adherent squares.

Divide $\Pi(4k+2;3)$ in the right corner of the initial board using $I(3;4)$ and $I(3;6)$. Then divide $\Pi(4k+2;2l-2)$. Note that $k \geq 1$, $l \geq 3$.

So we divided all boards where either m or n are even numbers except $\Pi(4k+2;5)$.

We can divide $\Pi(4;5)$ using $I(4;3)$ and $I(4;2)$. So $\Pi(4k+2;5)$ can be divided using $\Pi(4;5)$ and $I(6;5)$.■

Proposition 2. If both m and n are odd ($\Pi(2k+1;2l+1)$) and the first square is white then the first player wins.

Proof.

Note that the number of black squares is greater on one than the number of white squares and each player is making moves only on squares of the same colour.

Now show that the first player wins by dividing all squares of the board except the lower right one into adherent squares.

Divide $\Pi(5;3)_{(1;3)}$ in the lower right corner of the initial board. In this case it is elementary board:

1	2	3
5\`	4	1\`
2\`	3\`	7\`
4\`	5	6\`
6	7	*

(*Fig. 2*)

The excluded square is marked by * (note that the excluded square can be located in any corner of this board)

Also we can divide remaining $\Pi(2k-4;2l+1)$ and $\Pi(5;2l-2)$. Note that $k \geq 4$; $l \geq 3$. So we need to divide $\Pi(3;2l+1)_{(1;2l+1)}$; $\Pi(5;2l+1)_{(1;2l+1)}$ and $\Pi(7;2l+1)_{(1;2l+1)}$.

For $\Pi(3;2l+1)_{(1;2l+1)}$ divide $\Pi(3;5)_{(1;5)}$ in the right corner. Then divide $\Pi(3;2l\text{-}4)$. Note that $l \geq 4$. So we need to divide $\Pi(3;3)_{(2;2)}$; $\Pi(3;5)_{(1;5)}$ and $\Pi(3;7)_{(1;7)}$.

In the first case we exclude the central black square.

1	2	3
3`	*	4
4`	1`	2`

(*Fig. 3*)

$\Pi(3;5)_{(1;5)}$ is already divided and $\Pi(3;7)_{(1;7)}$ can be divided in the following way:

10`	6`	8	4`	2`	3	1
8`	7`	9	6	5	4	2
9`	10	5`	7	3`	1`	*

(*Fig. 4*)

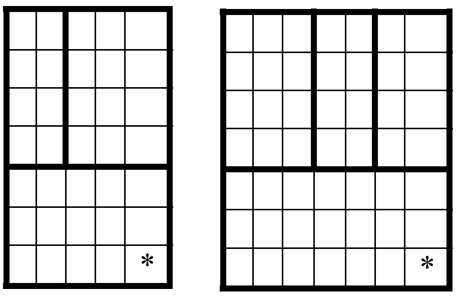
For $\Pi(5;2l+1)_{(1;2l+1)}$ divide $I(5;3)_{(1;3)}$ board in the right corner. Then divide $\Pi(5;2l\text{-}2)$. Note that $l \geq 3$. So we need to divide $\Pi(5;5)_{(1;5)}$:

11`	8`	12`	10`	9`
7`	10	11	2`	12
8	3`	6`	9	1`
6	7	5`	4`	2
5	4	3	1	*

(*Fig. 5*)

For $\Pi(7;2l+1)_{(1;2l+1)}$ divide $I(3;5)_{(1;5)}$ in the lower right corner. Then divide $\Pi(4;2l+1)$ and $\Pi(3;2l\text{-}4)$. Note that $l \geq 4$. So we need to divide $\Pi(7;3)_{(1;3)}$; $\Pi(7;5)_{(1;5)}$ and $\Pi(7;7)_{(1;7)}$.

The $\Pi(7;3)_{(1;3)}$ is already divided and $\Pi(7;5)_{(1;5)}$; $\Pi(7;7)_{(1;7)}$ can be divided in the following way:



(*Fig. 6*)

■

Proposition 3. If both *m* and *n* are odd and the first square is black then the second player wins.

Proof.

Divide all squares of such board except the first one into adherent squares.

1. If $(a\text{-}1)$ is odd then since a, b have the same parity numbers $(m\text{-}a)$; $(b\text{-}1)$; $(n\text{-}b)$ are also odd.

Divide a $\Pi(3;3)_{(2;2)}$ where the first square is located in the centre:

1	2	3
3`	f	4
4`	1`	2`

(*Fig. 7*)

Denote the first square by a letter «f».

Then divide remaining Π(a-2;b-2); Π(m-a-1;b-2); Π(a-2;n-b-1); Π(m-a-1;n-b-1); Π(3;b-2); Π(3;n-b-1); Π(a-2;3); Π(m-a-1;3). So we need to consider cases when some of the numbers (a-2); (m-a-1); (b-2) and (n-b-1) equal 2.

If only one of the equalities holds, for example for (n-b-1), then divide Π(a-2;n-b); Π(m-a+2;b-2) and Π(m-a-1;n-b+2). Thus we need to divide $Π(3;5)_{(2;2)}$:

3	4	5	6	7`
2	f	1`	4`	5`
1	3`	2`	7	6`

(*Fig. 8*)

If two of the equalities hold, for example for (n-b-1) and (b-2), then divide Π(a-2;n-b); and Π(m-a-1;n-b). Thus we need to divide $Π(3;7)_{(2;4)}$:

1	3`	4`	5`	8	10	7`
2	5	1`	f	7	6`	9`
3	4	2`	6	9	8`	10`

(*Fig. 9*)

If two another of the equalities hold, for example for (n-b-1) and (m-a-1), then divide Π(a-2;n-b); and Π(m-a+2;b-2). Thus we need to divide $Π(5;5)_{(2;2)}$:

5	6	4`	10	11`
4	2`	5`	6`	12`
3`	1`	7	11	10`
2	f	8	12	9`
1	3	9	7`	8`

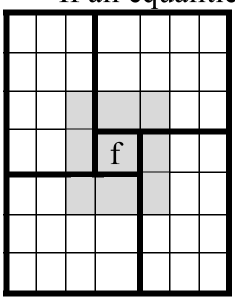
(*Fig. 10*)

If three of the equalities hold, for example for (n-b-1); (b-2) and (m-a-1), then divide Π(a-2;n-b). Thus we need to divide $Π(5;7)_{(2;4)}$:

7	5`	6`	8`	10`	9`	12`
4`	8	7`	9	12	15	16`
5	6	3`	10	11`	14`	17`
2`	4	1`	f	13	16	15`
1	3	2	11	14	17	13`

(*Fig. 11*)

If all equalities hold, then divide $Π(7;7)_{(4;4)}$:



(*Fig. 12*)

2. If (a-1) is even then numbers (m-a); (b-1); (n-b) are also even. Divide remaining Π(m-a;b); Π(m-a+1;n-b); Π(a-1;n-b+1) and Π(a;b-1). So we need to consider a case when some of the numbers (a-1); (m-a); (b-1) and (n-b) equal 2 and also a case when some of the numbers (a-1); (m-a); (b-1) and (n-b) equal 0.

2.1. The case for 2.

If only one equality holds, for example for (m-a), then divide $\Pi(m;n\text{-}b)$ and $\Pi(a\text{-}1;b)$. Thus we need to divide $\Pi(3;b)_{(1;b)}$. So this case is already considered for a board $\Pi(3;2l{+}1)_{(1;2l+1)}$.

If two of the equalities hold, for example for (b-1) and (n-b), then divide $\Pi(a\text{-}1;5)$. Thus we need to consider $\Pi(2l{+}1;5)_{(1;3)}$. For this purpose divide $\Pi(5;5)_{(1;3)}$ and $\Pi(5;7)_{(1;3)}$:

		8`	6`	7`
		7	5`	8
		6	4`	3`
		1`	2`	5
1	2	*	3	4

				9`
				8`
		8	9	7`
		7	5`	6`
		6	4`	3`
		1`	2`	5
1	2	*	3	4

(*Fig. 13*)

Also we can divide $\Pi(4;5)$ so we can divide $\Pi(2l{+}1;5)_{(1;3)}$.

If another two of the equalities hold, for example for (m-a) and (n-b), then divide $\Pi(a\text{-}1;b{+}2)$. Thus we need to consider $\Pi(3;2l{+}1)_{(1;2l-1)}$. For this purpose divide $\Pi(3;7)_{(1;5)}$ and $\Pi(3;9)_{(1;7)}$:

8`	10	9	4`	2`	1`	3
9`	6`	8	7	5	4	2
10`	7`	5`	6	*	3`	1

13`	12	11	7`	9	5`	2`	1`	3
11`	10`	9`	6`	8	4`	3`	5	2
12`	13	8`	10	7	6	*	4	1

(*Fig. 14*)

Also we can divide I(3;4) so we can divide $\Pi(3;2l{+}1)_{(1;2l-1)}$.

If three of the equalities hold, for example for (a-1); (n-b) and (b-1), then we need to consider $\Pi(2l{+}1;5)_{(3;3)}$. For this purpose divide $\Pi(5;5)_{(3;3)}$ and $\Pi(7;5)_{(3;3)}$:

8	9	7`	12`	11`
7	3`	8`	9`	10`
2`	1`	*	11	12
3	4	6`	10	5`
1	2	5	4`	6

3	4	9`	8`	6
		3`	4`	9
		7`	6`	8
		5`	2`	7
		*	1`	5
				2
				1

(*Fig. 15*)

Also we can divide $\Pi(4;5)$ so we can divide $\Pi(2l{+}1;5)_{(3;3)}$.

If all of the equalities hold, then we need to consider $\Pi(5;5)_{(3;3)}$(Fig. 15). So this case is already considered.■

2.2. The case for 0.

Firstly divide the following elementary boards: $I(5;7)_{(1;5)}$ and $I(5;9)_{(1;7)}$

15`	17	16	9	5`	8	7
16`	9`	15	8`	7`	6	5
17`	12`	14	4`	3`	1`	2`
14`	11`	13	10	6`	4	3
13`	10`	12	11	*	2	1

19`	18`	20	15	14	8`	7	6	5
20`	15`	19	12`	10`	6`	5`	3`	4`
22`	21	18	14`	13	7`	8	1`	2`
17`	16`	13`	11`	12	10	9	4	3
21`	22`	17	16	9`	11	*	2	1

(*Fig. 16*)

If only one of the equalities holds, for example for (a-1) then we need to consider the following cases. If both (b-1) and (n-b) are greater than 2, then divide $\Pi(m;b)_{(1;b)}$ and $\Pi(m;n-b)$. We can divide $\Pi(m;b)_{(1;b)}$ using the following boards: $\Pi(3;5)_{(1;5)}$(Fig. 2); $\Pi(3;7)_{(1;7)}$(Fig. 4); $\Pi(5;5)_{(1;5)}$(Fig. 5); $\Pi(5;7)_{(1;7)}$(Fig. 6) and also $\Pi(4;l)$ l>1.

If either (b-1) or (n-b) equals 2, for example (n-b)=2, then divide $\Pi(m;n)_{(1;n-2)}$ using the following boards: $\Pi(3;7)_{(1;5)}$(Fig.14); $\Pi(3;9)_{(1;7)}$(Fig. 14); $I(5;7)_{(1;5)}$; $I(5;9)_{(1;7)}$ and also $\Pi(4;l)$ l>1.

If both (b-1) and (n-b) equals 2 and (m-a) is greater than 2, then divide $\Pi(m;n)_{(1;3)}$ by using the following boards: $\Pi(5;5)_{(1;3)}$(Fig. 13); $\Pi(5;7)_{(1;3)}$(Fig. 13)and also $\Pi(4;5)$. So we need to consider $\Pi(3;5)_{(1;3)}$. The first player wins on this board:

		2		
1				3
		0		

(*Fig. 17*)

If two of the equalities hold, for example for (a-1) and (n-b) then divide $\Pi(m;n)_{(1;n)}$ using the following boards: $\Pi(3;5)_{(1;5)}$(Fig. 2); $\Pi(3;7)_{(1;7)}$(Fig. 4); $\Pi(5;5)_{(1;5)}$(Fig. 5); $\Pi(5;7)_{(1;7)}$(Fig. 6) and also $\Pi(4;l)$ l>1.

Putting together the results of previous prepositions we get:

Theorem 1. *(for a knight)* The second player wins on $\Pi(1;n)$. The first player wins on $\Pi(2;n)$ if either a-1 or n-a equals either 4u+2 or 4u+3, otherwise the second player wins. If either m or n is even, then the first player wins. If both m and n are odd and the first square is white then the first player wins. If both m and n are odd and the first square is black then the second player wins except $\Pi(3;5)_{(1;3)}$.

§4 General strategy for *a king*, *a queen*, *a rook*

In this case the elementary board is I(1;2):

1	1`

(*Fig. 18*)

First consider $\Pi(1;n)$ for a king. If either (b-1) or (n-b) is odd then the first player wins otherwise the second player wins.

Now consider $\Pi(1;n)$ for a queen and a rook. If n is even then the first player wins otherwise the second player wins.

Proposition 4. If either m or n is even, then the first player wins.

Proof.

Divide all squares of the given board into adherent squares using I(1;2).■

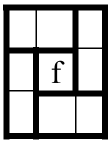
Proposition 5. *(for a king)* If both sides of the initial board are odd then the second player wins.

Divide all squares except the first one into adherent squares.

Assume the first square is black.

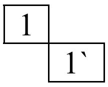
If (a-1) is even then we divide $\Pi(a-1;n)$; $\Pi(m-a;n)$; $\Pi(1;b-1)$ and $\Pi(1;m-b)$ using I(1;2).

If (a-1) is odd then we divide $\Pi(m-a-1;n)$; $\Pi(a+1;n-b-1)$; $\Pi(a-2;b+1)$ and $\Pi(3;b-2)$ using I(1;2). Now we need to divide $\Pi(3;3)_{(2;2)}$:



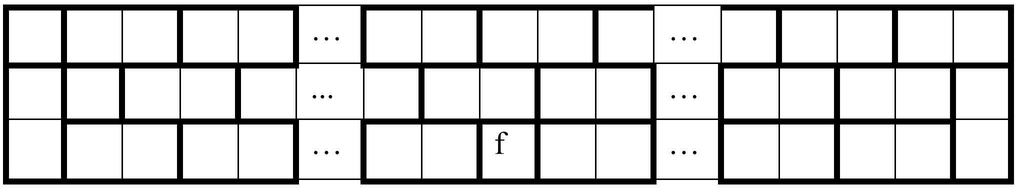
(*Fig. 19*)

Assume the first square is white. So a and b have the different parity. Without loss of generality let (a-1) and (m-a) be even and (b-1) and (n-b) be odd. Note that for a king we can also use the following figure as an elementary board:



(*Fig. 20*)

Divide $\Pi(a\text{-}1;n)$ and $\Pi(m\text{-}a\text{-}2;n)$ using I(1;2). So there is $\Pi(3;n)_{(1;(n+1)/2)}$ in the centre of the initial board. Divide it in the following way:



(*Fig. 21*)

Note that if a equals m then divide $\Pi(a\text{-}3;n)$ and $\Pi(3;n)_{(3;(n+1)/2)}$. ■

Proposition 6. *(for a queen and a rook)* If both sides of the initial board are odd then the second player wins.

Proof.

Choose $\Pi(m;3)$ where the first square is located. Moreover, we choose this board so that horizontal sides of two remaining boards are even. Then divide all squares of the board except the first one into adherent squares. Divide all $\Pi(m;2)$ (so we divide all columns except the chosen ones). Then divide the chosen $\Pi(m;3)$ in the following way:

1) Any square of the column which doesn`t contain the first square is adherent to the square in the same line of the column which also doesn`t contain the first square;

2) The number of squares in the column which contains the first square is even. So this column can be divided into adherent squares.■

Putting together the results of previous prepositions we get:

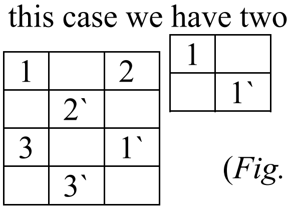
Theorem 2. *(for a king, a queen and a rook)*. For a king the first player wins on $\Pi(1;n)$ if either (b-1) or (n-b) is odd otherwise the second player wins. For a queen and a rook the first player wins on $\Pi(1;n)$ if n is even otherwise the second player wins. If either m or n is even, then the first player wins otherwise the second player wins.

§5 General strategy for a *bishop*

It`s clear that the second player wins on $\Pi(1;n)$.

All moves except the first one are uniquely specified on $\Pi(2;n)$. So if either (b-1) or (n-b) is odd then the first player wins otherwise the second player wins.

In this case we have two elementary boards. I(2;2) and I(4;3):



(*Fig. 22*)

Note that in this case squares of both colours can be divided into adherent squares. Also one can turn these elementary boards.

Proposition 7. If both numbers of black and white squares are even then the first player wins.

Proof.

Let the first square be black.

Let`s divide $\Pi(4k;l)$. We can divide $\Pi(4;2)$ using $I(2;2)$. And now we can divide $\Pi(4k;l)$ using $I(4;3)$ and $\Pi(4;2)$. Note that $k \geq 1$; $l \geq 2$.

Divide $\Pi(2k;2l)$ using $I(2;2)$. Note that k and l don`t divisible by four and $k \geq 1$; $l \geq 1$.

If the first square is white then we have analogous case.■

Proposition 8. If both numbers of black and white squares are odd then the second player wins.

Proof.

Assume the first square is black.

Divide $\Pi(2k;2l+1)_{(a;b)}$. Note that k not divisible by 2.

Choose $\Pi(2k;3)_{(a;r)}$ $r=\{1;2;3\}$. Moreover, we choose this board so, that both sides of two remaining boards are even. So we can divide these two boards.

Now consider the $\Pi(2k;3)_{(a;r)}$. Choose $\Pi(6;3)_{(t;r)}$ $t=\{1;2;\ldots;6\}$. Moreover, we choose this board so that both sides of two remaining boards divisible by 4. Clearly that if we choose any square on $\Pi(6;3)_{(t;r)}$, then all the other squares always can be divided into adherent squares. Note that $k \geq 3$; $l \geq 1$.

If the first square is white then we have analogous case.■

Proposition 9. Assume both sides of the initial board are odd and the first square is black. If one or three of the numbers $(a-2)$; $(m-a-1)$; $(b-2)$ and $(n-b-1)$ are not divisible by 4, then the first player wins otherwise the second player wins.

Proof.

Divide $\Pi(2k+1;2l+1)$ into adherent squares. Choose two crossing boards: $\Pi(2k+1;3)$ and $\Pi(3;2l+1)$. There is $\Pi(3;3)$ where the first square is located in the common part of these two boards. Moreover, we choose these board so that both sides of four remaining boards are even. After that we need to consider four boards: one side of all these boards equal 3 and the second side is even.

It is not difficult to make sure, that if one or three of the numbers $(a-2)$; $(m-a-1)$; $(b-2)$ and $(n-b-1)$ not divisible by 4, then the first player wins otherwise the second player wins.■

Proposition 10. Assume both sides of the initial board are odd and the first square is white. If one or three of the numbers $(a-2)$; $(m-a-1)$; $(b-2)$ and $(n-b-1)$ are not divisible by 4, then the second player wins otherwise the first player wins.

Proof.

As in the case where the first square is black we choose the same boards and consider remaining squares. It is not difficult to make sure, that if one or three of the numbers $(a-2)$; $(m-a-1)$; $(b-2)$ and $(n-b-1)$ not divisible by 4, then the second player wins otherwise the first player wins.■

Putting together the results of previous prepositions we get:

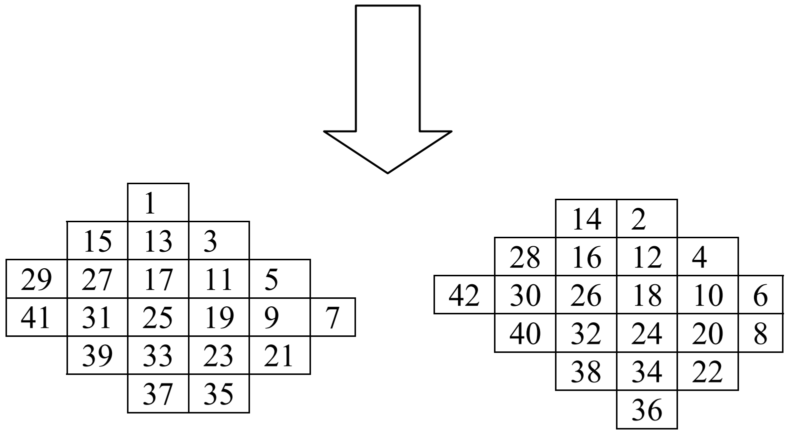
Theorem 3. *(for a bishop)* The second player wins on $\Pi(1;n)$. The first player wins on $\Pi(2;n)$ if either $(b-1)$ or $(n-b)$ is odd otherwise the second player wins. If both numbers of black and white squares are even then the first player wins. If both numbers of black and white squares are odd then the second player wins. Assume both sides of the initial board are odd and the first square is black. If one or three of the numbers (*a*-2); (*m*-*a*-1); (*b*-2) and (*n*-*b*-1) not divisible by 4, then the first player wins otherwise the second player wins. If the first square is white and one or three of the numbers (*a*-2); (*m*-*a*-1); (*b*-2) and (*n*-*b*-1) not divisible by 4, then the second player wins otherwise the first player wins.

§6 Some unusual boards and pieces

Note that the presented winning strategies can also work on unusual boards. For example, it works on boards in the form of cross as in the case for bishop. Also we can consider boards with holes using this method.

Point 6.1. Let`s turn the initial board 45º clockwise. Note that if we now copy squares of the only one colour then we will get a «stepped» board. So if the initial board was considered for a bishop then the «stepped» board can be considered for a rook. For example:

1	2	3	4	5	6	7
14	13	12	11	10	9	8
15	16	17	18	19	20	21
28	27	26	25	24	23	22
29	30	31	32	33	34	35
42	41	40	39	38	37	36



(*Fig. 23*)

As we have found winning strategies for a bishop so we have also found winning strategies for a rook on the unusual board.

Point 6.2. A *camel* is a piece with the following move:

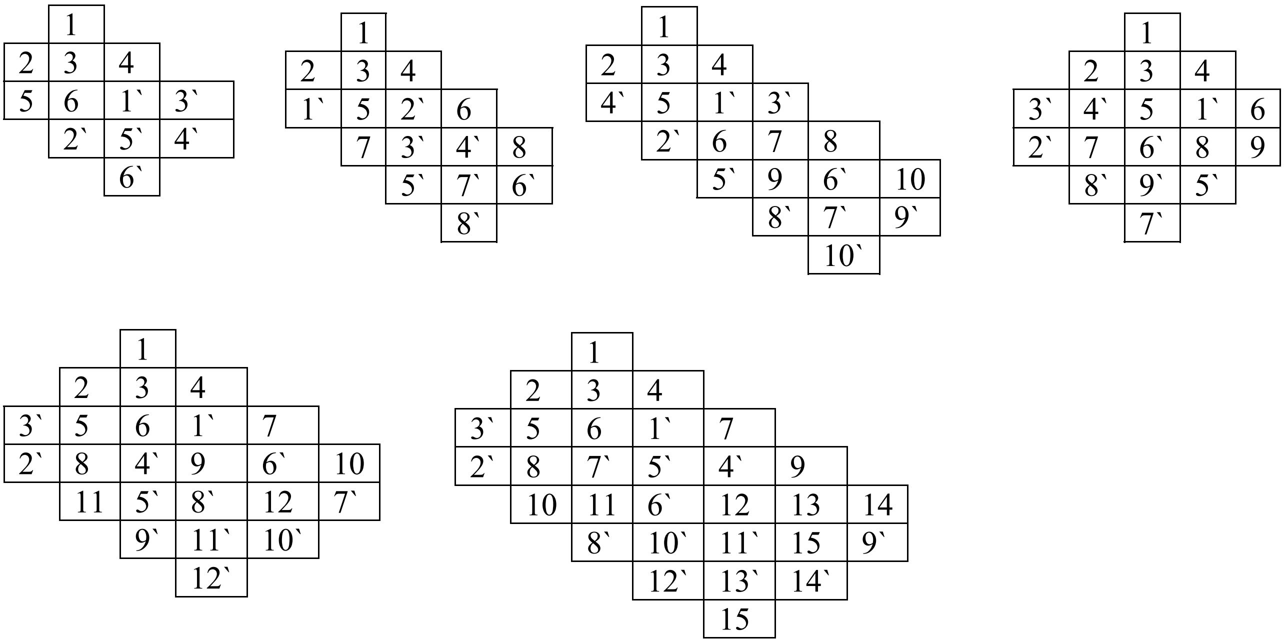
(*Fig. 24*)

Proposition 11. *(for a camel)* If both sides of the initial board are even (Π(2*k*;2*l*)), then the first player wins.

Consider a knight on «stepped» boards. Note that if we consider a knight on such boards then we also consider a *camel* on usual boards.

Assume the first square is black.

In this case elementary boards are:



(*Fig. 25*)

So if we reconstruct these boards to usual, then they are I(6;4); I(8;4); I(10;4); I(6;6); I(8;6) and I(10;6). Thus one can get Π(2*k*;2*l*) using these elementary boards. Note that $k \geq 3$; $l \geq 2$. So we need to consider Π(2;2*l*) and Π(4;4).

All moves except the first one are uniquely specified on Π(2;2*l*), and after any move *b* either decreases or increases by 3. Thus if either (*b*-1) or (2*l*-*b*) equals (6*u*+*p*) where p={3;4;5}, then the first player wins, otherwise the second player wins.

Divide Π(4;4) in the following way:

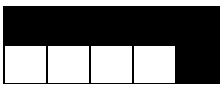
1		3	
	4		2`
2		5	
	1`		3`

(*Fig. 26*)

If the first square is located in the 4th or 5th one then the second player wins otherwise the first player wins.

If the first square is white then we have analogous case.■

Point 6.3. A *giraffe* is a piece with the following move:



(*Fig. 27*)

Proposition 12. *(for a giraffe)* If either *m* or *n* divisible by 8 then the first player wins. On Π(2*k*;8*l*+7) $k \neq 7$; $l \geq 1$.

It`s clear that the second player wins on Π(1;*n*). On Π(*k*;*n*) *k*={2;3;4} the first player wins if either (*b*-1) or (*n*-*b*) equals (8*u*+*p*), where *p*={4;5;6;7}.

In this case elementary boards are I(2;8); I(5;8); I(7;10); I(7;12):

1	3	6	7	2`	4`	5`	8`
2	4	5	8	1`	3`	6`	7`

1	6	10	13`	11	7`	17	12
2	7	17`	14`	3`	6`	18	13
3	8	18`	15`	2`	9`	19	14
4	9	19`	16`	5`	8`	20	15
5	1`	20`	10`	4`	11`	12`	16

1	5`	9	16	22	28	10`	17`	23`	29`
2	6`	10	17	23	29	11`	16`	22`	28`
3	7`	11	18	24	30	12`	34`	35	31`
4	8	12	19	25	31	13`	18`	24`	30`
5	1`	13	9`	4`	8`	14`	19`	25`	32`
6	2`	14	20	26	32	15`	21`	27`	33`
7	3`	15	21	27	33	34	20`	26`	35`

1	5`	9	16			10`	17`				
2	6`	10	17			9`	16`				
3	7`	11	15`			12`	18`				
4	8	12	18			11`	19`				
5	1`	13	19	4`	8`						
6	2`	14	20			15`	21`				
7	3`	15	21			14`	20`				

(*Fig. 28*)

The second part of the board can be divided symmetrically.

We can divide Π(8*k*;*l*) using I(2;8) and I(5;8). Note that $k \geq 1$; $l \neq 3$. Π(2*k*;8*l*+7) $k \neq 7$; $l \geq 1$ can be divided using I(7;8); I(7;10); I(7;12) and I(2*k*;8*l*).■

The most important result here is that we have constructed a method that allows us to study boards with a complex structure («stepped» ones; not simply connected ones etc.) and another pieces (unusual pieces with non-standard moves).

§7 General strategy for a *rook*, which leaves a «trace»

Lemma 3. If we are given a $k \times l$ board ($k>l$) and the first player makes a first move from one of the corners along smaller side, then the second player wins. Note that on the first square first player can pass a move.

Call not visited squares *free*.

Proof.

Without loss of generality the first player makes a move from the upper left square. After the first player`s move the second player moves to the bottom as long as possible. Then one get the analogous position. Notice that after the first pair of turns the number of squares where the second player moves hasn't decreased with respect to the number of squares where the first player moves. So the second player always wins using such moves because initially $k>l$.■

Theorem 4. For a rook which leaves «trace» the first player wins.

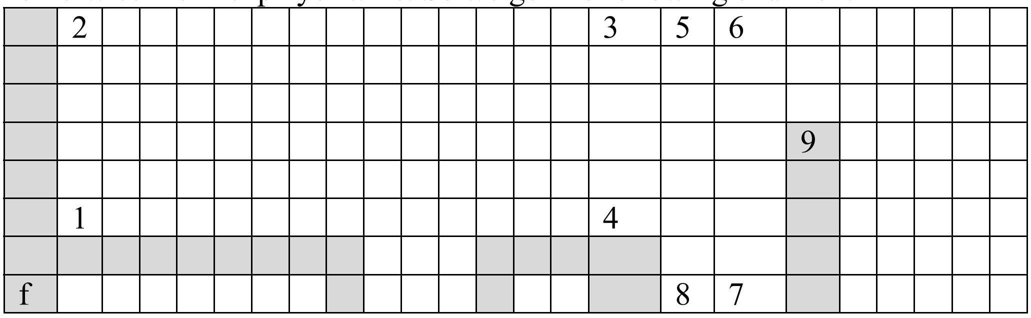
Proof.

Without loss of generality assume (a-1) is the biggest number among (a-1); (b-1); (m-a) and (n-b). So in this case the first player moves from the first square to the bottom as long as possible. Then the second player moves to the right or to the left in the lower line. If the number of free squares from the first square to the top of the board equals one, then the first player goes to the a-th line, otherwise he goes to the top as long as possible. Now the first player makes moves according to the third lemma. So if the second player makes moves in this rectangle (from the right or from the left of the first square) then the first player wins.

If the second player goes to the opposite rectangle then we have analogous case. Thus the second player has to make a move to the square in the b-th column. Then the first player goes to the square in the (a+1)-th line. Note that if the number of free squares from the first one to the top of the board equals one, then the first player compels the second player to give him a chance to make a move on this square using such strategy, otherwise the first player wins.

Then the second player goes on the (a+1)-th line (for example to the right). Note that the number of free squares in this line where the second player can make a move is less than a number of free squares from the (a+1)-th line to the bottom of the board. So the first player make moves according to the third lemma, but he can also go upper than the (a+1)-th line. Moreover, he can either make this move or let the second player make this move. So the first player always wins.

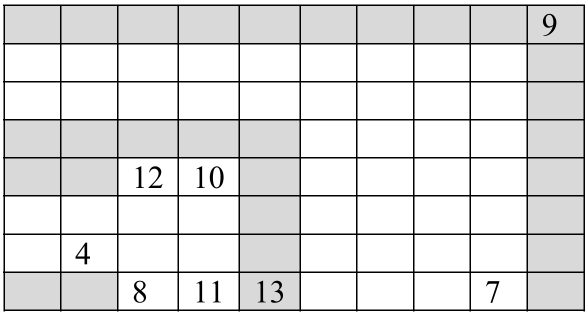
Now we need to determine when the first player should go upper than the (a+1)-th line. Assume the length of second player's first move in the (a+1)-th line equals g and the number of free squares from the (a+1)-th line to the top equals h. If $g>h$, then the first player make moves according to the third lemma. Thus the second player goes upper than the (a+1)-th line, otherwise the first player wins. So we get the following situation:



(*Fig. 29*)

Then the first player makes moves according to the third lemma on the board with vertices 1;2;3;4 (see Fig. 29).

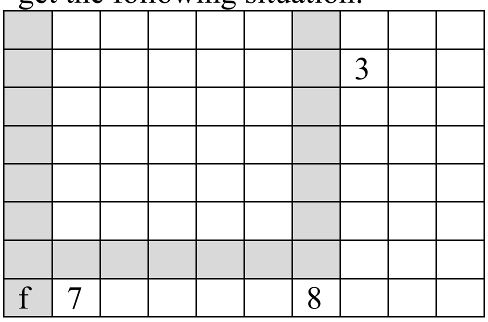
Note that if the second player goes on the board with vertices 5;6;7;8, then the first player goes to the bottom as long as possible. So we get the following situation:



(*Fig. 30*)

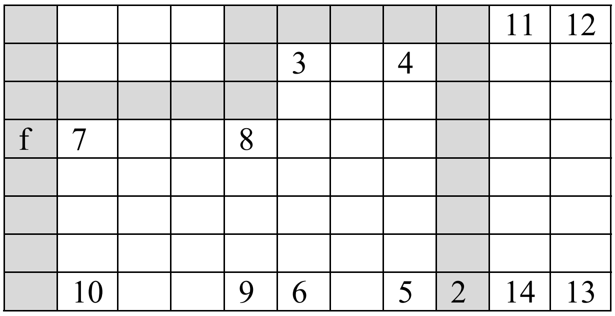
So the first player made the last move to the 13-th square. If the second player goes to the left then the first player makes moves according to the third lemma on the board with vertices 12;10;11;8 (see Fig. 30). If the second player goes to the right then we have analogous situation.

If $g \leq h$, then the first player makes moves by the third lemma on the rectangle $g \times h$. So we get the following situation:



(*Fig. 31*)

If the second player goes to the left then the first player wins. If the second player goes to the right, then the first player goes to the bottom as long as possible. So we get the following situation:



(Fig. 32)

So the first player made the last move to the second square. If the second player goes to the right then the first player makes moves according to the third lemma on the board with vertices 11;12;13;14 (see Fig. 32). If the second player goes to the left then we have two cases. If the second player goes on the board with vertices 7;8;9;10 then the first player makes moves according to the third lemma. If the second player goes to the board with vertices 3;4;5;6, then the first player goes to the top as long as possible. So we get the analogous situation.

Thus this strategy shows that the first player wins on any board except Π(1;1).■