Problem 3: Cyclic Inequality

Team: *Belarus*

Abstract

All points of the initial statement were considered.

- Point 1 is solved completely (see theorem 1).
- Point 2 is solved completely (see theorems 2-3).
- In point 3 the existence of required tuple for $C_{k,a}(x) > A(x)$ was proved in case $\frac{k-1}{k+1} < a < k-1$ (see proposition 1).
- In point 4 the following estimations were obtained: if $k \le l$, then inequality holds for $\forall a \in \mathbb{R}_+$; if $k > l$, $a_{min} \le \frac{k}{l} - 1$ (see theorem 4).
- The following inequalities were studied:
  $\frac{x_1^{k+l}+bx_2^{k+l}}{x_1^k+ax_2^k} + \frac{x_2^{k+l}+bx_3^{k+l}}{x_2^k+ax_3^k} + \cdots + \frac{x_n^{x+l}+bx_1^{k+l}}{x_n^k+ax_1^k} \ge \frac{b+1}{a+1}\left(x_1^l + x_2^l + \cdots + x_n^l\right)$ (problem 2)
  $\frac{ax_1^{k+l}+x_2^{k+l}}{x_1^k+ax_2^k} + \cdots + \frac{ax_n^{k+l}+x_1^{k+l}}{x_n^k+ax_1^k} \ge x_1^l + \cdots + x_n^l$ (problem 3)

Statement and basic denotation

Let a – positive real number, n – natural number. Denote:

$$A(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$A_l(x) = \frac{{x_1}^l + {x_2}^l + \cdots + {x_n}^l}{n}$$

$$C_{k,l,a}(x) = \frac{a+1}{n}\left(\frac{{x_1}^{k+l}}{{x_1}^k + a{x_2}^k} + \frac{{x_2}^{k+l}}{{x_2}^k + a{x_3}^k} + \cdots + \frac{{x_n}^{k+l}}{{x_n}^k + a{x_1}^k}\right)$$

$$C_{k,a}(x) = \frac{a+1}{n}\left(\frac{{x_1}^{k+1}}{{x_1}^k + a{x_2}^k} + \frac{{x_2}^{k+1}}{{x_2}^k + a{x_3}^k} + \cdots + \frac{{x_n}^{k+1}}{{x_n}^k + a{x_1}^k}\right)$$

Where $x = (x_1, x_2, \dots, x_n)$.

Prove, that:

1) If $a \ge k-1$, then $C_{k,a}(x) \ge A(x)$, $\forall\, x_i \in R_+$
2) If $0 < a < \frac{k-1}{k+1}$, then exist 2 sets *x* and *y,* such that: $C_{k,a}(x) > A(x)$ and $C_{k,a}(y) < A(y)$
3) Investigate case $\frac{k-1}{k+1} < a < k-1$

4') Investigate inequality: $C_{k,l,a}(x) \ge A_l(x)$

4) Find min *a*, such that $C_{k,l,a}(x) > A^l(x)$ for all n-tuples *x*

Proofs:

Lemma 1. If $a \ge k-1$, then $\frac{x_1({x_1}^k - {x_2}^k)}{{x_1}^k + a{x_2}^k} \ge \frac{k(x_1 - x_2)}{a+1}$.

Proof.

Multiplying both sides of inequality by multiple of denominators and collecting terms, we obtain:

$(a+1-k){x_1}^{k+1} + ka{x_2}^{k+1} + kx_2{x_1}^k \ge (ka+a+1)x_1{x_2}^k$, which is true by **weighted AM-GM inequality**. □

Theorem 1.

If $a \ge k-1$, then $C_{k,a}(x) \ge A(x)$, $\forall\, x_i \in R_+$

Proof.

Transpose $A(x)$ to the LHS and multiply the inequality by *n.* Then regroup terms:

$$(a+1)\left(\frac{{x_1}^{k+l}}{{x_1}^k + a{x_2}^k} + \frac{{x_2}^{k+l}}{{x_2}^k + a{x_3}^k} + \cdots + \frac{{x_n}^{k+l}}{{x_n}^k + a{x_1}^k}\right) - (x_1 + x_2 + \cdots + x_n) =$$

$$= \left(\frac{(a+1){x_1}^{k+l}}{{x_1}^k + a{x_2}^k} - x_1\right) + \cdots + \left(\frac{(a+1){x_n}^{k+l}}{{x_n}^k + a{x_1}^k} - x_n\right) = a\left(\frac{x_1({x_1}^k - {x_2}^k)}{{x_1}^k + a{x_2}^k} + \cdots + \frac{x_n({x_n}^k - {x_1}^k)}{{x_n}^k + a{x_1}^k}\right) \ge 0 \qquad \textbf{(1)}$$

By **lemma 1**

$$\frac{x_1({x_1}^k - {x_2}^k)}{{x_1}^k + a{x_2}^k} + \frac{x_2({x_2}^k - {x_3}^k)}{{x_2}^k + a{x_3}^k} + \cdots + \frac{x_n({x_n}^k - {x_1}^k)}{{x_n}^k + a{x_1}^k} \ge \frac{k(x_1 - x_2)}{a+1} + \frac{k(x_2 - x_3)}{a+1} + \cdots + \frac{k(x_n - x_1)}{a+1} = 0 \qquad □$$

From now on we consider the inequality in form **(1)**.

Theorem 2.

There exists *x* such that $C_{k,a}(x) > A(x)$, if $0 < a < \frac{k-1}{k+1}$.

Proof.

Let us find x_1 and x_2 such that:

$\frac{x_1({x_1}^k-{x_2}^k)}{{x_1}^k+a{x_2}^k}+\frac{x_2({x_2}^k-{x_1}^k)}{{x_2}^k+a{x_1}^k}>0$ ***(2)***

$\Leftrightarrow x_1({x_1}^k-{x_2}^k)(a{x_1}^k+{x_2}^k)+x_2({x_2}^k-{x_1}^k)(a{x_2}^k+{x_1}^k)>0$

Since inequality is symmetric, w.l.o.g. we can suppose that $x_1>x_2$. Divide by positive $({x_1}^k-{x_2}^k)$ and collect terms.

(2) $\Leftrightarrow a({x_1}^{2k+1}+{x_2}^{2k+1})>{x_1}^{2k}x_2+x_1{x_2}^{2k}+(a-1)({x_1}^{k+1}{x_2}^k+{x_1}^k{x_2}^{k+1})$ **(3)**

$LHS \xrightarrow[x_2\to 0]{} ax_1^{2k+1}\neq 0$ and $RHS \xrightarrow[x_2\to 0]{} 0$. Therefore $LHS>RHS$.

Thus, there exists required x_1 and x_2 such that ***(2)*** holds.

Let $x_2=x_3=\cdots=x_n$. Then **(1)**=**(2)**.So, there exists set $(x_1,x_2,\dots,x_2)$, which satisfies condition. □

Theorem 3.

There exists *x* such that $C_{k,a}(x)<A(x)$, if $0<a<\frac{k-1}{k+1}$.

Proof.

Similarly to the proof of **theorem 2**, consider the case *n* = 2.

$\frac{x_1({x_1}^k-{x_2}^k)}{{x_1}^k+a{x_2}^k}+\frac{x_2({x_2}^k-{x_1}^k)}{{x_2}^k+a{x_1}^k}<0$ **(4)**

$(\mathbf{4})\Leftrightarrow x_1({x_1}^k-{x_2}^k)(a{x_1}^k+{x_2}^k)+x_2({x_2}^k-{x_1}^k)(a{x_2}^k+{x_1}^k)<0$

Regroup terms. Obtain:

$(\mathbf{4})\Leftrightarrow a({x_1}^{k+1}-{x_2}^{k+1})({x_1}^k-{x_2}^k)<x_1x_2({x_1}^{k-1}-{x_2}^{k-1})({x_1}^k-{x_2}^k).$ **(5)**

W.l.o.g. (inequality is symmetric) one can assume that $x_1>x_2$. Reduce both sides of **(5)** by nonnegative term $({x_1}^k-{x_2}^k)$. Finally we get:

$a({x_1}^{k+1}-{x_2}^{k+1})<x_1x_2({x_1}^{k-1}-{x_2}^{k-1})$ **(6)**

Divide both sides by ${x_1}^{k+1}$. Denote $y=\frac{x_2}{x_1}$. Now we can rewrite **(6)** in the following form:

$$a(1-y^{k+1})<(y-y^k)$$

$\lim_{y\to 1}\frac{a(1-y^{k+1})}{y-y^k}=a\lim_{y\to 1}\frac{1-y^{k+1}}{y-y^k}<\frac{k-1}{k+1}\lim_{y\to 1}\frac{1-y^{k+1}}{y-y^k}=\left[\frac{0}{0}\right]=1$ by L'Hopital's rule.

It follows that there exists x_1 and x_2, such that inequality **(4)** is true. Now, to find the required *n*-tuple, we may take $(x_1,x_2,\dots,x_2)$. Then the required inequality can be represented as **(4)**, which is true. □

Proposition 1.

There exists *x* such that $C_{k,a}(x)>A(x)$, if $\frac{k-1}{k+1}<a<k-1$

Proof.

Similarly to the **theorem 2**.

Remark: Also we put forward a hypothesis: for some values of *a* there exists *n*-tuple *z* such that $C_{k,a}(z)<A(z)$.

Lemma 2 **(generalization of lemma 1)**

There exists *a* ≥ 0 such that $\frac{x_1^l(x_1^k-x_2^k)}{x_1^k+ax_2^k}\geq\frac{k(x_1^l-x_2^l)}{l(a+1)}$

Proof.

Multiply both sides of inequality by multiple of denominators and collect terms. Therefore, $(la+l-k)x_1^{k+l}+kx_1^kx_2^l+kax_2^{k+l}\geq(la+l+ka)x_1^lx_2^k$. From **weighted AM-GM inequality** ($la+l-k>0$ and $a>0$) it is true, if the mentioned conditions hold.

Consider when the conditions implement. If $k>l$, then $a\geq\frac{k}{l}-1$. If $k<l$, then $a\geq 0$. □

Theorem 4.

For some values of *a* inequality $C_{k.l.a}(x)\geq A_l(x)$ holds.

Proof.

$$(a+1)\left(\frac{{x_1}^{k+l}}{{x_1}^k+a{x_2}^k}+\frac{{x_2}^{k+l}}{{x_2}^k+a{x_3}^k}+\cdots+\frac{{x_n}^{k+l}}{{x_n}^k+a{x_1}^k}\right)\geq{x_1}^l+\cdots+{x_n}^l$$

Transpose $A_l(x)$ to the LHS and regroup terms (similarly lemma 1):

$$\frac{x_1^l(x_1^k-x_2^k)}{x_1^k+ax_2^k}+\frac{x_2^l(x_2^k-x_3^k)}{x_2^k+ax_3^k}+\cdots+\frac{x_n^l(x_n^k-x_1^k)}{x_n^k+ax_1^k}\ge 0 \qquad \textbf{(7)}$$

From **lemma 2** (use to every summand) follows that:

$$\frac{x_1^l(x_1^k-x_2^k)}{x_1^k+ax_2^k}+\frac{x_2^l(x_2^k-x_3^k)}{x_2^k+ax_3^k}+\cdots+\frac{x_n^l(x_n^k-x_1^k)}{x_n^k+ax_1^k}\ge\frac{k(x_1^l-x_2^l)}{l(a+1)}+\frac{k(x_2^l-x_3^l)}{l(a+1)}+\cdots+\frac{k(x_n^l-x_1^l)}{l(a+1)}=0$$

Conditions to *a* are the same as in **lemma 2**. □

Problem 1.

Find min *a*, such that $C_{k,l,a}(x)>A^l(x)$ for all *n*-tuples *x*

Solution.

From Power Mean $A_l(x)\ge A^l(x)$. Then, by ***Theorem 4***:

Case 1: if $k\le l$, then inequality holds for $\forall a\in\mathbb{R}_+$.

Case 2: if $k>l$, $a_{min}\le\frac{k}{l}-1$. □

Problem 2

Let's find conditions on *a*, *b* such that the following inequality hold for any positive x_i (*k*, *l* – fixed positive real numbers).

$$\frac{x_1^{k+l}+bx_2^{k+l}}{x_1^k+ax_2^k}+\frac{x_2^{k+l}+bx_3^{k+l}}{x_2^k+ax_3^k}+\cdots+\frac{x_n^{x+l}+bx_1^{k+l}}{x_n^k+ax_1^k}\ge\frac{b+1}{a+1}(x_1^l+x_2^l+\cdots+x_n^l)$$

Lemma 3.

$\frac{x_1^{k+l}+ax_2^{k+l}}{x_1^k+ax_2^k}\ge\frac{x_1^l+ax_2^l}{a+1}$, for all $a\in\mathbb{R}_+$and $k,l\in\mathbb{N}_+$

Proof.

Multiply both sides of inequality by multiple of denominators and collect terms. Obtain:

$$(a+1)x_1^{k+l}+a(a+1)x_2^{k+l}\ge x_1^{k+l}+a^2x_2^{k+l}+ax_1^lx_2^k+ax_1^kx_2^l\Leftrightarrow x_1^{k+l}+x_2^{k+l}\ge x_1^kx_2^l+x_1^lx_2^k$$

From the Muirhead inequality it is true. Hence, $(k+l,0)\succ(k,l)$ and inequality holds. □

Lemma 4.

$\frac{x_2^l(x_2^k-x_1^k)}{x_1^k+ax_2^k}\ge\frac{k(x_2^l-x_1^l)}{l(a+1)}$ with condition: $\begin{cases} if\ k>l, then\ a\le\frac{l}{k-l}\\ if\ k\le l, then\ a\ge 0\end{cases}$

Proof.

Multiply both sides of inequality by multiple of denominators and collect terms. Obtain:

$$(l(a+1)-ka)x_2^{k+l}+kx_1^{k+l}+kax_1^lx_2^k\ge(la+l+k)x_1^kx_2^l$$

From ***weighted AM-GM*** it is true. Hence, $(l(a+1)-ka)(k+l)+k^2a=(la+l+k)l$ and $k(k+l)+kal=(la+l+k)k$. □

Theorem 5.

$$\frac{x_1^{k+l}+bx_2^{k+l}}{x_1^k+ax_2^k}+\frac{x_2^{k+l}+bx_3^{k+l}}{x_2^k+ax_3^k}+\cdots+\frac{x_n^{x+l}+bx_1^{k+l}}{x_n^k+ax_1^k}\ge\frac{b+1}{a+1}(x_1^l+x_2^l+\cdots+x_n^l) \qquad \textbf{(8)}$$

With condition b>a and : $\begin{cases} if\ k>l, then\ a\le\frac{l}{k-l}\\ if\ k\le l, then\ a\ge 0\end{cases}$

Proof.

Regroup the terms. Obtain:

$$\left(\frac{x_1^{k+l}+ax_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{x_n^{x+l}+ax_1^{k+l}}{x_n^k+ax_1^k}\right)+\left(\frac{(b-a)x_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{(b-a)x_1^{k+l}}{x_n^k+ax_1^k}\right)\ge\left(x_1^l+\cdots+x_n^l\right)+\frac{b-a}{a+1}(x_1^l+\cdots+x_n^l)$$

Now assume $A=\frac{x_1^{k+l}+ax_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{x_n^{x+l}+ax_1^{k+l}}{x_n^k+ax_1^k}$, $B=\frac{(b-a)x_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{(b-a)x_1^{k+l}}{x_n^k+ax_1^k}$, $C=x_1^l+\cdots+x_n^l$ and $D=\frac{b-a}{a+1}(x_1^l+\cdots+x_n^l)$. Let us prove that $A\geq C$ and $B\geq D$.

1) $A\geq C$.
   Transpose *C* to the LHS and regroup terms. Then we obtain:
$$A-C=\left(\frac{x_1^{k+l}+ax_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{x_n^{x+l}+ax_1^{k+l}}{x_n^k+ax_1^k}\right)-\left(x_1^l+\cdots+x_n^l\right)=$$
$$=\left(\frac{x_1^{k+l}+ax_2^{k+l}}{x_1^k+ax_2^k}-\frac{x_1^l+ax_2^l}{a+1}\right)+\cdots+\left(\frac{x_n^{x+l}+ax_1^{k+l}}{x_n^k+ax_1^k}-\frac{x_n^l+ax_1^l}{a+1}\right)\geq 0$$
   From **lemma 3** each summand is nonnegative. Then inequality holds.
2) $B\geq D$
   Multiply both sides of inequality by *(a+1)*, divide by nonnegative *(b-a)*, transpose *D* to the LHS and regroup terms. Now, we have:
$$B-D=\left(\frac{(a+1)x_2^{k+l}}{x_1^k+ax_2^k}-x_2^l\right)+\cdots+\left(\frac{(a+1)x_n^{k+l}}{x_{n-1}^k+ax_n^k}-x_n^l\right)=\frac{x_2^l(x_2^k-x_1^k)}{x_1^k+ax_2^k}+\cdots+\frac{x_1^l(x_1^k-x_n^k)}{x_n^k+ax_1^k}$$
   From **lemma 4** we have:
$$\frac{x_2^l(x_2^k-x_1^k)}{x_1^k+ax_2^k}+\cdots+\frac{x_1^l(x_1^k-x_n^k)}{x_n^k+ax_1^k}\geq\frac{k(x_2^l-x_1^l)}{l(a+1)}+\cdots+\frac{k(x_1^l-x_n^l)}{l(a+1)}=0$$

□

Problem 3.
Let's find conditions on *a* such that the following inequality hold for any positive x_i (*k*, *l* – fixed positive real numbers).

$$\frac{ax_1^{k+l}+x_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{ax_n^{k+l}+x_1^{k+l}}{x_n^k+ax_1^k}\geq x_1^l+\cdots+x_n^l$$

Lemma 5.
$\frac{ax_1^{k+l}+x_2^{k+l}}{x_1^k+ax_2^k}\geq\frac{ak-k+al}{al+l}x_1^l-\frac{ak-k-l}{al+l}x_2^l$ if $\begin{cases}a\geq 1+\frac{l}{k}\\ a^2l-ak+k\geq 0\end{cases}$

<u>Proof.</u>
Multiply both sides of inequality by multiple of denominators and collect terms. Obtain:
$$\frac{a^2l-ak+k}{al+l}x_1^{k+l}+\left(\frac{a^2k-ak+a^2l}{al+l}-a+1\right)x_2^{k+l}+\frac{ak-k-l}{al+l}x_1^kx_2^l\geq\frac{a^2l-ak+a^2k}{al+l}x_1^lx_2^k$$
From <u>***weighted AM-GM***</u> it is true. Coefficients of inequality are nonnegative. Therefore $\begin{cases}a\geq 1+\frac{l}{k}\\ a^2l-ak+k\geq 0\end{cases}$
□

Theorem 6.
$\frac{ax_1^{k+l}+x_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{ax_n^{k+l}+x_1^{k+l}}{x_n^k+ax_1^k}\geq x_1^l+\cdots+x_n^l$ if $\begin{cases}a\geq 1+\frac{l}{k}\\ a^2l-ak+k\geq 0\end{cases}$

<u>Proof.</u>
From **lemma 5** we have:
$\frac{ax_1^{k+l}+x_2^{k+l}}{x_1^k+ax_2^k}+\cdots+\frac{ax_n^{k+l}+x_1^{k+l}}{x_n^k+ax_1^k}\geq\frac{ak-k+al}{al+l}x_1^l-\frac{ak-k-l}{al+l}x_2^l+\cdots+\frac{ak-k+al}{al+l}x_n^l-\frac{ak-k-l}{al+l}x_1^l=x_1^l+..+x_n^l$
Inequality holds if: $\begin{cases}a\geq 1+\frac{l}{k}\\ a^2l-ak+k\geq 0\end{cases}$ □