LCME

Problem 9.

Point 1. We proved that the numbers , which satisfies condition of the first point of the problem, are Fibonacci's numbers, beginning from the forth Fibonacci number 3. The proof is based on the induction.

Induction's basis. Let's consider the second and the third Fibonacci numbers– 3и 5. Let's show, that $a_1 ... a_m$ of this numbers equals 1.

$$3 = 1 + 2; \qquad \frac{1}{2} = 0 + \frac{1}{1 + \frac{1}{1}} = [0;1,1]$$

$$5 = 2 + 3; \qquad \frac{2}{3} = 0 + \frac{1}{\frac{3}{2}} = 0 + \frac{1}{1 + \frac{1}{2}} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = [0;1,1,1]$$

Induction's step. Let's prove the accuracy of the (n+1)th step if we have accuracy of all previous steps. We have a set $[0;\underbrace{1,1 \dots 1}_{n}]$, which defines Fibonacci number F_{n+2}. At the same time $F_{n+2}=F_{n+1}+F_n$, where F_{n+1} and F_n – Fibonacci numbers and we have the following equality $\frac{F_n}{F_{n+1}} = [0;\underbrace{1,1 \dots 1}_{n}]$. Let's form a set $[0;\underbrace{11 \dots 1}_{n+1}]$ and prove that it defines Fibonacci number F_{n+3}.

$$\left[0;\underbrace{1,1 \dots 1}_{n+1}\right] = \frac{1}{1 + \frac{F_n}{F_{n+1}}} = \frac{F_{n+1}}{F_{n+1} + F_n} = \frac{F_{n+1}}{F_{n+2}}$$

Finally $F_{n+1}+F_{n+2} = F_{n+3}$. That's what we had to prove.

Some facts for points 2-5.

Further we show some general facts, which can be applied to the different points of the problem.

Let's consider any set $[a_0;a_1,a_2,\dots,a_m]$. Let's count this fraction with the beginning in a_m and equate the result to some $\frac{x_i}{y_i}$, as in example

$$\frac{x_{m-1}}{y_{m-1}} = \frac{1}{a_{m-1} + \frac{1}{a_m}}, \qquad \frac{x_{m-2}}{y_{m-2}} = \frac{1}{a_{m-2} + \frac{1}{a_{m-1} + \frac{1}{a_m}}}, \dots$$

On some step we obtain the fraction $\frac{x_i}{y_i}$. Let's organize fraction $\frac{x_{i-1}}{y_{i-1}}$:

$$\frac{x_{i-1}}{y_{i-1}} = \frac{1}{a_{i-1} + \frac{x_i}{y_i}} = \frac{y_i}{a_{i-1} * y_i + x_i}.$$

And we shouldn't forget about the coefficient, which can be organized as the result of cancel factors of the $\frac{x_i}{y_i}$.

Thus,

$$\frac{x_{i-1}}{y_{i-1}} = t_{i-1} * \frac{y_i}{a_{i-1} * y_i + x_i}, \quad where \ 1 \leq t \leq \infty.$$

Adding. At first, if at least one of a_i will be more then k (only if it isn't the last a_m), then the "full number", which is exspressed by the set $[a_0;a_1,a_2,\ldots,a_m]$, won't be k-good number.

Secondly, we can always select such t_i, that the "full number" wouldn't be k-good number. For example, $\frac{1}{2+\frac{1}{3+\frac{1}{2}}} = \frac{7}{16}$, $7 + 16 = 23$, and 23 – the first not-2-good number.