LCME, Russia, Saint-Petersburg. Problem 8.

Point 1. Let P(x, y)=$x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that P(x, y)>0 for all x, y>0 if and only if a>-1

Proof. Let's equate P(x, y)=(x+y)($x^2 + (a-1)xy + y^2$). x+y>0, so positivity of P(x, y) is equal to positivity of $x^2 + (a-1)xy + y^2$=xy($\frac{x}{y}$+a-1+$\frac{y}{x}$) (it's correct because x, y≠0). xy>0, $\frac{x}{y}+\frac{y}{x}$≥2(according to Cauchy's inequality), so positivity of original polynomial is equal to inequality a>-1.

Point 2. Let P(x, y)=$x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4. Prove that P(x, y)>0 (a) for all x, y>0 if and only if b>$\frac{a^2+8}{4}$ or a≥-4, b>-2a-2

Proof. P(x, y)=$x^2y^2((\frac{x}{y})^2 + a\frac{x}{y} + b + a\frac{y}{x} + (\frac{y}{x})^2)$,(it's correct because x, y≠0), so P(x, y)>0 for all x, y>0 if and only if Q(x, y)=$(\frac{x}{y})^2 + a\frac{x}{y} + b + a\frac{y}{x} + (\frac{y}{x})^2$>0 . Let t=$\frac{x}{y} + \frac{y}{x}$. So Q(x, y)=Q(t)=$t^2 + at + b - 2$. It's a quadratic trinomial with definitional domain [2, +∞). So it's positive if and only if D<0 or f(2)>0, f'(2)≥0. It means that b>$\frac{a^2+8}{4}$ or a≥-4, b>-2a-2.

(b) for all x, y≠0 if and only if b>$\frac{a^2+8}{4}$ or |a|≥-4, b>2|a|-2

1)P(x,y)=P(-x, -y), so if x, y<0, it has the same conditions.

2) P(x, -y)=(-x, y), so let x<0, y>0. So by the same transformations with t=-($\frac{-x}{y} + \frac{y}{-x}$)≤-2, so P(x,y)>0 for x<0, y>0 is equal to D<0 or f(-2)>0, f(-2)≤0(It means that b>$\frac{a^2+8}{4}$ or a≥4, b>2a-2).

To sum up, for P(x, y)>0 for x, y≠0 if and only if b>$\frac{a^2+8}{4}$ or |a|≥4, b>2|a|-2

Remark 1. For all symmetric polynomial of odd degree(2n+1), for x, y>0 there is a proper polynomial of even degree(2n) such, that positivity of it is equal to positivity of an original polynomial.

It is because polynomial of odd degree is divisible by x+y and x+y>0. Proper polynomial is a result of dividing original polynomial by x+y.

Remark 2. For all symmetric polynomial of odd degree for x, y<0 positivity is impossible.

It's because x+y<0, so positivity of original polynomial is equal to negativity of polynomial with even degree, but its leading coefficient is more than 0, so it's more than 0 on the open ray from the biggest root to +∞ for example.

Point 3. 1)Let P(x, y)= $x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5$ be a symmetric polynomial of degree 5. P(x, y)>0 (a)for x, y>0 if and only if b>$\frac{a^2+2a+5}{4}$ or a≥5, b>3a-1

(b) for x,y: x+y>0 if and only if b>$\frac{a^2+2a+5}{4}$ or a≥5, b>3a-1

Proof. According to remark 1, for x+y>0 we can divide P(x, y) by x+y and get Q(x, y)=$x^4 + (a-1)x^3y + (b-a+1)x^2y^2 + axy^3 + y^4$. Positivity of P(x,y) is equal to positivity of Q(x, y).

But according to point 2 we can say, that P(x, y)>0 for all x, y>0 if and only if b>$\frac{a^2+2a+5}{4}$ or a≥5, b>-a-1. If x<0, y>0, b>$\frac{a^2+2a+5}{4}$ or a≥5, b>3a-1. So positivity for all x,y: x+y>0 is equal to b>$\frac{a^2+2a+5}{4}$ or a≥5, b>3a-1.

2)Let P(x, y)= $x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6$ be a symmetric polynomial of degree 6. P(x, y)>0 for (a)x, y>0 if and only if D>0, 2a+2b+c+2>0 or D≤0, 4a+b+9>0, where D=$-4a^3(c-2a) + a^2(b-3)^2 - 4(b-3)^3 + 18a(b-3)(c-2a) - 27(c-2a)^2$

Proof. P(x, y)=$x^3y^3(\frac{x^3}{y^3} + a\frac{x^2}{y^2} + b\frac{x}{y} + c + b\frac{y}{x} + a\frac{y^2}{x^2} + \frac{y^3}{x^3})$.(it's correct because x, y≠0). Let t=$\frac{x}{y} + \frac{y}{x}$. So, positivity of P(x, y) is equal to positivity of Q(t)=$t^3 + at^2 + (b-3)t + c - 2a$. It is cubic polynomial with definition domain [2, +∞). So its positivity is equal to D>0, Q(2)>0 or D≤0, Q'(2)>0, where D=$-4a^3(c-2a) + a^2(b-3)^2 - 4(b-3)^3 + 18a(b-3)(c-2a) - 27(c-2a)^2$, Q'(t)=3$t^2$+2at+b-3. It means D>0, 2a+2b+c+2>0 or D≤0, 4a+b+9>0

(b) P(x, y)>0 for x, y≠0 if and only if D>0 2a+2|b+1|+c>0 or D≤0 4|a|+b+9>0

Proof. For x, y<0 the proof is similarly.

For x>0, y<0(and for x<0, y>0) condition will be changed on D>0 2a-2b+c-2>0 or D≤0 -4a+b+9>0, because the definition domain will be (-∞,-2].(t=-($\frac{x}{-y} + \frac{-y}{x}$))

To sum up this results, P(x, y)>0 for x, y≠0 if and only if D>0 2a+2|b+1|+c>0 or D≤0 4|a|+b+9>0

3)Let P(x, y)= $x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^3 + bx^2y^5 + axy^6 + y^7$ be a symmetric polynomial of degree 7. P(x, y)>0 for all x, y>0 if and only if D'>0, 2a'+2b'+c'+2>0 or D≤0, 4a'+b'+9>0, where D'=$-4a'^3(c'-2a') + a'^2(b'-3)^2 - 4(b'-3)^3 + 18a'(b'-3)(c'-2a') - 27(c'-2a')^2$, a', b', c' are the coefficients from proper symmetric polynomial of even degree, a'=a-1, b'=b-a+1, c'=c-b+a-1. It's equal to D'>0, a+b+c+1>0 or D'≤0 3a+b+6>0 (a)

(b) For x, y<0 according to remark positivity is impossible.

For x>0, y<0(or x<0, y>0) (x+y>0)conditions will be: D'>0 5a-3b+c-7>0 or D≤0, -5a+b+14>0. (t=-($\frac{x}{-y} + \frac{-y}{x}$) where D'=$-4a'^3(c'-2a') + a'^2(b'-3)^2 - 4(b'-3)^3 + 18a'(b'-3)(c'-2a') - 27(c'-2a')^2$, a', b', c' are the coefficients from proper symmetric polynomial of even degree, a'=a-1, b'=b-a+1, c'=c-b+a-1

To sum up, P(x, y)>0 for all x,y: x+y>0 if and only if D'>0, max(a+b+c+1,5a-3b+c-7) or D≤0 max(3a+b+6, -5a+b+14), where D'=$-4a'^3(c'-2a') + a'^2(b'-3)^2 - 4(b'-3)^3 + 18a'(b'-3)(c'-2a') - 27(c'-2a')^2$, a', b', c' are the coefficients from proper symmetric polynomial of even degree, a'=a-1, b'=b-a+1, c'=c-b+a-1.

Point 4. For n>7 for x, y>0 we should search for conditions of not existing of roots, or their being less than 2. For symmetric polynomials of even degree for x, y<0 conditions will be similarly, of odd degree-it is impossible.

Some methods are expressed in remark 1.

For n=8, 9 all the computing can be done by elementary methods.