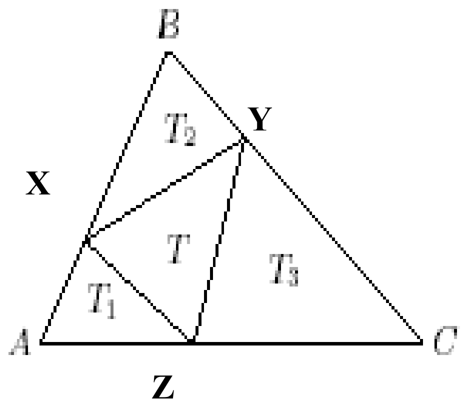
LCME, Russia, Saint-Petersburg, Problem 4.

Let's mark X,Z,Y edges of triangle T on the triangle ABC.|AB|=q,|BC|=v,|AC|=t-lengths of sides of triangle ABC.|AX|=x,|BY|=y,|CZ|=z=>|BX|=q-x,|CY|=v-y,|AZ|=t-z



We can count up area of triangle ABC and triangles, being included into it :

area(ABC)= $(1/2)*\sin(A)*q*t$ = $(1/2)*\sin(B)*q*v$ =
= $(1/2)*\sin(C)*t*v$

area(AXZ)= $(1/2)*\sin(A)*x*(t-z)$ = $area(ABC)*x*(t-z)/(q*t)$ =
$area(ABC)*a*(1-c)$

area(BXY)= $(1/2)*\sin(B)*y*(q-x)$ = $area(ABC)*y*(q-x)/(q*v)$ =
$area(ABC)*b*(1-a)$

area(CYZ)= $(1/2)*\sin(C)*z*(v-y)$ = $area(ABC)*z*(v-y)/(t*v)$ =
$area(ABC)*c*(1-b)$

area(XYZ)=area(ABC)-area(CYZ)-area(BXY)-area(AXZ)=
$area(ABC)*(1-(a+b+c)+(ab+ac+bc))$

where a,b,c in (0,1) $a=x/q$, $b=y/v$, $c=z/t$

1.

(a)Let's prove, that $area(T)\geq min\, area(T_1), area(T_2), area(T_1)$ .Let it be false. It equals that area(T)<area(T_1),area(T)<area(T_2),area(T)<area(T_3) at the same time. Let's write it as a formula.

$area(ABC)*(1-(a+b+c)+(ab+ac+bc))$ < $area(ABC)*a*(1-c)$

$area(ABC)*(1-(a+b+c)+(ab+ac+bc))$ < $area(ABC)*b*(1-a)$

$area(ABC)*(1-(a+b+c)+(ab+ac+bc))$ < $area(ABC)*c*(1-b)$

Let's sum these inequality's and abbreviate on area(ABC).We'll get a inequality:

$3*(1-(a+b+c)+(ab+ac+bc))<(a+b+c)-(ab+ac+bc)$, which equals

$3/4<(a+b+c)-(ab+ac+bc)$ .It will be true if $F(a)=(a+b+c)-(ab+ac+bc)-3/4$ would be positive on (0,1).F'(a)=1-(b+c):

1)b+c=g>1 F'(a)<0. sup F(a)= $\lim F(a)=b+c-b*c-3/4$,where a goes to 0. Let's prove, that sup F(a)<0, i.e. for every b,c in (0,1) that b+c>1, b+c -b*c-3|4<0
b+c -b*c-3/4=g-b(g-b)-3/4=b^2-b*g+g-3/4. D=g^2-4(g-3/4)<0 as 1<g<2. Consequently sup F(a)>0 and F(a)>0 for every a,b,c in (0,1). It means that $area(T)\geq min\, area(T_1), area(T_2), area(T_1)$ is true. Area(T) can be equal min{area(T_1), area(T_2), area(T_3)}. For example, there is a triangle ABC with edges:q=v=t;a=b=c=0.5.

Inequality $area(T)\leq max\, area(T_1), area(T_2), area(T_1)$ isn't always true. For example, there is a triangle ABC with equals edges q=v=t. Let's take sequence of a,b,c:a_n=b_n=c_n=1/n. We will get sequence of area(T)= $\sqrt{3/4}*a^2-(1/2\mathrm{n})*a^2$ .For n goes to endlessness, area(T) goes to area(ABC) and min{area(T_1), area(T_2), area(T_3)} goes to 0.So there is N, that for every n>N $\sqrt{3/4}*a^2-(1/2\mathrm{n})*a^2$ > $(1/2\mathrm{n})*a^2$ will be true.

2) $b+c=g\leq 1$ is similar.

(b) $height(T)\geq min(height(T_1), height(T_2), height(T_3))$

$area(T)\geq min\, area(T_1), area(T_2), area(T_1)$ is true. It means that there is a triangle T' from T_1,T_2,T_3,for which $area(T)\geq area(T')$ .But T and T' have one general side r. So $(1/2)*r*h(T)\geq(1/2)*r*h(T')$ is true=> $h(T)\geq h(T')$ where h(T) -height of T,h(T')-height of T to side r.