LCME, Russia, Saint-Petersburg, problem 2.

1. f(x) can be constant f(x)=0

(a) Let f: $\mathbb{R} \rightarrow \mathbb{R}$ and f(x)-polynomial with the property that $f(f(x)+k*x)=x*f(x)$ for all real numbers x. Let's look 3 cases.

1) $deg(f(x))>1$

$$deg(f(f(x)+k*x))=deg(f(x))*max(deg(f(x)),1)=(deg(f(x)))^2>deg(f(x))+1=deg(x*f(x))$$

2) $deg(f(x))=1$

- higher coefficient is -k. So f(x)=-kx+g(x),where deg(g(x))<1

$$deg(f(f(x)+k*x))=deg(f(x))*deg(g(x))<deg(f(x))+1=deg(x*f(x))$$

- higher coefficient isn't -k.

$$deg(f(f(x)+k*x))=deg(f(x))*max(deg(f(x)),1)=1<deg(f(x))+1=deg(x*f(x))$$

3) $1>deg(f(x))$

$$deg(f(f(x)+k*x))=deg(f(x))*max(deg(f(x)),1)=deg(f(x))*1<deg(f(x))+1=deg(x*f(x))$$

So $f(f(x)+k*x)=x*f(x)$ wouldn't be always true for all real numbers x, where f(x)-polynomial,becouse of degrees of right and left parts of equality are different.

2. special case n=2 in part 3.

3. f(x) can be constant f(x)=0.

(a) f(x) cann't be polynomial.

Let f: $\mathbb{R} \rightarrow \mathbb{R}$ and f(x)-polynomial with the property that

$$f(f(x_1)+...f(x_n)+k*x_1*...*x_n)=x_1*f(x_2)+x_2*f(x_3)+...+x_n*f(x_1)$$

for all real numbers x. Let's look 4 cases.

1) $deg(f(x))>1$

$$deg(f(f(x_1)+...f(x_n)+k*x_1*...*x_n))=deg(f(x))*max(deg(f(x)),n)>deg(f(x))+1$$
$$deg(f(x))+1=deg(x_1*f(x_2)+x_2*f(x_3)+...+x_n*f(x_1))$$

2) $deg(f(x))=1$

- higher coefficient is -k. So f(x)=-kx+g(x),where deg(g(x))<1

$$deg(f(x))+1=deg(f(f(x_1)+...f(x_n)+k*x_1*...*x_n))=n>deg(f(x))+1$$

$$deg(x_1*f(x_2)+x_2*f(x_3)+...+x_n*f(x_1))$$

- higher coefficient isn't -k.

$$deg(f(f(x)+k*x))=deg(f(x))*max(deg(f(x)),1)=1<deg(f(x))+1=deg(x*f(x))$$

3) $1>deg(f(x))$

$$deg(f(f(x_1)+...f(x_n)+k*x_1*...*x_n))=deg(f(x))*n>deg(f(x))+1$$
$$deg(f(x))+1=x_1*f(x_2)+x_2*f(x_3)+...+x_n*f(x_1)$$

4) f(x)-constant(deg(f(x))=0). It's easy to understand that f(x) can be only constant 0.

So $f(f(x_1)+...f(x_n)+k*x_1*...*x_n)=x_1*f(x_2)+x_2*f(x_3)+...+x_n*f(x_1)$ wouldn't be always true for all real numbers x,where f(x)-polynomial,becouse of degrees of right and left parts of equality are different.