LCME, Russia, Saint-Petersburg, Problem 10 “Centro-symmetric shadows”

Definitions:

$S = \{s_i\}$; -A set of points.

S_i; - a point of a set S with an index i

$S', S'', S^{(n)}, S^{(a)}$ – Shadow of the set S on the first line, on the second, on the n’th , on the line a.

$s_i', s_i'', s_i^{(n)}, s_i^{(a)}$ – Shadow of the point with the index i of the set s on the first line, on the second, on the n’th lin, on the line a

$O, O', O^{(n)}, O^{(a)}$ – the center of symmetry of the initial set, of the shadow on the first line, on the n’th line, on the line a

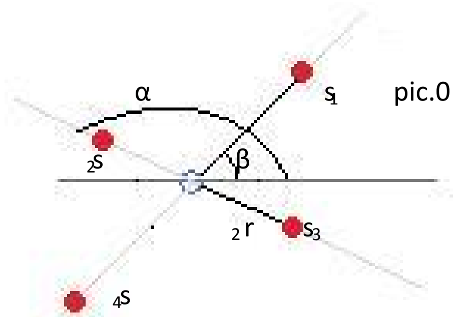
Remark 1.

If a set is centro-symmetric, then every it’s shadow is centro-symmetric too.

And Shadow’s center of symmetry will be shadow of the center of symmetry of initial set. It’s Obviously because of the fact, that orthogonal projection transfers equal distances to equal distances.

Remark 2.

Centro-symmetric set consists of pairs (or three,if one of them lies on it’s center of symmetry) of points, located on the line, passing through the center of symmetry and every pair of points may be defined by the radius (distance between any point of in pair and center of symmetry) and the angle between horizontal line, passing through the center of symmetry and the line, which the pair of points lies on.(pic.0)



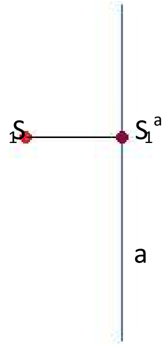
pic.0

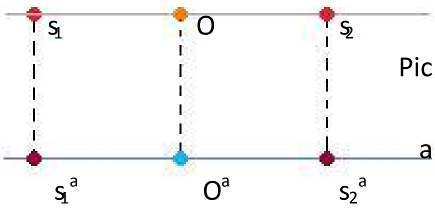
1. Condition, which is put on k(n) «For any set S of any points in the plane, if there exist k lines, no two parallel, such that for each line the shadow of S on this line is centro-symmetric, then the initial set S is also centro-symmetric » may be equally rewrote this way: «for any NON centro-symmetric set of n points, for any k lines, there exist at least one line, with a non centro-symmetric shadow of the set on it» or “For any non centro-symmetric set of n points there couldn’t be k lines with centro-symmetric shadows on them”, because, if it was another way, then there exists non centro symmetric set of n points and there exist k lines, such that shadows of that set on them are centro-symmetric, but by the definition of k(n) that would mean that the initial set is centro-symmetric, but it’s not centro-symmetric.

Firstly, We'll consider elementary cases: for sets of 1 and of two points.

1. A set of one point.(pic.1)
   $S=\{s_1\}$; S' – shadow on the first line.
   Such set is always centro-symmetric.
   It's only point is symmetric to itself with respect to itself.
   So It is the entire center of symmetry.

pic.1



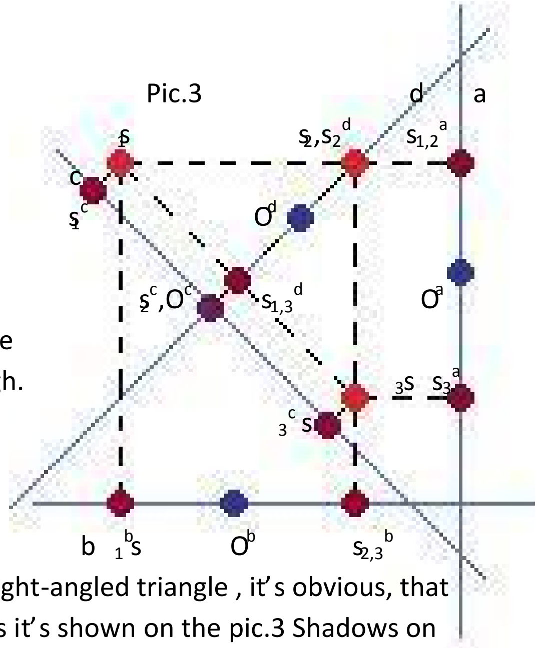


Pic.2

2. A set of two points.(pic 2.2)
   Points of this set always lies on one line, connecting them and are equidistantfrom the middle of the segment, which has those points as it's ends.
   That way, this set is always centro-symmetric with a respect to the middle of the segment between them.

3. A set of three points. (pic.3)
   obviously, that the only centro-symmetric set of three points is a set of three point, lying on one line, with two of them, being equidistant from the third. It seems, that in this case one line is enough, if it is parallel to the line, which contains all the set, however, there exists an example when one, two, three and even 4 lines are not enough.

Pic.3



   In this case we can make a set of edges of isosceles right-angled triangle , it's obvious, that this set is not centro-symmetric, then draw 4 lines, as it's shown on the pic.3 Shadows on them are centro-symmetric. But any other line(fifth) either would be parallel to one of the first four, or Shadow on it would be not centro-symmetric. Thus k(3)=5.
4. So, if there are some points of the set S, which lies on the rays of the right angle, so then we need 2 more lines to prove that the set is centro-symmetric, for any right angle, if there are no of them, then we just need

$$k(n) = \begin{cases} \frac{n}{2} + 2t, n = 2m, t - number\ of\ possible\ right\ angles \\ \frac{n-1}{2} + 2t, n = 2m + 1, t - number\ of\ possible\ right\ angles \end{cases}$$