ITYM

8 : Positivity of Symmetric Polynomials

Introduction:

Theorem : Every symmetric polynomial with n variables, can be expressed as a polynomial with the n fondamentals symetrical fonctions, so when n=2 as a polynomial with the quantites $s=(x+y)$ and $p=xy$.

1) For every real x and y,

$$P(x,y)=x^3+ax^2y+axy^2+y^3=(x+y)\times(x^2+(a-1)\times xy+y^2)$$

As $x+y>0$ when $x>0$ and $y>0$, $P(x,y)>0 \Leftrightarrow x^2+bxy+y^2>0$ where $b=a-1\in\mathbb{R}$

But $x^2+bxy+y^2>0 \Leftrightarrow \frac{x^2+bxy+y^2}{xy}>0 \Leftrightarrow \frac{x}{y}+\frac{y}{x}+b>0$ because $xy>0$, so substituting $Z=\frac{x}{y}+\frac{y}{x}$ ($x>0$ and $y>0 \Rightarrow Z\geq 2$)then for all $(x,y)\in\mathbb{R}^+-\{0\}\times\mathbb{R}^+-\{0\}$:

$$P(x,y)>0 \Leftrightarrow Z+b>0 \Leftrightarrow b>-2 \Leftrightarrow a>-1$$

so $P(x,y)>0 \Leftrightarrow a>-1$

2) (a) For all $(x,y) \in \mathbb{R}-\{0\}\times\mathbb{R}-\{0\}$

$$P(x,y)>0 \Leftrightarrow x^4+ax^3y+bx^2y^2+axy^3+y^4>0$$
$$\Leftrightarrow \frac{x^4+ax^3y+bx^2y^2+axy^3+y^4}{x^2y^2}>0$$
$$\Leftrightarrow \frac{x^2}{y^2}+\frac{y^2}{x^2}+a\left(\frac{x}{y}+\frac{y}{x}\right)+b>0$$

Substituting $Z=\frac{x}{y}+\frac{y}{x}$ ($x>0 \wedge y>0 \Rightarrow Z\geq 2$), we have also $Z^2=\frac{x^2}{y^2}+\frac{y^2}{x^2}+2$. Then, for all $(x,y), x>0, y>0$ $P(x,y)>0 \Leftrightarrow Z^2+aZ+b-2>0$ where $Z\geq 2$ (and $Z\in\mathbb{R}$), so

$P(x,y)=Q(Z)$ for some polynomial $Q(Z)$ of degree 2.

$\Delta=a^2-4b+8$

If $\Delta<0$ then $Q(Z)$ has no real roots, so $Q(z)>0$

Otherwize $\Delta\geq 0$ $Q(z)\leq 0$ between the 2 roots,

We get : if $\Delta<0 \Leftrightarrow a^2-4b+8<0 \Leftrightarrow b>\frac{a^2+8}{4}$ then $P(x,y)=Q(Z)>0$ for all positive x and y.

Otherwise, $\Delta \geq 0$ we must have $\frac{-a+\sqrt{(a^2-4b+8)}}{2}<2$ so that no roots can be reached with x > 0 and $y>0$, for $P(x,y)=Q(Z)$ and Z will be greater or equal than 2.

Therefore:

$$\frac{-a+\sqrt{(a^2-4b+8)}}{2}<2 \qquad \text{(where } a^2-4b+8\geq 0 \text{)}$$
$$\Leftrightarrow \sqrt{(a^2-4b+8)}<4+a$$

$$\Leftrightarrow 16+8a>-4b+8 \quad \text{and} \quad a\geq -4$$
$$\Leftrightarrow b>-2a-2 \quad \text{and} \quad a\geq -4$$

From $\frac{a^2+8}{4}\geq -2a-2$ as $\frac{a^2+8}{4}\geq -2a-2 \Leftrightarrow a^2+8+8+8a>0 \Leftrightarrow (a+4)^2\geq 0$, for every $a\in\mathbb{R}$

we get $P(x,y)>0$ for every real $x>0, y>0 \Leftrightarrow \left\{ \begin{array}{lll} a<-4 & \text{and} & b>\frac{a^2+8}{4} \quad \text{or} \\ (a\geq -4 & \text{and}) & b>-2a-2 \end{array} \right\}$

2) (b)

As $P(x,y)=P(-x,-y), P(-x,y)=P(x,-y)$ we just have to study the case $x>0$

so if $y>0$ we have the condition $b>\frac{a^2+8}{4}$ *if* $a<-4$ else $b>-2a-2$

and if $y<0$ we can get the condition, (by the same way) (Substituting Y = - y and A= - a)

$$b>\frac{A^2+8}{4} \text{ if } A<-4 \text{ else } b>-2A-2 \qquad \Leftrightarrow b>\frac{a^2+8}{4} \text{ if } a>4 \text{ else } b>2a-2$$

so $P(x,y)>0$ *for all* $x\neq 0$ $y\neq 0 \Leftrightarrow b>\frac{a^2+8}{4}$ *if* $|a|>4$ else $b\geq 2|a|-2$

3) (a) (n=5)

$$P(x,y)=x^5+ax^4y+bx^3y^2+bx^2y^3+axy^4+y^5$$
$$\Leftrightarrow P(x,y)=(x+y)(x^4+(a-1)x^3y+(b-a+1)x^2y^2+(a-1)xy^3+y^4)$$

So $P(x,y)>0$ *for all real* $x>0$ $y>0 \Leftrightarrow x^4+Ax^3y+Bx^2y^2+Axy^3+y^4>0$

where $A=a-1$ and $B=b-a+1$

So

$P(x,y)>0$ *for every real* $x>0, y>0 \Leftrightarrow B>\frac{A^2+8}{4}$ *if* $A<-4$ else $B>-2A-2$

$\Leftrightarrow \{b-a+1>\frac{a^2-2a+9}{4} \Leftrightarrow b>\frac{a^2+2a+5}{4}\ if\ a<-3\}\ else\ \{b-a+1>-2a \Leftrightarrow b>-a-1\}$

(b) (n=5) There is no way for P(x,y) to take only positives values on $\mathbb{R}-\{0\}\times\mathbb{R}-\{0\}$ because $P(x,y)=-P(-x,-y)$

(a) (n=6) $P(x,y)=x^6+ax^5y+bx^4y^2+cx^3y^3+bx^2y^4+axy^5+y^6$

$P(x,y)>0 \Leftrightarrow \frac{x^3}{y^3}+\frac{y^3}{x^3}+a(\frac{x^2}{y^2}+\frac{y^2}{x^2})+b(\frac{x}{y}+\frac{y}{x})+c>0$ (dividing by x^3y^3 (> 0))

Substituting $Z=\frac{x}{y}+\frac{y}{x}$ we get $P(x,y)>0 \Leftrightarrow Z^3+aZ^2+(b-2)Z+c-2a>0$
Let F be the polynomial $F(Z)=Z^3+aZ^2+(b-2)Z+c-2a$ and his discriminant

$$\Delta=(a^2-4)(b-3)^2+(18a(b-3)-27(c-a)-4a^3)(c-2a) \quad .$$

If $\Delta<0$ the polynomial has only one real root, so $P(x,y)>0 \Leftrightarrow F(2)>0$

else $\Delta\geq 0$ the polynomial has 3 real roots , so $P(x,y)>0 \Leftrightarrow F(2)>0$ and $X2<2$ where X2

is the biggest root of F'(x) ($F'(x)=3Z^2+2aZ+b-2$ and $\Delta\geq 0$ ==> $\Delta'\geq 0$) (Since F has 3 roots and we want all of them to be smaller than 2, and X2 is between the two biggest roots, this is seen to be the good condition upon F)

$$X2=\frac{-2a+\sqrt{(4a^2-12b+24)}}{6}=\frac{-a+\sqrt{(a^2-3b+6)}}{3}$$

From $\frac{-a+\sqrt{(a^2-3b+6)}}{3}<2 \Leftrightarrow \sqrt{(a^2-3b+6)}<6+a$ where $a^2-3b+6\geq 0$

$\Leftrightarrow -3b+6<12a+36\ if\ a>-6\ \ (else\ (a,b)\in\emptyset)$

$\Leftrightarrow b>-4a-10\ \ if\ a>-6$

and from $F(2)>0 \Leftrightarrow 8+4a+2b-4+c-2a>0 \Leftrightarrow 4+2a+2b+c>0$

We get :

if $\Delta<0$ $P(x,y)>0$ $\Leftrightarrow c>-2a-2b-4$

else $P(x,y)>0$ $\Leftrightarrow a>-6$ and $b>-4a-10$ and $c>-2a-2b-4$

(b) (n=6) As $P(x,y)=P(-x,-y), P(-x,y)=P(x,-y)$ so we just have to study the case $x>0$

so if $y<0$ substituting $Y=-y$ and $A=-a$ and $C=-c$ we get

$$P(x,y)=x^6+Ax^5Y+bx^4Y^2+Cx^3Y^3+bx^2Y^4+AxY^5+Y^6$$

We can observe that the discriminant remains the same

$$\Delta' = (A^2-4)(b-3)^2+(18\mathrm{A}(b-3)-27(C-A)-4A^3)(C-2\mathrm{A})$$
$$\Leftrightarrow \Delta' = (a^2-4)(b-3)+(-18\mathrm{a}(b-3)+27(c-a)+4a^3)(2\mathrm{a}-c) = \Delta$$

So

if $\Delta<0$ then $P(x,y)>0 \quad \Leftrightarrow c>-2\mathrm{a}-2\mathrm{b}-4$ and $C>-2\mathrm{A}-2\mathrm{b}-4 \Leftrightarrow c<-2\mathrm{a}+2\mathrm{b}+4$
so $P(x,y)>0 \quad \Leftrightarrow |c+2\mathrm{a}|<2\mathrm{b}+4$

else $P(x,y)>0 \quad \Leftrightarrow |a|<6$ and $b>|4\mathrm{a}|-10$ and $|c+2\mathrm{a}|<2\mathrm{b}+4$

(a) (n=7)

$$P(x,y)=x^7+ax^6y+bx^5y^2+cx^4y^3+cx^3y^4+bx^2y^5+axy^6+y^7$$
$$\Leftrightarrow P(x,y)=(x+y)(x^6+(a-1)(x^5y+xy^5)+(b-a+1)(x^4y^2+y^4x^2)+(c-b+a-1)(x^3y^3)+y^6)$$

Substituting $A=a-1$; $B=b-a+1$ and $C=c-b+a-1$ we get the discriminant

$$\Delta=(A^2-4)(B-3)^2+(18A(B-3)-27(C-A)-4A^3)(C-2\mathrm{A})$$
$$\Leftrightarrow \Delta=(a^2-2\mathrm{a}-3)(b-a-2)^2+(18(a-1)(b-a-2)-27(c-b)-4(a-1)^3)(c-b-a+1)$$

if $\Delta<0 \quad P(x,y)>0$

$$\Leftrightarrow C>-2A-2B-4 \Leftrightarrow c-b+a-1>-2a+2-2b+2a-2-4 \Leftrightarrow c>-b-a-3$$

else $P(x,y)>0 \quad \Leftrightarrow A>-6$ and $B>-4\mathrm{A}-10$ and $C>-2A-2B-4$

$\Leftrightarrow a>-5$ and $b>-3a-7$ and $c>-b-a-3$

(b) (n=7) There is no way for P(x,y) to take only positives values on $\mathbb{R}-\{0\}\times\mathbb{R}-\{0\}$

because $P(x,y)=-P(-x,-y)$

4) (a) $x>0$, $y>0$

Let P_n be the homogenous polynomial :

$$P_n(x,y)=\sum_{i=0}^{\left[\frac{n}{2}\right]} a_i(x^{n-i}y^i+x^i y^{n-i})$$ where each $a_i \in \mathbb{R}$ and $a_0=1$

(if n is heaven, $a_{\left[\frac{n}{2}\right]}$ is divided by 2 in comparaison with question 1,2,3)

4a.1 There is two trivial conditions, we can see first : $P_n(x,y)>0$

- if each $ai\geq 0$ (except $i=0$)
- if $a_i > \binom{n}{i}\times(-1)^i$ for every heaven (and $a_{\left[\frac{n}{2}\right]} > \dfrac{\binom{n}{\frac{n}{2}}\times(-1)^{(\frac{n}{2})}}{2}$) , then

  $P_n(x,y)>(x-y)^n$ (because $x>0$ $y>0$)
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We can also denote that for any odd, ($n>7$) the questions depends on the conditions on P_{n-1} .

Because if n is odd then :

$$P_n(x,y)=\sum_{i=0}^{\frac{n-1}{2}} a_i(x^{n-i}y^i+x^i y^{n-i})=(x+y)\times\sum_{i=0}^{\frac{n-2}{2}}\left(\sum_{j=0}^{i} a_j\times(-1)^{i-j}\right)\times(x^{n-i}y^i+x^i y^{n-i})$$

So $P_n(x,y)=(x+y)\times P_{n-1}$ where the coeffiscient of P_{n-1} are such $Ai=\sum_{j=0}^{i} a_j\times(-1)^{i-j}$

4a.2 As we made before, we can also say for any heaven $n>7$

$$P_n(x,y)=\sum_{i=0}^{\left[\frac{n}{2}\right]} a_i(x^{n-i}y^i+x^i y^{n-i})=\frac{1}{x^{\frac{n}{2}}y^{\frac{n}{2}}}\times\sum_{i=0}^{\left[\frac{n}{2}\right]} ai\left(\left(\frac{x}{y}\right)^{\frac{n}{2}-i}+\left(\frac{y}{x}\right)^{\frac{n}{2}-i}\right)$$

So $$P_n(x,y)>0 \Leftrightarrow \sum_{i=0}^{\left[\frac{n}{2}\right]} ai\left(\left(\frac{x}{y}\right)^{\frac{n}{2}-i}+\left(\frac{y}{x}\right)^{\frac{n}{2}-i}\right)>0$$

Substituting $Z=\frac{x}{y}+\frac{y}{x}$ As $Z^n=\sum_{i=0}^{n}\left(\frac{x}{y}\right)^i\times\left(\frac{y}{x}\right)^n\times\binom{n}{i}=\sum_{i=0}^{\frac{n}{2}}\binom{n}{i}\times\left(\left(\frac{x}{y}\right)^{n-2i}+\left(\frac{y}{x}\right)^{n-2i}\right)$

We can find a polynomial $Q(Z)=\sum_{i=0}^{\frac{n}{2}} b_i Z^i$ such that $Q(Z)>0 \Leftrightarrow P(x,y)>0$ for $Z\geq 2$

(b) First of all, as we have seen before, for every odd integer n,

$$P_n(-x,-y)=\sum_{i=0}^{\left[\frac{n}{2}\right]} a_i((-x)^{n-i}(-y)^i+(-x)^i(-y)^{n-i})=-\sum_{i=0}^{\left[\frac{n}{2}\right]} a_i(x^{n-i}y^i+x^i y^{n-i})=-P_n(x,y)$$

so there isn't any way for P_n to take only positives valueson $\mathbb{R}-\{0\}\times\mathbb{R}-\{0\}$. So the question remains for every heaven integer.

4b.1 So n heaven is necessary,

we have also seen before that $P(x,y)=P(-x,-y), P(-x,y)=P(x,-y)$ so we just have to study the case $x>0$.

So substituting by every a_i with i odd, by $A_i=-a_i$ and $Y=-y$ we get exactly the same polynomial P_n , so if we have any necessary conditions on P_n to be positiv when $x>0$ and $y>0$, we can find conditions for every real $x\neq0$ and $y\neq0$, without losing precision.

5)

Let P'_n be the non-homogenous polynomial :

then P'_n can be written has a sum of homogenous polynomial of degré $j\leq n$

$P'_n(x,y)=\sum_0^n P_n(x,y)$ but, where every ai ,defining P_n can be different.

We can also see, that for every odd n, P'_n can't take only positive value, because the positivity depends directly on P_n . In particular for the limites of $P(x,y)$ when $x\to\pm\infty$ or $y\to\pm\infty$, then $\lim P'_n(x,y)=\lim P_n(x,y)$. So the question remains for every heaven integer.

5.1 n=2

Using the theorem seen in introduction, let $s=(x+y)$ and $p=xy$, for every real $x\neq0$, $y\neq0$.

$P'_2(x,y)=a(x^2+y^2)+bxy+c(x+y)+d$ (a, b, c in $\mathbb{R}$)
$\Leftrightarrow P'_2(x,y)=a\times s^2+(b-2a)p+cs+d$

As p can ever take a smaller value when s is constant (Substituting $x=X+k$ and $y=Y-k$) we have $p(k)=(xy)=-k^2+xy+k(y-x)$ (continues on $\mathbb{R}$ and if $k=y$ $p=0$)

So $P'_2(x,y)>0\Leftrightarrow as^2+(b-2a)p+cs+d>0\Leftrightarrow p>\frac{-as^2+cs+d}{(b-2a)}\, if\, b-2a>0$
which is impossible, because, p a smaller value, so we can find two real x,y where $P'_2(x,y)\leq0$.

But if $b \leq 2a$ as p is maximal when x = y (when s is constant) then

$$P'_2(x,y)>0 \Leftrightarrow p<\frac{-as^2+cs+d}{b}-2a \Leftrightarrow x^2<\frac{-2x^2+2cx+d}{(b-a)} \Leftrightarrow (-2-b+a)x^2+2cx+d>0$$
$$\Leftrightarrow -2-b+a>0 \wedge 4c^2-4d\times(-2-b+a)<0$$

5.2 n=4

$$P'_4(x,y)=a(x^4+y^4)+b(x^3y+y^3x)+c(x^2y^2)+d(x^3+y^3)+e(x^2y+y^2x)$$
$$+f(x^2+y^2)+g(xy)+h(x+y)+i$$
$$\Leftrightarrow P'_4(x,y)=a(s^2-2p)^2+bp(s^2-2p)+(c-2a)p^2+ds^3+(e-3d)sp+fs^2+(g-2f)p+hs+i$$
$$\Leftrightarrow P'_4(x,y)=a(s^4)+ds^3+(b-4a)s^2p+(2a-2b+c)p^2+fs^2(e-3d)sp+(g-2f)p+hs+i$$

So
$$P'_4(x,y)>0$$
$$\Leftrightarrow p((b-4a)s^2+(2a-2b+c)p+(e-3d)s+g-2f)>as^4+ds^3+fs^2+hs+i$$

As p can always take a smaller value when s is constant, if we can find 2 real x,y such that

$(b-4a)s^2+(2a-2b+c)p+(e-3d)s+g-2f>0$ this is imposible.

So $(b-4a)s^2+(2a-2b+c)p+(e-3d)s+g-2f \leq 0$ for all x,y is necessary.

And
$$(b-4a)s^2+(2a-2b+c)p+(e-3d)s+g-2f \leq 0$$
$$\Leftrightarrow (2a-2b+c)p \leq (b-4a)s^2+(e-3d)s+g-2f$$

which is impossible if $2a-2b+c<0$ else

As p is maximal when x = y (when s is constant) then we get

$$(2a-2b+c)p \leq (b-4a)s^2+(e-3d)s+g-2f$$
$$\Leftrightarrow (2a-2b+c)x^2 \leq (4b-16a)x^2+(2e-6d)x+g-2f$$
$$\Leftrightarrow (6b-18a-c)x^2+(2e-6d)x+g-2f \geq 0$$
$$\Leftrightarrow 6b-18a-c>0 \wedge (2e\text{-}6d)^2-4(g-2f)\times(6b-18a-c) \leq 0$$

or
$$6b-18a-c=0 \wedge e-3d=0 \wedge g-2f \geq 0$$

These conditions are necessary.

5.3 n

Using the fact that $P'_n(x,y)$ can be expressed with the quantites s and p, and that p can take every value between $\frac{s}{2}$ and $-\infty$, when we know necessary and suffisant conditions on P'_n-2 to be always negative, we can find suffisant conditions on P'_n to be always positiv.