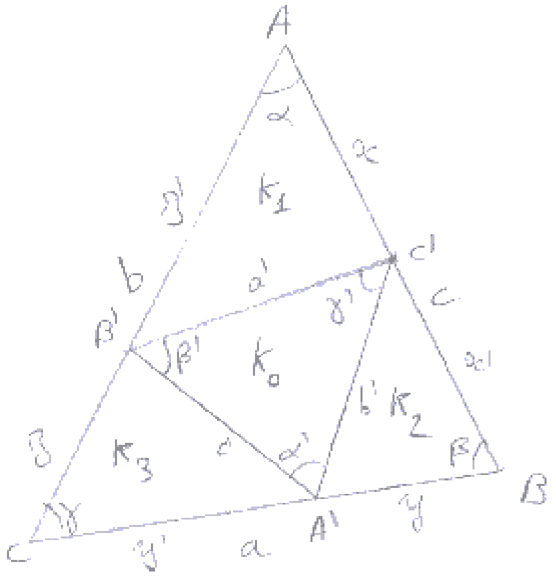
Aera



Let K denote the area of ABC.

Let x, x' be the ratio C' divides AB in. x+ x'= 1

Define the same way z, z', y and y'

Then $K_1 = xz'*K$, $K_2 = x'y*K$, $K_3 = y'z*K$

$K_0 = K\,(1-xz'-x'y-y'z) = K\,(xyz + x'y'z')$

Let us prove the stronger result: $K_0^2(K_0+ K_1 + K_2+ K_3) – 4K_1*K_2*K_3 \geq 0$

Indeed, $K_0^2K – 4xyzx'y'z'\ K^3 = K^3\ (\ (xyz + x'y'z')^2 - 4xyzx'y'z'\ ) = K^3\ (xyz – x'y'z')^2 \geq 0$.

Circumscribed circle.

Let R_0 denote the radii of the circumscribed circle of A'B'C'

$R_0 = a'b'c' / (4*K_0)$ $R_0<R_1$ **ó** sin α' > sin α

For R_0 to be the least among R_1, R_2, R_3 and R_0, we must have sin α'>sin α and sin β'>sin β and sin ɣ'> sin ɣ

If ABC is an acute triangle, the latter implies that α'>α, β'>β and ɣ'> ɣ which cannot holds.

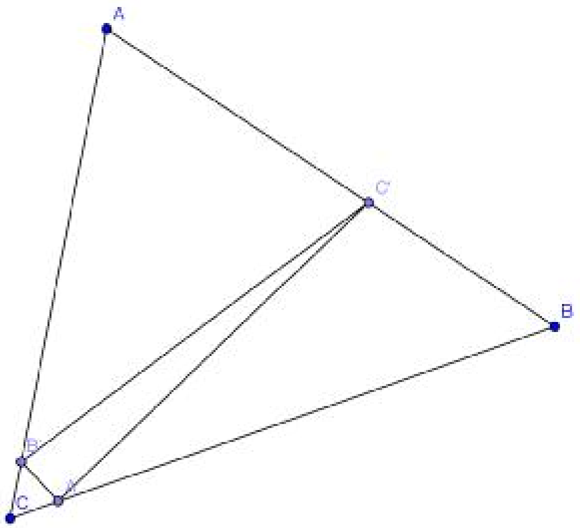
If ABC has an obtuse angle, one can find counterexamples.

We can also consider the heights, we can see that given any height of the triangle A'B'C' , one of the two nearby triangle will have a smaller height.

We'll give a proof for the perimeter in the 4th question.

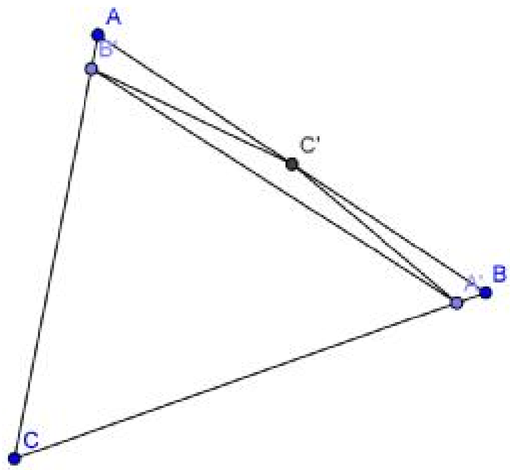
As for angles, bisectors and medians the answer turns out to be positive. We will show ways to get a counter example in any triangle by having an element tending to 0.

Angle:



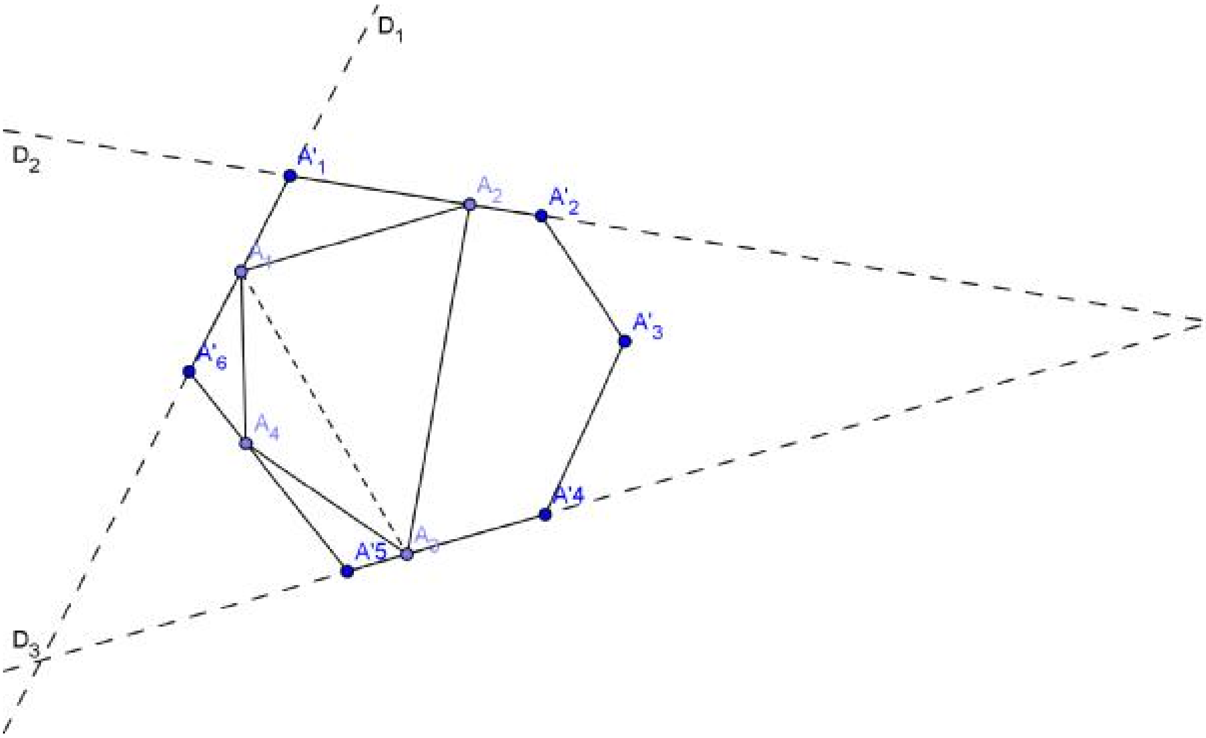
As D(A'B',C)→ 0

Median and bisector:



C' is the middle of AB, A'B'//AB and D(A'B',C')→0

3) **Area**



Let us consider a convex polygon $A_1A_2…A_n$ inscribed in another convex polygon $A'_1A'_2…A'_m$

Suppose for the sake of contradiction that $A_1A_2…A_n$ has the smallest area among all the external polygons.

Consider $A_1A_2A_3$. It is inscribed in $A'_1A'_2…A'_m$ and has the smallest area among all the new external polygons. (Because $A_1A_2…A_n$ is convex)

Let's call the segments A_1, A_2 and A_3 are on respectively 1 , 2 and 3.

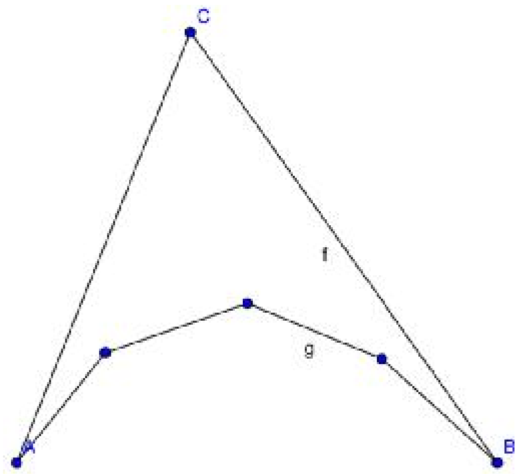
Draw the lines D_1, D_2 D_3. They form a triangle (assume that 2 parallel lines cut themselves at infinity).

Then $A_1A_2A_3$ is inscribed in that triangle and the area of $A_1A_2A_3$ is the smallest among the three other triangles (because $A'_1A'_2…A'_m$ is convex). But this is impossible as we proved in 1).

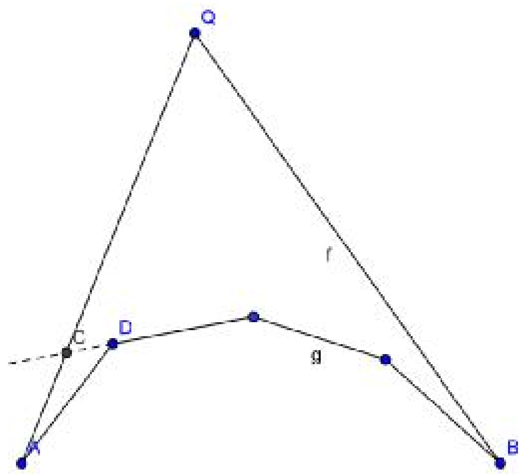
As for the perimeter, we will use exactly the same argument but we will need a lemma.

Lemma 1:

Given two piecewise linear concave curves such that f(0) = g(0), f(1) = g(1) and f(x)≥g(x) for all x in [0,1]. The perimeter of the curve f is greater than or equal to the one of g.



We will use induction on the number of pieces of g. If g has one piece, the result is trivial. Suppose the result holds for n pieces and consider g with n+1 pieces.



Extend the 2^{nd} piece to the left, then the length of the curve CDB is smaller than the one of the curve CQB (induction hypothesis) and by the triangle inequality, AC + CD > AD. This finishes the proof of the lemma.

The inscribed polygon cannot have the smallest perimeter among all the external polygons.

4) We will present a generalization of the questions about the inscribed triangle having the minimal perimeter/area among the external triangles.

Consider a convex polygon $A_1A_2 \ldots A_n$ inscribed in a polygons $A'_1A'_2 \ldots A'_n$ such that A_i is on the side $A'_{i-1}A_i$. Denote by B_i the intersection of $A_{i-1}A_{i+1}$ and A_iA_{i+2}.

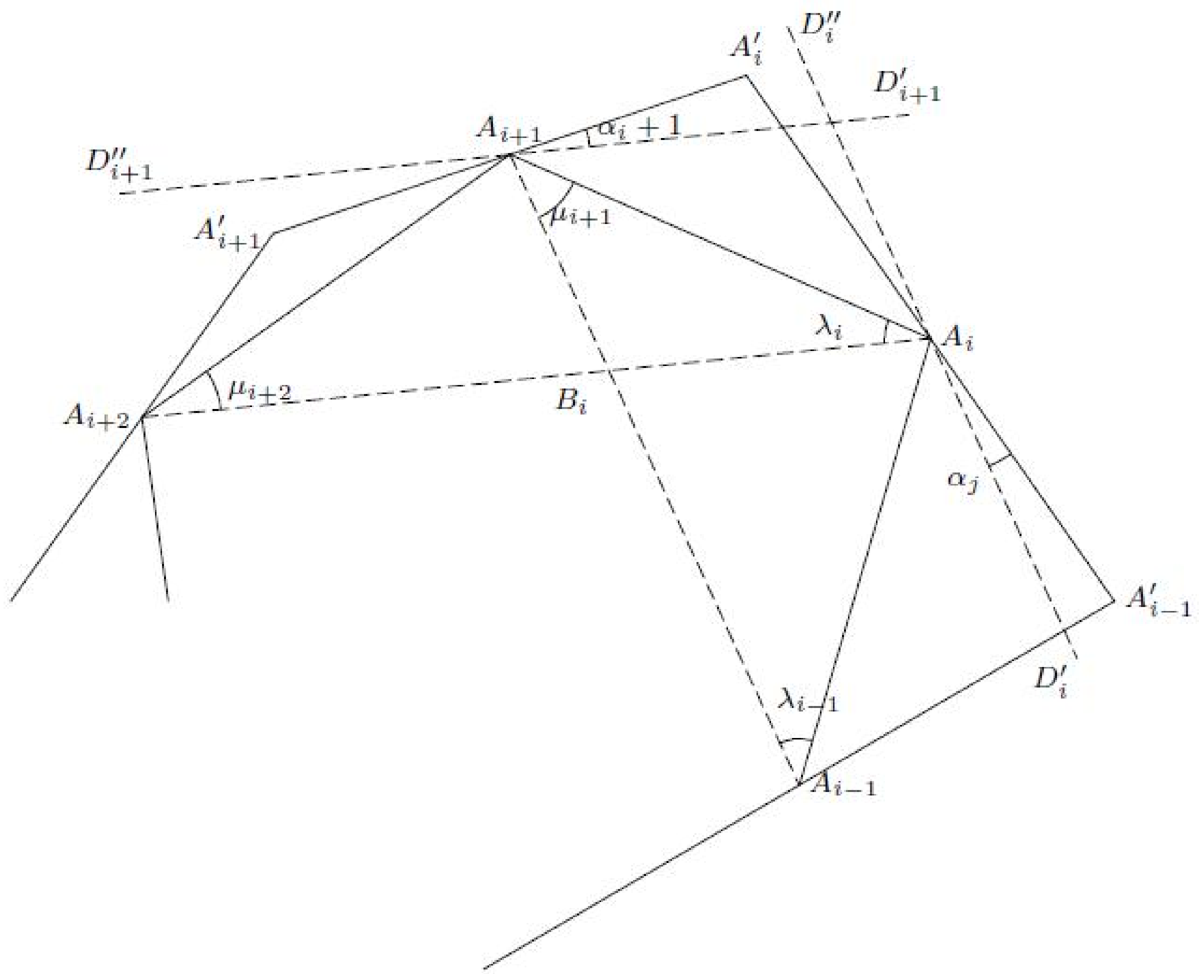
Then the following $K(A_iB_iA_{i+1}) < K(A_iA_{i+1}A'_i)$ cannot hold for all $1 \leq i \leq n$.

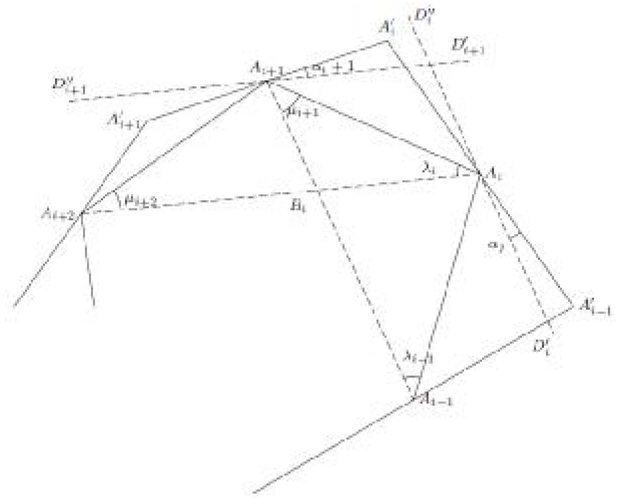
Where K(ABC) denote the perimeter of ABC. The same result holds if K denote the area.

Define α_i as the angle the side $A'_{i-1}A'_l$ makes with the side $A_{i-1}A_{i+1}$ such that $(A_{i-1}A_{i+1})$ has to be rotated counterclockwise by α_i to become $(A'_{i-1}A'_i)$

AB will denote the signed length of the segment AB, ABC the unsigned angle if nothing else is stated.

$\lambda_i = A_{i+2}A_iA_{i+1}$; $\mu_i = A_{i-2}A_iA_{i-1}$. And $\Delta_i = P(A_iB_iA_{i+1}) - P(A_iA_{i+1}A'_i)$





Using law of sines, we get

$(A_iB_i + B_iA_{i+1})/ (A_iA_{i+1}) = (\sin \lambda_i + \sin \mu_{i+1}) / \sin (\lambda_i + \mu_{i+1}) = \cos (\lambda_i - \mu_{i+1}) /2 / \cos (\lambda_i + \mu_{i+1})/2$ (1)

As $A_iA_{i+1}A'_I = \lambda_i + \alpha_{i+1}$ and $A_{i+1}A_iA'_I = \mu_{i+1} - \alpha_{i,}$

$(A_iA'_I + A'_iA_{i+1})/A_iA_{i+1} = (\sin(\mu_{i+1} - \alpha_i) + \sin(\lambda_i + \alpha_{i+1}))/ \sin(\lambda_i + \mu_{i+1} + \alpha_{i+1} - \alpha_i)$

$= \cos((\lambda_i - \mu_{i+1} + \alpha_{i+1} + \alpha_i)/2/ \cos ((\lambda_i + \mu_{i+1} + \alpha_{i+1} - \alpha_i)/2)$ (2)

We have $-\lambda_{i-1} < \alpha_i < \mu_{i+1}$. Hence the denominators of (1) and (2) are positive and

Δ_I has the same sign as

$\cos (\lambda_i - \mu_{i+1})/2*\cos (\lambda_i + \mu_{i+1} + \alpha_{i+1} - \alpha_i)/2 - \cos (\lambda_i + \mu_{i+1})/2*\cos (\lambda_i - \mu_{i+1} + \alpha_{i+1} + \alpha_i)/2$

$= \sin(\lambda_i + \alpha_{i+1}/2)*\sin(\alpha_i/2) - \sin(\mu_{i+1} - \alpha_i/2)*\sin(\alpha_{i+1}/2)$ (3)

Now we distinguish 3 cases :

1) The α_i are all equals to zero, then $\Delta_I = 0$ for all i.
2) There is an I such that $\alpha_i > 0$ and $\alpha_{i+1} \leq 0$ (or $\alpha_i \geq 0$ and $\alpha_{i+1} < 0$) , so that $\mu_{i+1} - \alpha_i/2 > 0$ and $\lambda_i + \alpha_{i+1}/2 > 0$ and $\Delta \geq 0$.
3) All the α_i are different from zero and have the same sign. (Suppose they are all > 0)

Divide (3) by $\sin \alpha_i/2 * \sin \alpha_{i+1}/2$, we get that Δ_I has the same sign as

$\sin \lambda_i \operatorname{cotan} (\alpha_{i+1}/2) - \sin \mu_{i+1} \operatorname{cotan} (\alpha_i/2) + \cos \lambda_i + \cos \mu_{i+1}$

Assume for the sake of contradiction that $\Delta_i < 0$ for all i. As $\cos \lambda_i + \cos \mu_{i+1} > 0$,

$\sin \lambda_i \operatorname{cotan} \alpha_{i+1}/2 < \sin \mu_{i+1} \operatorname{cotan} \alpha_i/2$. Dividing by $\sin \lambda_i \operatorname{cotan} \alpha_i /2$ yields

$0 < \operatorname{cotan}(\alpha_{i+1}/2)/ \operatorname{cotan} (\alpha_i/2) < \sin \mu_{i+1} / \sin \lambda_i$ (4)

Multiplying (4) for i = 1,…,n and using the fact that

$\prod(\sin \mu_{i+1}/\sin \lambda_{i-1}) = \prod (\sin \mu_{i+1} / \sin \lambda_{i-1})$ $(\mu_0 = \mu_n , \mu_{n+1} = \mu_1)$

Yields: $1 < \prod \sin \mu_{i+1}/\sin \lambda_{i-1}$

$$= \prod (\sin (\lambda_{i-1} + \mu_{i+1})/2*\cos (\lambda_{i-1} - \mu_{i+1})/2 - \cos (\lambda_{i-1} + \mu_{i+1})/2*\sin(\lambda_{i-1} - \mu_{i+1})/2) /A_i$$

Where $A_i = \sin (\lambda_{i-1} + \mu_{i+1})/2*\cos (\lambda_{i-1} - \mu_{i+1})/2 + \cos (\lambda_{i-1} + \mu_{i+1})/2*\sin(\lambda_{i-1} - \mu_{i+1})/2$

Diving this by $\sin (\lambda_{i-1} + \mu_{i+1})/2*\cos (\lambda_{i-1} - \mu_{i+1})/2$ and using the theorem of tangents yields a contradiction.

As for Area let's now define $\Delta_i= S(A_iB_iA_{i+1}) – S(A_iA_{i+1}A'_i)$

$S(A_iB_iA_{i+1}) = ½ * A_iA_{i+1}^2*\sin\mu_{i+1}\sin\lambda_i/\sin (\mu_{i+1} + \lambda_i)$

$S(A_iA_{i+1}A'_i) = = ½ * A_iA_{i+1}^2 * \sin(\mu_{i+1} – \alpha_i)* \sin(\lambda_i + \alpha_{i+1})/ \sin(\lambda_i + \mu_{i+1} + \alpha_{i+1} – \alpha_i)$

Using the same argument as before, Δ_I has the same sign as :

$\sin \mu_{i+1} \sin\lambda_i \sin(\mu_{i+1}+ \lambda_i + \alpha_{i+1} – \alpha_i) – \sin(\mu_{i+1} – \alpha_i) \sin(\lambda_i +\alpha_{i+1}) \sin(\mu_{i+1} + \lambda_i)$
$= \sin \mu_{i+1} \sin \lambda_i *[\sin(\mu_{i+1} – \alpha_i) \cos(\lambda_i + \alpha_{i+1})+\cos(\mu_{i+1} –\alpha_i) \sin(\lambda_i + \alpha_{i+1})] – \sin(\mu_{i+1} – \alpha_i) \sin(\lambda_i + \alpha_{i+1}) \times[\sin \mu_{i+1} \cos \lambda_i + \cos \mu_{i+1} \sin \lambda_i]$ (5)

This has the same sign as $\cot (\lambda_i + \alpha_{i+1}) – \cot \lambda_i + \cot (\mu_{i+1} – \alpha_i) – \cot \mu_{i+1}$ (5.1)

3 cases:

1) All α_i are 0 then $\Delta_I = 0$
2) There is an i such that $\alpha_i> 0$ and $\alpha_{i+1}\leq 0$ (or $\alpha_i \geq 0$ and $\alpha_{i+1}< 0$) then $\Delta_I > 0$
3) Else assume that all $\alpha_i> 0$, developpe (5), and dividing by $\sin \alpha_i \sin \alpha_{i+1}$ we achieve that Δ_i has the same sign as

$\cot \alpha_{i+1} \sin^2\lambda_i – \cot \alpha_i \sin^2 \mu_{i+1} + \sin \mu_{i+1} \cos \mu_{i+1} + \sin \lambda_i \cos \lambda_i$

Assume for the sake of contradiction that $\Delta_I <0$ for all i.
As $\sin \mu_{i+1} \cos \mu_{i+1} + \sin \lambda_i \cos \lambda_i > 0$, we get that $\cot \alpha_{i+1} \sin^2 \lambda_i < \cot \alpha_i \sin^2 \mu_{i+1}$
That is $\cot \alpha_{i+1}/ \cot \alpha_i < \sin^2 \mu_{i+1}/ \sin^2 \lambda_i$ for all i.
Multiplying this relation for all I and using the law of sines, we get :

$1 < \prod \sin^2 \mu_{i+1}/ \sin^2 \lambda_{i -1} = \prod A_{i-1}A_i^2/A_{iA}i+1 = 1$

Contradiction.