We want to solve the following functional equation :
(1) $f(f(x_1)+f(x_2)+...+f(x_n)+kx_1x_2...x_n) = x_1f(x_2)+x_2f(x_3)+...+x_nf(x_1)$
where n is an integer such that $n \geq 1$.

Part I

Cas $n = 1$

The functional equation becomes :
$f(f(x) + kx) = xf(x)$
We didn't solve this equation. The zero function is solution.
For $k = 0$, the function $f(x) = x^{\frac{1+\sqrt{5}}{2}}$ for $x \in \mathbb{R}^+$ and $f(x) = 0$ for $x \in \mathbb{R}^-$ is a solution.
The function $f(x) = x^{\frac{1-\sqrt{5}}{2}}$ for $x \in \mathbb{R}^+$ and $f(x) = 0$ for $x \in \mathbb{R}^-$ is a solution too for $k = 0$.

Part II

Cas $n = 2$

1 Cas $f(0) \neq 0$

Let $a = f(0)$. We assume that $a \neq 0$.
Let $y = 0$ in (1), we have : $f(f(x) + a) = ax$ **(2)**, which implies that f is injective.
Using relation (2) with $x = f(y)+a$, (for all real y), we get $f(ay+a) = a(f(y)+a)$ **(3)** .
Then, we take $y = 0$ in (3) and we obtain :
$f(a) = 2a^2$
Now put $x = 2a$ in (2),
We get : $f(f(2a)+a) = f(a) = 2a^2$. Because f in injective, we have : $f(2a) = 0$.
By (1) and $x = y = 2a$ we have : $f(4ka^2) = 0 = f(2a)$ therefore $2a = 4ka^2$. If $k = 0$, $a = 0$.
Else if $a = \frac{1}{2k}$ then $k = \frac{1}{2a}$.
In the original equation, we take $y = 2a$. Then $f(f(x) + x) = 2af(x) = f(f(2f(x)) + a)$. By injectivity $f(2f(x)) + a = f(x) + x$. **(4)**
Finally, for $x = a$ in (4), we get : $f(2f(a)) = f(a)$. By injectivity : $a = 2f(a) = 4a^2$ consequently $a = \frac{1}{4}$ therefore $k = 2$.

1.1 Case $k = 2$ and $f(0) = \frac{1}{4}$

First, let's calculate $f(1)$ and $f(\frac{-1}{2})$.
We have $f(f(1) + \frac{1}{4}) = \frac{1}{4} = f(0)$. By injectivity of f, $f(1) = \frac{-1}{4}$
Also, we have previously seen that $f(2f(x)) + \frac{1}{4} = f(x) + x$.
Setting $x = 1$, we obtain $f(\frac{-1}{2}) = \frac{1}{2}$.
Let x a real and $h_x(y) = f(y) + 2xy$ for all $y \in \mathbb{R}$.
Let $A = \{x/f(x) = \frac{1}{4} - \frac{x}{2}\}$ and $F = \{x/f(x) = 2x^2\}$.

It is clear that $A \cap F = \{\frac{-1}{2}, \frac{1}{4}\}$.
Let $g(x) = \frac{f(x)}{2x-1}$.
We have seen previously that $f(f(x)+x) = \frac{f(x)}{2}$.
Then setting $x = g(y)$, we have :
$f(x) + f(y) + 2xy = f(x) + x$
Applying f : $xf(y) + yf(x) = \frac{f(x)}{2}$.
Simplifying : $f(x) = 2x^2$. Then $g(y) \in F$.

We're going to show 3 lemma :

Lemma 1 :
h_x is injective or $x \in F$.
Proof :
We assume that $h_x(y) = h_x(z)$. Then $f(x) + f(y) + 2xy = f(x) + f(z) + 2xz$.
Applying f : $xf(y) + yf(x) = xf(z) + zf(x)$.
These two equations implie that $f(x) = 2x^2$ ou $y = z$.
The conclusion follows.

Lemma 2 :
If a and $b \in F$ $(a \neq b)$, then $-a-b \in F$.
Proof :
We set $x = \frac{-1}{2} . \frac{f(a)-f(b)}{a-b}$ where $a, b \in F$ and $a \neq b$.
we get : $h_x(a) = h_x(b)$. By lemme 1, $x \in F$.
But $f(a) = 2a^2$ and $f(b) = 2b^2$. Therefore $x = -a-b \in F$.
The conclusion follows.

Lemma 3 :
$F = \{\frac{-1}{2}, \frac{1}{4}\}$.
Proof :
We assume that there exist $x \in F$ such that $x \neq \frac{-1}{2}$ and $x \neq \frac{1}{4}$.
By lemma 2, $\frac{1}{2} - x \in F$ therefore $\frac{-1}{4} - (\frac{1}{2} - x) = \frac{-3}{4} + x \in F$.
By lemma 2, $-(\frac{-3}{4} + x) - (\frac{-1}{4} - x) = 1 \in F$. Contradiction.
The conclusion follows.

We have previously seen that for all y, $g(y) \in F$.
By lemma 3, forall y, $g(y) = \frac{-1}{2}$ or $g(y) = \frac{1}{4}$.
Then $f(y) = y - \frac{1}{2}$ or $f(y) = \frac{-y}{2} + \frac{1}{4}$.
We check that $f(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2} = 0$
We assume that there exist a such that $f(a) = a - \frac{1}{2}$.
We have seen previously that $f(f(x)+\frac{1}{4}) = \frac{x}{4}$. With $x = a$ we get $f(a-\frac{1}{4}) = \frac{a}{4}$.
But $f(a - \frac{1}{4}) = (a - \frac{1}{4}) - \frac{1}{2} = a - \frac{3}{4}$ or $f(a - \frac{1}{4}) = \frac{1}{4} - \frac{a-\frac{1}{4}}{2} = \frac{3}{8} - \frac{a}{2}$.
Therefore $a = 1$ or $a = \frac{1}{2}$.
We check that the case $a = \frac{1}{2}$ gives $f(a) = \frac{1}{4} - \frac{a}{2}$.
Case $a = 1$ gives : $f(1) = \frac{1}{2} = f(\frac{-1}{2})$ impossible by injectivity.
Conclusion : $f(x) = \frac{1}{4} - \frac{x}{2}$ is the only solution to the original functional equation for $k = 2$ and $f(0) \neq 0$.

2 Case $f(0) = 0$

Puting $x = y = 0$ in the original functional equation, we get : $f(f(x)) = 0$.

2.1 Case k=0

We put $y = f(z)$ in the original equation. Then : $f(f(x)) = yf(x) = f(z)f(x)$. But $f(f(x)) = 0$. Then : $f(z)f(x) = 0$. Therefore, f is the zero function.

2.2 Case k$\neq 0$

Let : $A = \{x/f(x) = 0\}$, $B = \{f(x), x \in \mathbb{R}\}$ and $F = \{x/f(x) = kx^2\}$. It is clear that $A \cap F = \{0\}$.
Let $g(x) = \frac{-f(x)}{kx}$ where $x \neq 0$.
In the original functional equation, we put $y = g(x)$. We obtain : $0 = f(f(y)) = g(x)f(x) + f(g(x))x$.
Then $f(g(x)) = kg(x)^2$. Therefore $g(x) \in F$.
We are going to show 2 lemma.
Lemma 1 :
$\frac{-2}{k} \in A$
Proof :
In the original functional equation, we set $y = f(z)$. We get $f(f(x)+kxf(z)) = f(x)f(z)$.
In this last relation, let $x = z = \frac{-1}{k}$. Then we have : $f(0) = f(\frac{-1}{k})^2$. Therefore $f(\frac{-1}{k}) = 0$.
Always with the same relation, by symetry : $f(f(x) + kxf(z)) = f(f(z) + kzf(x))$.
Then setting $x = \frac{-1}{k}$ we obtain $f(-f(z)) = f(f(z)) = 0$.
Finally, we take $x = z = \frac{-2}{k}$. Then we get $f(-f(\frac{-2}{k})) = f(\frac{-2}{k})^2$. Therefore $f(\frac{-2}{k}) = 0$.
By definition, $\frac{-2}{k} \in A$.

Lemma 2 :
If $x \in A$ and $y \in B$ then $xy \in A$.
Proof :
By definition : $f(x) = 0$ and $y = f(z)$ for z real.
Applying f both sides of the equation, we have :
$f(xf(z) + zf(x)) = 0$
But here we have $f(x) = 0$ and $y = f(z)$, $f(xy) = 0$.
Therefore $xy \in A$.

Lemma 3 :
$y \in A$ if and only if $\frac{-1}{ky} \in A$.
Proof :
We assume that $\frac{-1}{ky} \in A$. By lemma 2, for all $x \in A$, $xf(y) \in A$. If $x = \frac{-1}{ky}$, $xf(y) = g(y) \in F$.
But $A \cap F = \{0\}$, then we have $f(y) = 0$. Then $y \in A$.
Reciprocally, $\frac{-1}{ky} \in A$ implies that $\frac{-1}{k\frac{-1}{ky}} = y \in A$.
The conclusion follows.

Lemma 4 :
If $x \in F$, then $2kx^3 \in B$.
Proof :
We put $x = y$ in the original functional equation, we get :
$f(2f(x) + kx^2) = 2xf(x)$.
But we have $x \in F$, then $f(x) = kx^2$. Therefore $f(3kx^2) = 2kx^3$. By definition, $2kx^3 \in B$.

Now, to obtain a contradiction, we assume that there exist $b \in F$ such that $b \neq 0$.
We see that $f(b) = kb^2$ then $kb^2 \in B$.
By lemma 1, $\frac{-2}{k} \in A$. By lemma 2, $\frac{-2}{k}.kb^2 \in A$. Then $-2b^2 \in A$.
Using lemma 3, $\frac{-1}{-2b^2k} \in A$. Therefore $\frac{1}{2kb^2} \in A$.
Finally, thanks to lemma 2 and lemma 4 $\frac{1}{2kb^2}.2kb^3 \in A$. Therefore $b \in A$.
Contradiction because $b \neq 0$ and $A \cap F = \{0\}$.
Then $F = \{0\}$
But for all real x non zero, $g(x) \in F$, then $g(x) = 0$ forall x non zero.
Then $\frac{-f(x)}{kx} = 0$ forall x non zero.
Therefore f is the zero function.

Part III

Case $n > 2$

3 Case $f(0) = 0$

In the original equation we take $x_2 = x_3 = ... = x_n = 0$. Then we have $f(f(x_1)) = 0$.
But $n \geq 3$, then, we can put $x_3 = ... = x_n = 0$, we obtain : $f(f(x_1) + f(x_2)) = x_1 f(x_2)$.
This is true for $x_1 = f(z)$. Then $f(f(f(z)) + f(x_2)) = f(z)f(x_2)$. But $f(f(z)) = 0$, then $f(f(x_2)) = 0 = f(z)f(x_2)$ for all z and x_2 real. Therefore f is the zero function.

4 Case $f(0) \neq 0$

Let $a = f(0) \neq 0$.
In the original equation, puting $x_3 = ... = x_n = 0$, we get, because of the symetry :
$x_1 f(x_2) + ax_2 = x_2 f(x_1) + ax_1$. Then $h(x_1) = h(x_2)$ où $h(x) = \frac{f(x)-a}{x}$. Therefore f a linear function.
It is easy to see that there are no linear solutions for $n \geq 3$.

Part IV

Generalization

A generalisation of the original functional equation is :
$f(af(x) + f(y) + kxy) = xf(y) + yf(x)$
Where a is a constant.
With the same arguments (just puting coefficients in front of the variables), we can prove that, if $a \neq -1$,
For $k \neq \frac{a+1}{a}$, f is the zero function
For $k = \frac{a+1}{a}$, $f(x) = \frac{a}{(a+1)^2} - \frac{x}{a+1}$ or f is the zero function.

For $a = -1$, f is the zero function.