ITYM Problem 1
English

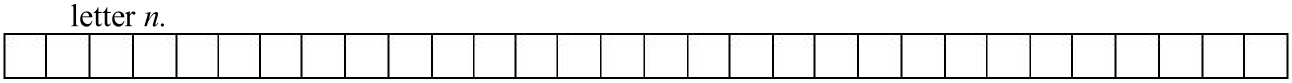
<u>Question 1 :</u>

We cover the m x n rectangle of 1x2 dominoes. There will be exactly $\left[\frac{n\times m}{2}\right]$. Each domino can't contain more than one blue cell, otherwise there would be two blue cells side by side and so a symmetric ax. So by the pigeonhole principle, there can be at the maximum $\left[\frac{n\times m+1}{2}\right]$ blue cells.

So the optimal case is the chessboard, in which there's no symmetric ax, and the number of blue cells is $\left[\frac{n\times m+1}{2}\right]$.

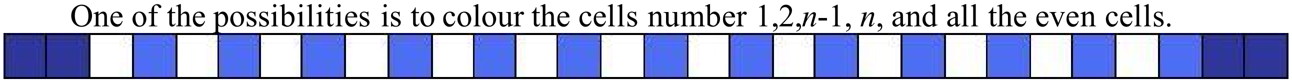
<u>Question 2 :</u>

We will study first the 1xn case. We number the cells from the left to the right with the letter n.

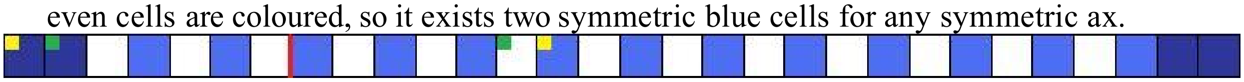


Upper bound :

One of the possibilities is to colour the cells number 1,2,n-1, n, and all the even cells.

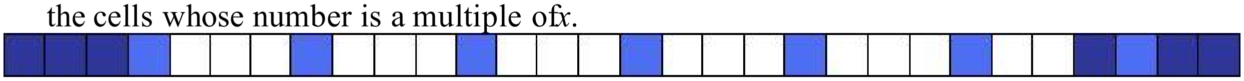


For a given symmetric ax, we draw the symmetric of the cells 1 and 2 if the ax is in the left part of the line, and the symmetric of the cells n-1 and n if the ax is the right part of it. Their symmetric are two consecutive cells, so an odd one and an even one. And all the even cells are coloured, so it exists two symmetric blue cells for any symmetric ax.



So, the number of coloured cells is $\left[\frac{n}{2}\right]+2$.

We can generalize the algorithm.
Let x be an integer. We colour the cells 1,2, to x and the cells n-x+1 to n. We colour then the cells whose number is a multiple of x.



If we choose whatever symmetric ax, the symmetric of the beginning or ending consecutive cells will be a set of x consecutive cells, with x consecutive numbers. And within x consecutive numbers, one can be divided by x.

So there's at least two symmetric blue cells for each symmetric ax.

The number of coloured cells is $\left[\frac{n}{x}\right]+2\times x-2$.

So $S(n,1)\leqslant\left[\frac{n}{x}\right]+2\times x-2$ for all $x\in\mathbb{N}$.

To find the best x, we approximate $\left[\frac{n}{x}\right]$ with $\frac{n}{x}$.

To find the minimum of $\frac{n}{x}+2\mathrm{x}-2$ we differentiate at x, so we obtain $2-\frac{n}{x^2}$.

The minimum is when this derivative equals to 0, so when $x=\sqrt{\frac{n}{2}}$.

Then we have $$\frac{n}{\sqrt{\frac{n}{2}}}+2\times\sqrt{\frac{n}{2}}-2=\sqrt{2n}+\sqrt{2n}-1=\sqrt{8n}-2$$.

And , because it's an upper bound, we choose the upper integer, we have :

$S(n,1)\leqslant min(\left[\frac{n}{x}\right]+2\mathrm{x}-2)\approx\sqrt{8n}-1$.

Lower bound :

In a row of n cells there is exactly n-1 symmetric axes. Then we colour optimally the row. Let a be the number of even coloured cells and b the number of odd ones. Each even cell can be the symmetric of an odd one. So every even cell can be "useful" for only b axes. There can be at the maximum only $a\times b$ symmetric axes with this configuration. So we have $a\times b\geqslant n-1$. Besides $a+b$ has to be minimal. So we have to seek the two integers a and b so $\boldsymbol{a\times b\geqslant n-1}$ **and** $\boldsymbol{a+b}$ **is minimal.** . $S(n,1)$ can't then be less than $a+b$.

We can first forget the condition $(a,b)\in\mathbb{N}^2$. In this case we obtain an undervaluation (because $a+b$ is the less possible) but not the best over valuation.

$$a \times b = n-1 \Rightarrow b = \frac{n-1}{a}$$

So $a+b = a + \frac{n-1}{a}$.

To find the minimum, we differentiate at a and we obtain the derivative : $1 - \frac{n-1}{a^2}$

This derivative equals zero and changes of sign when $a = \sqrt{n-1}$.
We finally obtain: $a = b = \sqrt{n-1}$ et $a+b = 2\sqrt{n-1} = \sqrt{4n-4}$

$$\sqrt{4n-4} \leqslant S(n,1)$$

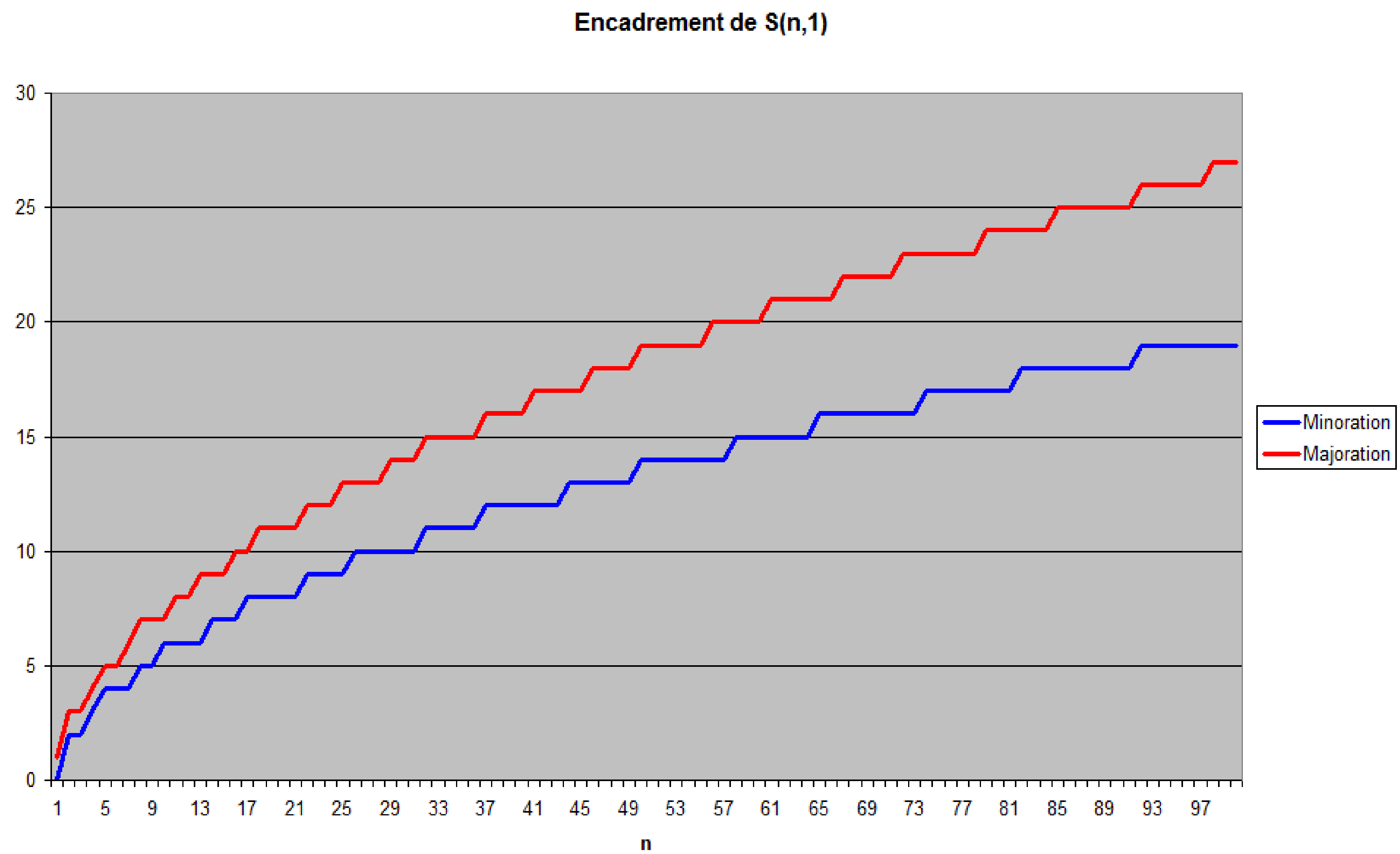
We draw the next graph with estimating $S(n,1)$ between $[\sqrt{4n-4}]$ et $[\sqrt{8n}-1]$

.

In the plan, we obtain :

With $m \leqslant n$

$$[\sqrt{4n-4}] \leqslant S(n,m) \leqslant [\sqrt{8m}] + [\sqrt{8n}] - 3$$



Encadrement de S(n,1)
30
25
20
15
10
5
0
1
5
9
13
17
21
25
29
33
37
41
45
49
53
57
61
65
69
73
77
81
85
89
93
97
n
Minoration
Majoration