Good numbers

In the following :

A *k-good* number n is a number such that there exists a, b integer numbers, $a<b$ and there is a writing of $\frac{a}{b}$ as a continuous fraction with all his partials quotients integers in [1,n] .
The set Gk is the set of all k-good numbers.

Let f(x,y) be a function such that for every x,y in IN , $f(x,y)=\frac{1}{y+x}$

Preliminary remark : as a/b = ka/kb for every k in IN* , if a+b is n-good, k(a+b) is n-good too because the fraction ka/kb can be decomposed as the fraction a/b . So if a number is n-good, his multiples are n-good too.

1. Find the 1-good numbers

The n-th 1-good number is the number associated to the fraction $[0;1,1,1,...1]$ (n digits 1).

Denote by [n]1 the n-th good fraction, then [n+1]1 = f(1,[n]1)

The 1-good numbers are the Fibonnaci numbers other than 1 and 2 and their multiples, because :

- [2]1 = $\frac{1}{1+\frac{1}{1}}=\frac{1}{2}$ → F(3)=2+1 is 1-good (where F(n) are Fibonnaci numbers : F(0)=1 , F(1)=1 , F(n+2) = F(n+1)+F(n)) ; same way, F(4)=5 is 1-good, associated to [0,1,1,1] =2/3 .
- if the fraction [n]1 has numerator F(n-1) and denominator F(n) (which is associated to the number F(n+1)) , [n+1]1 = f(1,[n]1) and so

$$[n+1]1=\frac{1}{1+[n]1}=\frac{1}{1+\frac{F(n-1)}{F(n)}}=\frac{1}{\frac{F(n)}{F(n)}+\frac{F(n-1)}{F(n)}}=\frac{1}{\frac{F(n)+F(n-1)}{F(n)}}=\frac{1}{\frac{F(n+1)}{F(n)}}=\frac{F(n)}{F(n+1)}$$

and [n+1]1 has numerator F(n) and denominator F(n+1) and is associated to F(n+2).

An induction proves that the Fibonnaci numbers, excepted 1and 2, are the only numbers associated to irreductible continuous fractions with all partial quotients equal to one ; minding the preliminary remark, their multiples are 1-good too.

2. general properties of sets G_k

- $k<k' \Rightarrow G_k \subset G_k{}'$

- $\lim G_k = \mathbb{N} \setminus \{0,1,2\}$ which means that for every integer m greater than 2, we can find k such that m is k-good. Proof : As $\frac{F(n-1)}{F(n)}$ is a 1-good fraction and so k-good too, f(k, $\frac{F(n-1)}{F(n)}$) is a k-good fraction. But $f(k,\frac{F(n-1)}{F(n)})=\frac{1}{k+\frac{F(n-1)}{F(n)}}=\frac{F(n)}{F(n-1)+kF(n)}$ and so the number F(n)+F(n-1)+kF(n) = F(n+1)+kF(n) is k-good too. In particulary, taking $n=2$, the number 2k+3 is always k-good ; as every odd number can be written this way, and every even number strictly greater than 2 can be written as the product of a power of 2 and an odd number different of 1, and this odd number is k-good so his multiples are k-good too ; or if the number is a power of 2, 4=1+3 is 2-good so his multiples are 2-good too.

This being given, if only a finite number of odd numbers are out of G_2 , you only need to take the highest of them (M), find k great enough such that M belongs to G_k with the precedent algorithm : then all odd numbers (out of 1) are in G_k , and all even numbers (out of 0 and 2) as they are multiples of odd numbers. So the answer « no » to the question 2a implies an answer « yes » to the question 3.

We call a k-good number « primitive » if he is in G_k , while every of his divisors are not in G_k .

As we can draw a tree with the continuous fractions (fig 1), we can do the same with their associated primitive k-good numbers (fig.2) :

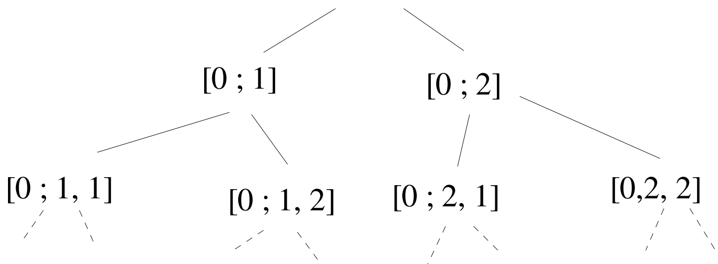


FIGURE 1 (n=2)

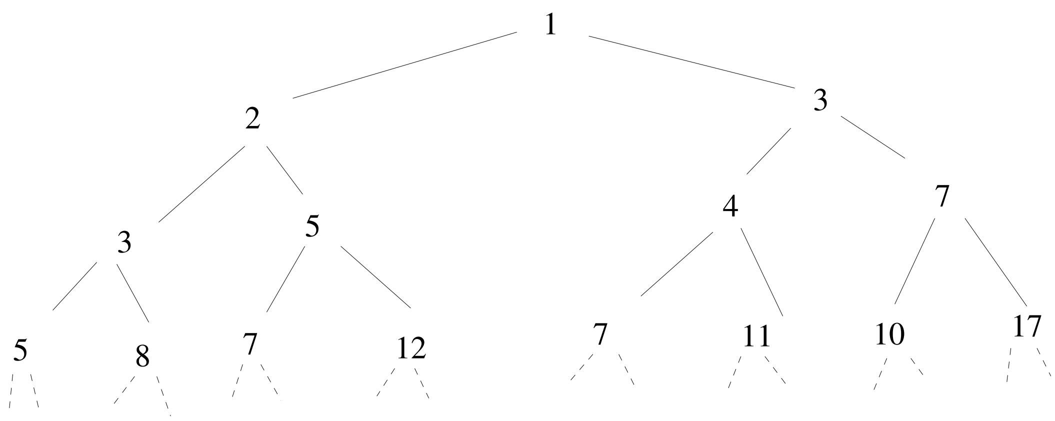
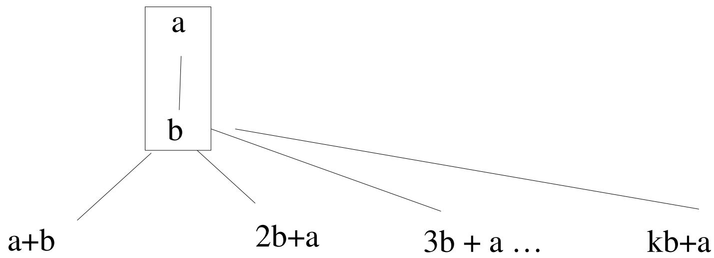


FIGURE 2 (n=2)

(The Fibonnacci numbers appear on the left). Any k-tree obeys to the following rules of construction :



Any prime k-good number is primitive, so it appears at least one time in the tree.

This indicates an algorithmic method to test if a number is k-good or not.

Empirical results :

- The first prime numbers that are not 2-good are : 2, 23, 37, 53, 59, 61, 83, 103, 107, 113, 127, 137, 139, 149, 151, 197, 211, 223, 227, 229, 331, 347, 349, 353, 359, 383, 421, 439, 461, 479, 491, 509, 523, 541, 557, 563, 569, 607, 631 …
- All the prime numbers at least until 1453 are 3-good.

4. Denoting the fine numbers numbers such that they can be written as a+b or b-a with $a<b$ and $\frac{a}{b}$ a 2-good fraction, this is evident that all numbers greater or equal to 1 are fine , since $2n>n$,

2n-n = n and n/(2n) = $\dfrac{1}{1+\dfrac{1}{1}}$