PROBLEM 8 - A Positivity of Symmetric Polynomials

1. **We denote now $P(x,y)=x^3+ax^2y+axy^2+y^3$. Let us show that: P(x,y) > 0 $(IR_+^2) \Leftrightarrow$ a>-1**

P(x,y) > 0 $(IR_+^2) \Rightarrow P(1,1) > 0$

$\Rightarrow 1 + a + a + 1 > 0$

$\Rightarrow 2 + 2a > 0$

$\Rightarrow a > -1.$

Mutually

$\forall(x,y) \in IR_+^2 \quad a > -1 \Rightarrow P(x,y) = x^3 + ax^2y + axy^2 + y^3 > x^3 - x^2y - xy^2 + y^3$

We can factorize: $x^3 - x^2y - xy^2 + y^3 = (x+y)(x-y)^2$.

$\forall(x,y) \in IR_+^2 \quad a > -1 \Rightarrow P(x,y) = x^3 - ax^2y - axy^2 + y^3 > (x+y)(x-y)^2 \geq 0$

a>-1 $\Rightarrow$ P(x,y) > 0 (IR_+^2)

So, indeed, we have the equivalence

P(x,y) > 0 $(IR_+^2) \Leftrightarrow$ a>-1

2. We denote : P(x,y) = $x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$

***Let us show that:* P(x,y) > 0 $(IR_+^2) \Leftrightarrow b > \frac{a^2+8}{4}$**

or

$$\begin{cases} a \geq -4 \\ b > -2a - 2 \end{cases}$$

If b>$\frac{a^2+8}{4}$ we have

P(x,y) = $x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$

P(x,y) > $x^4 + ax^3y + \frac{a^2+8}{4}x^2y^2 + axy^3 + y^4$

We can factorize:

$$x^4 + ax^3y + x^2y^2(\frac{a^2+8}{4}) + axy^3 + y^4$$
$$= \left(x^2 + \frac{a}{2}xy + y^2\right)^2.$$

So, P(x,y)> $\left(x^2 + \frac{a}{2}xy + y^2\right)^2 \geq 0$

And we have demonstrated that

$$b > \frac{a^2+8}{4} \Rightarrow \mathrm{P(x,y)} > 0\ (IR_+^2),$$

Let us suppose now $\{\begin{matrix} a \geq -4 \\ b > -2a-2 \end{matrix}$. In this case we have

$$\mathrm{P(x,y)} > x^4 + ax^3y + (-2a-2)x^2y^2 + axy^3 + y^4$$
$$= (x-y)^2(x^2 + (a+2)xy + y^2).$$

For $a \geq -4$ we have

$$x^2 + (a+2)xy + y^2 \geq x^2 - 2xy + y^2 = (x-y)^2 \geq 0.$$

So far we have implications:

$$\{\begin{matrix} a \geq -4 \\ b > -2a-2 \end{matrix} \Rightarrow \mathrm{P(x,y)} > 0\ (IR_+^2)$$

and

$$\{b > \frac{a^2+8}{4} \Rightarrow \mathrm{P(x,y)} > 0\ (IR_+^2),$$

thus

$$\{\begin{matrix} a \geq -4 \\ b > -2a-2 \end{matrix}\ or\ \{\begin{matrix} a < -4 \\ b > \frac{a^2+8}{4} \end{matrix} \Rightarrow \mathrm{P(x,y)} > 0\ (IR_+^2).$$

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<u>Let us show the contrary</u>

If P(x,y) > 0 ($(x, y) \in IR_+^2$),

then in particular:

$P(1,1) = 1 + a + b + a + 1$ >0

⇒ b>-2a-2

We denote: b=$\alpha + \frac{a^2+8}{4}$

P(x,y) =$x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$

$=x^4 + ax^3y + x^2y^2(\alpha + \frac{a^2+8}{4}) + axy^3 + y^4$

$=x^4 + ax^3y + x^2y^2\frac{a^2+8}{4} + axy^3 + y^4 + \alpha x^2y^2$

$= \left(x^2 + \frac{a}{2}xy + y^2\right)^2 + \alpha x^2y^2$

If $b > \frac{a^2+8}{4}$ we have $\alpha > 0$ and in this case P(x,y)>0 for all positive values of x and y.

If $b \leq \frac{a^2+8}{4}$, or in other words $\alpha \leq 0$, we can write $\alpha = -\beta\ with\ \beta \geq 0$

P(x,y) = $\left(x^2 + \frac{a}{2}xy + y^2\right)^2 - \beta x^2y^2$

$= (x^2 + \frac{a}{2}xy + y^2 + \sqrt{\beta}xy)(x^2 + \frac{a}{2}xy + y^2 - \sqrt{\beta}xy)$

$= \left(x^2 + xy\,(\frac{a}{2} + \sqrt{\beta}) + y^2\right)\left(x^2 + xy\,(\frac{a}{2} - \sqrt{\beta}) + y^2\right) > 0$

Let us denote

$$x^2 + xy\,(\frac{a}{2} + \sqrt{\beta}) + y^2 = P_1\,(x,y)$$

$$x^2 + xy\,(\frac{a}{2} - \sqrt{\beta}) + y^2 = P_2\,(x,y)$$

and

$$Q_1(t) = P_1\,(t,1)$$

$$Q_2(t) = P_2\,(t,1)$$

Q_1 and Q_2 have the same sign on IR_+. Thus Q_1 and Q_2 share the same positive roots. Q_1 and Q_2 are of degree 2 and have the product the free coefficient equal to 1, so, in particular, both roots of each of them have the same sign and thus either $Q_1 = Q_2$, or they do not nullify on IR_+..

We have :

$$\begin{cases} Q_1(t) > 0\ (IR_+) \\ Q_2(t) > 0\ (IR_+) \end{cases}$$

Or

$$\begin{cases} Q_1(t) < 0 \; (IR_+) \\ Q_2(t) < 0 \; (IR_+) \end{cases}$$

In fact the second possibility is in fact impossible because the dominant term of Q_1 and Q_2 is positive.

[$T(x,y) = x^2+cxy+y^2$]

It is easy to verify that $T(x,y) > 0 \; (IR_+^2) \Leftrightarrow c>-2$

We obtain:

$$\begin{cases} \frac{a}{2} + \sqrt{\beta} > -2 \\ \frac{a}{2} - \sqrt{\beta} > -2 \end{cases}$$

$$\Rightarrow \begin{cases} a \geq -4 \\ \frac{a}{2} - \sqrt{\beta} > -2 \end{cases}$$

$$\sqrt{\beta} = \sqrt{-\alpha} = \sqrt{\frac{a^2+8}{4} - b} \qquad (in\ case\ \alpha < 0)$$

Thus if $b \leq \frac{a^2+8}{4}$, we necessary have $a \geq -4$ and at the same time we have demonstrated that the positivity implies b>-2a-2.

(b) Let us show that :

$$P(x,y) > 0 \; (IR^{*2}) \Leftrightarrow b > \frac{a^2+8}{4}$$

Or

$$|a| \leq 4 \quad b > 2|a| - 2$$

$$P(x,y) > 0 \; (IR^{*2}) \Leftrightarrow \begin{cases} P(x,y) > 0 \; (IR_+^2) \\ P(x,y) > 0 \; (IR_+ \times IR_-) \quad (P(x,y) > 0 \; (IR_+ \times IR_-) \Leftrightarrow P(x,y) > 0 \; (IR_- \times IR_+) \\ P(x,y) > 0 \; (IR_-^2) \end{cases}$$

$$\Leftrightarrow \begin{cases} P(x,y) > 0 \; (IR_+^2) \\ P(x,-y) > 0 \; (IR_+^2) \\ P(-x,-y) > 0 \; (IR_+^2) \end{cases}$$

In fact if $b > \frac{a^2+8}{4}$ then in all the cases above we have a sufficient condition for P to be positive. If that is not true than the positivity is equivalent to the système of two following inequalities :

$$a \geq -4\,, b > -2a - 2$$

$$-a \geq -4\,, b > -2a - 2$$

Which is in its turn equivalent to the condition

$$[|a| \leq 4 \quad and\; b > 2|a| - 2\,].$$

On the other hand, if P(x,y) is positive for all values (x,y) from IR^{*2}, again in view of point (a) of this question, the part IR_+^2 gives us the condition that either $b > \frac{a^2+8}{4}$ or $a \geq -4\,, b > -2a - 2$,

and the part $IR_+ \times IR_-$ adds to it that either $b > \frac{a^2+8}{4}$ or $-a \geq -4\,, b > -2a - 2$. So finally we get the equivalence we was going to demonstrate.

3.

P(x,y) = $x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5$

$=(x + y)(x^4 + (a - 1)x^3y + (b + 1)x^2y^2 + (a - 1)xy^3 + y^4)$

In view of Q2(a) we have the following criterion :

$$P(x,y) > 0\ (IR_+^2) \Leftrightarrow \left[\begin{array}{l} \boldsymbol{b > \frac{(a-1)^2+4}{4}} \\ \begin{cases} \boldsymbol{a \geq -5} \\ \boldsymbol{b > -2a - 1} \end{cases} \end{array}\right.$$

P(x,y) = $x^5 + ax^4y + bx^3y^2 + bx^{2y^3} + axy^4 + y^5 > 0\ (IR_+^2)$

$$\Leftrightarrow (x^4 + (a-1)x^3y + (b - a + 1)\, x^2y^2 + xy^3(a-1) + y^4 > 0\ (IR_+^2)$$

With the analogous factorisation we can deduce the question about positivity of

$$P(x,y) = x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7$$

on IR_+^2 to the same question for a symmetric polynomial of degree 6.

Let us concider the polynomial

$$P(x,y) = x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6.$$

Evidently we can rewrite it as

$$P(x,y) = (x^3 - y^3)^2 + axy(x^4 + \frac{b}{a}x^3y + \frac{c+2}{a}x^2y^2 + \frac{b}{a}xy^3 + y^4).$$

If a is positive, then and the polynomial of the fourth degree is always positive (for this we have a criterion from point 2) it gives us the necessary condition for positivity of P(x,y).

We shall explain now the general strategy which we shall apply for polynomials of even degree in the point 5.

First of all let us define $Q(t) = P(t, 1)$. It is easy to see that positivity of $P(x, y)$ on IR_+^2 is équivalent to positivity of $Q(t)$ for t>0, and the same for semi-positivity (≥ 0). Now, let us consider the space of coefficients of polynomial $Q(t)$. It is clear that this polynomial is positive for t>0 if and only if it is strictly increasing for this domain or if it is strictly positive at all its local minima. Local minima being roots of polynomial $Q'(t)$, by the well-known theorem they depend continuously on the coefficients of $Q'(t)$, and sot they depend continously on the coefficients of $Q(t)$. So, if for some values of parameters the polynomial $Q(t)$ is positive, it rests positive for a small changes of these parametres. So, the set S of coefficients of positive polynomial is open in the space of all coefficients with evident topology and we can study its borne (in fact it will be defined with the equalities which corresponds to the inequalities for coefficients we are looking for).

For a point at a borne of S we have the following situation : it gets value zero for some $t_0 > 0$ but never gets negative value. Furthemore, the polynomial $Q(t)$ is reciprocal (or «symmetric ») one, so the value $\frac{1}{t_0}$ is also a root of $Q(t)$. Now we have two possibilities, either $t_0 = 1$, either $t_0 \neq \frac{1}{t_0}$. In the first case $Q(t)$ is divisible by $u(t) = t - 1$, and in the second by $v(t) = t^2 + at + 1$. Let us denote k the maximal integer k such that $Q(t)$ is divisible by the mentioned polynomial to the power k. Then in the case of $u(t)$ k is pair and in the case of $v(t)$ either k is pair or $v(t)$ is positive for all t>0. In the both cases k-th power is symmetric and so the rest of $Q(t)$ is also symmetric (and of the smaller degree). It immediately gives us the estimations analogous to those of the power 4.

For the point (b), it is impossible for the polynomials

$$P(x,y) = x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5$$

and

$$P(x,y) = x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7$$

To be positive for all the non-zero values of variables (x,y). That is so just because we can substitute, for example, y=1 obtaining a polynomial in x of odd degree which certainly gets negative values. That is the same for all the symmetric polynomials of negative degree, which partially answers the questions 4 and 5.

For the polynomial P(x,y) = $x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6$ the question of point (b) is reduced by the same procedure as in point 2 to the case of point (a).

4. **a)** It is easy to verify that if all the coefficients of a symmetric polynomial P are positive (or even non-negative if we presuppose that the coefficients of monomials x^{degP} and y^{degP} equals to 1 ; in fact we always presuppose it implicitly in this texte following the spirit of the statement, if the contrary is not stated explicitly), it always gets strictly positive values for $(x, y) \in IR_+^2$.
b) If the degree of polynomial is odd, it always gets values of both signs (easy to see just fixing the value y=1). For the even degree n=2k the sufficient condition is the système of inequalities (for polynomial P(x,y)= $x^7 + a_1x^6y + a_2x^5y^2 + a_3x^4y^3 + \cdots + a_1xy^6 + y^7$) : $a_i > \binom{i}{n}$ for the same reasons as presented in the previous points (just direct application of inequalities and afterthat a Newton's formula). To get finer sufficient condition we can consider Newton's formula for the n'th power of a linear form with negative coefficients, so permitting some coefficients to be negative still preserving the positivity of the symmetric polynomial.
5. We have already presented sufficient condition. To get necessary conditions we can specialize our variables x,y. For example just substitutins x=y=1 gives us the following necessary condition of positivity with $(x, y) \in IR_+^2$: the sum of all polynomials should be greater than -2. Furthermore, we have a necessary condition for positivity with $(x, y) \in IR^2$ that the degree of polynomial should be even. Other necessary conditions we obtain just with grouping terms in symmetric way to obtain many binomials of Newton.
Furthemore, we can always factor out (x+y) for all the symmetric polynomials of odd degree deducing the question of it positivity with $(x, y) \in IR_+^2$ to the same question of polunomial of degree 1 lower.
For the polynomial of even degree we do the same procedure as described in the point 3 for polynomial of the 6th degree, obtaining the necessary estimates.