Placements of Pentominoes

June 25, 2009

1)

We are going to prove that T(m,n) obeys in the general case to the estimation :

$$\frac{m \cdot n}{14.8} \leqslant T(m,n) \leqslant \frac{m \cdot n}{12}$$

(the lower bound is here asymptotic, while the upper bound is exact)

Upper bound

We can suppose $m \geqslant n$

A) If $n < 6$

An upper bound can be obtained by covering the rectangle $m \times n$ with respect to a one-dimensional regular pattern. Examples are given here (fig. 1) :

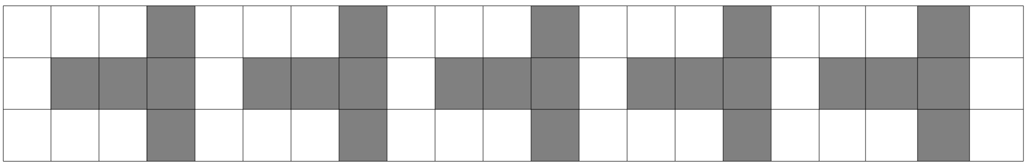


FIGURE 1-1 : $T(m,3) = \left\lceil \frac{n-2}{4} \right\rceil$ (considering the n-2 center cells, a pentomino cannot empeach more than 4 of them to be the center of an other pentomino)

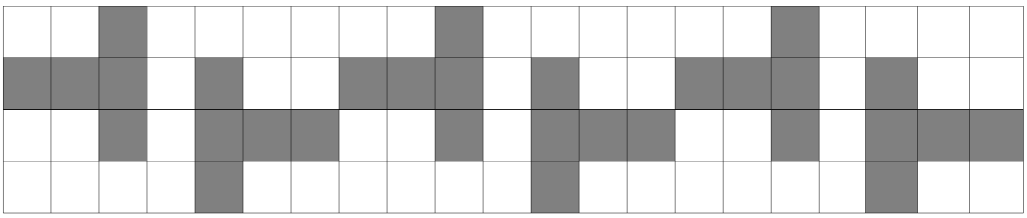


FIGURE 1-2 : with n=4

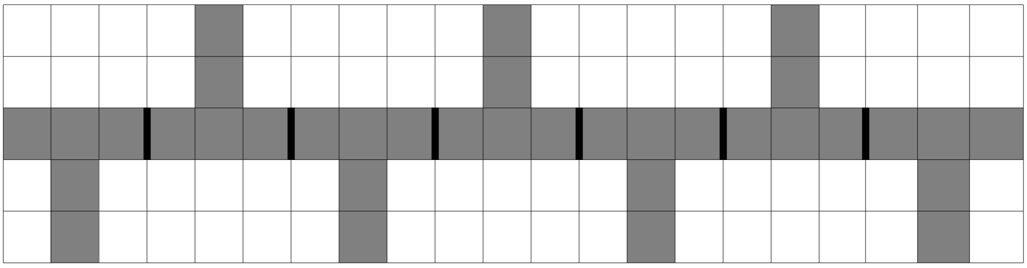


FIGURE 1-3 : with n=5

B) If $n \geqslant 6$

An upper bound can be obtained by covering the rectangle m*n with respect to a two-dimensional regular pattern.
Here is a first example : we denote the pattern by (P1)

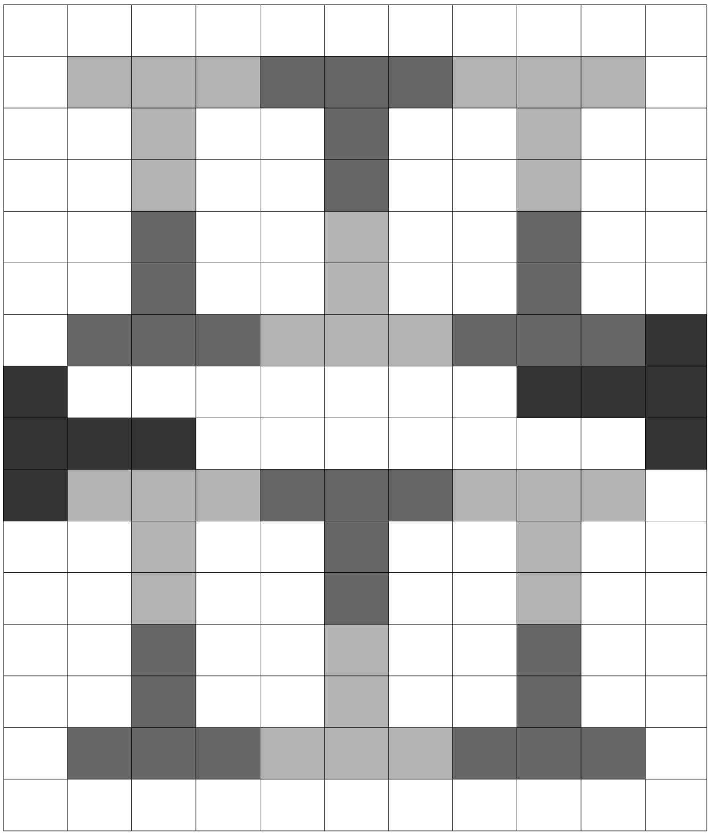


FIGURE 2 : $T(11,16) \leq 14$ (several colours are used to distinguish beetween pentominoes. The pattern (P1) is composed with the vertical pentominoes)

In the pattern P1, each pentamino is placed in a 4×3 rectangle. We see that covering with respect to (P1) provides an upper bound of :

$$T(m,n) \leqslant \frac{m \cdot n}{12}$$

(a table can be written to establish it properly according to the congruences of m and n modulus 3 and 4)

Another one (P2) with the same density of covered cells :

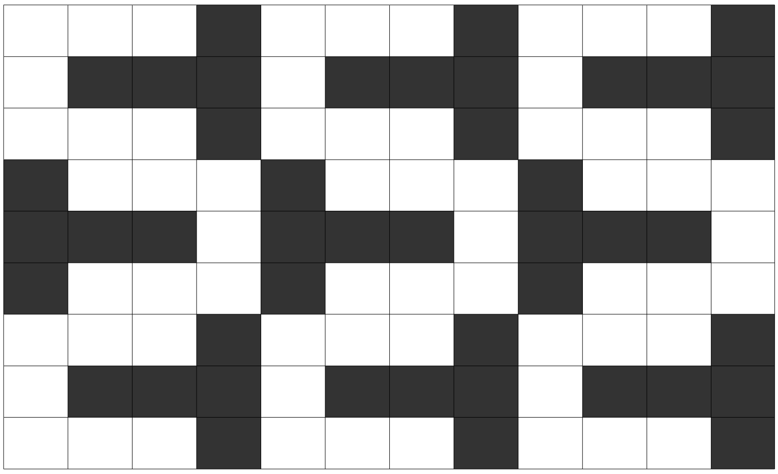


FIGURE 3 : $T(9\ ,\ 12)\ \leqslant\ 9$ Pattern (P2)

Lower Bound

Let R be a m*n rectangle, with $m \geqslant 6$ and $n \geqslant 6$. We are going to count, for each cell in the i-th row and j-th column, the number $C_{i,j}$ of T-pentominoes that we could place on the board, covering this cell. On a « central cell » (ie with a length of at least three cells to the nearest border), this number is 4*5 =20.
Indeed, there are 4 possibilities when choosing the direction of the pentamino, and 5 when choosing the place of the cell in the pentomino (for example, the center). Near the borders and the corners, this number is smaller, but in fact very few particular cases are to be encountered. They are all presented in the following figure (fig.3) :

...	...	...	...	...
6	14	20	20	...
6	14	20	20	...
4	10	14	14	...
2	4	6	6	...

FIGURE 4 : corner (green), borders (yellow) and center (white) cells

Once the values are known on each cell, we sum up all the results. We obtain :

$$S_{m,n} = \sum_{i,j} C_{i,j}$$

Separating center, borders and corners, we have :

$$S_{m,n} = 20(n-4)(m-4)+(6+14)\cdot 2(n+m-4)+4(10+4+4+2)$$

Finally,

$$S_{m,n} = 20\cdot n\cdot m-40\cdot(n+m)-80$$

Let's consider a pentomino T_0 placed on the empty board. On every cell in the neighborhood of T_0 (that is, in the blue area representes in fig.4), we count the number of T-pentominoes that could have been placed on the board, passing on the cell and the pentomino « reference ». This work leads to fig.4

0	2	5	6	5	2	0
2	5	12	12	12	5	2
4	9	20	20	20	9	4
2	8	15	20	15	8	2
0	6	12	20	12	6	0
0	2	5	12	5	2	0
0	0	2	4	2	0	0

FIGURE 5 (the pentomino T_0 appears in the center, its neighborhood in blue. The values $C_0(i,j)$ are written in each cell)

Then we sum up the values on each cell and we obtain :

$$S_0 = \sum_{i,j} C_0(i,j) = 316$$

We call this quantity the weight of the T-pentomino.

Now, we consider a minimal configuration, with exactly T(m,n) pentaminoes placed on the board such that there is no room for any other pentomino. We denote the pentominoes by $T_1, T_2, \ldots, T_{T(m,n)}$. We can place them successively on the board. For each pentamino T_r placed on the board, we add the values $C_r(i,j)$ to the cells.
At the end of the process, we have necessarily, on each cell :

$$\forall (i,j) \in [\![1;n]\!] \times [\![1;m]\!] \ , \ S(i,j) = \sum_{k=1}^{T(m,n)} C_k(i,j) \geqslant C_{i,j}$$

(if not, one more pentomino could be placed)

Also :

$$S_0 \cdot T(m,n) \geqslant \sum_{i,j} S(i,j) \geqslant \sum_{i,j} C_{i,j} = S_{m,n} \qquad (1)$$

(The left inequality comes from the fact that some pentaminoes are on the borders, and so that their whole neighborhood and weight S_0 are not on the rectangle's cells)

Hence, we get the lower bound :

$$T(m,n) \geqslant \frac{S_{m,n}}{S_0} = \frac{20 \cdot n \cdot m - 40 \cdot (n+m) - 80}{316}$$

Which means asymptotically :

$$T(m,n) \geqslant \frac{m \cdot n}{15.8}$$

But we can still improve this lower bound. Looking again at the fig. 4, we observe that 12 diffferent T-pentominoes can be placed entirely in the neighborhood of T_0 . They are represented in the fig.5

		2	5	6	5	2		
	2	5	11	12	11	5	2	
	4	9	20	20	20	9	4	
	2	7	15	20	15	7	2	
		6	12	20	12	6		
		2	5	12	5	2		
			2	4	2			

		2	5	6	5	2		
	2	5	11	12	11	5	2	
	4	9	20	20	20	9	4	
	2	7	15	20	15	7	2	
		6	12	20	12	6		
		2	5	12	5	2		
			2	4	2			

FIGURE 6 : pentamino-shaped areas in the neighborhood of T_0 (only 6 are represented, but the 6 other can be obtained by symetry with respect to the vertical axis passing through T_0)

Now, if T_r is a pentamino of a convenient configuration (ie such that there is no more room), each coloured area of the fig.5 must have one of its cells (i_0 , j_0) covered by a pentamino $T_{r'}$, strictly different from T_r and making part of the configuration (if not, an additional pentomino could be placed on the coloured area, which is absurd because the configuration is already convenient)

$$S(i_0 , j_0) \geqslant C_r(i_0 , j_0) + C_{r'}(i_0 , j_0) = C_r(i_0 , j_0) + 20 \geqslant 22 \geqslant C_{i,j} + 2$$

(which will allow us to get a narrower lower bound)

We call the cells of the (i_0 , j_0) type N-cells. A N-cells set is said convenient if each coloured area is satisfied (ie, owns a N-cell) Several possibilities can be considered in order to minimize the number and especially the weight (sum of the quantities $C_r(i,j)$) of N-cells in the neighborhood of T_r . One N-cell (i,j) can make part of several coloured aeras (until 4), but if it does, its weight $C_r(i,j)$ according to T_r will be higher than if is situated on a border of the neighborhood, as shown in fig. 6, which repertories the number of coloured aera per cell in the neighborhood of T_r . We will denote generaly by W the weight of a convenient set of N-cells.

	1	0	0	0	1	
1	2	2	2	2	2	1
0	2				2	0
1	3	3		3	3	1
	1	1		1	1	
	1	3	4	3	1	
		1	0	1		

FIGURE 7 : Number of coloured aera per cell in the neighborhood of T_r . We can see that this number is higher near the pentamino T_r , like the quantities $C_r(i,j)$. This explains the difference between weight and number optimisation [of the N-cells]

•<u>Weight optimisation</u> : The N-cells are all the furthest ones (see fig.7 below)

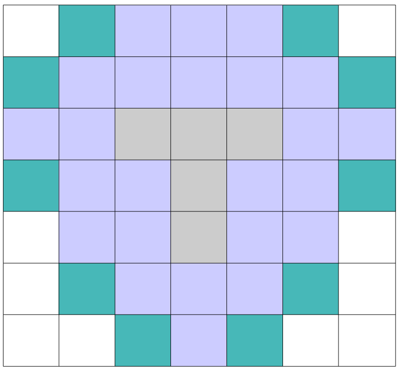


FIGURE 8 : The N-cells (green) of the weight-optimisation

In the weight optimisation, we have : $W = \sum_{p=1}^{10} C_r(i_p, j_p) = 10 \times 2 = 20$ (where W is the weight of the N-cells)

We deduce from there that the constant $S_0 = 316$ acting in the lower bound can be replaced by :

$$S'_0 = S_0 - W = 296$$

Providing a new asymptotic lower bound :

$$T(m,n) \geqslant \frac{m \cdot n}{14.8}$$

(only asymptotical)

(note that the two last relations, even if established from the particular weight optimisation, apply in all cases)

Consequences of the weight optimisation in the configuration :
Given the positions of N-cells of the weight optimisation (fig.7), we can look to the possibilities placement of pentominoes around T_r , in order to cover them all. It would not be interessant to use a single pentomino for each N-cell, because they would be to close from each other. Even with this reduction, the possibilities are too numerous to be all considered.
However, we can remark that starting from a convenient N-cells set very close to the weight optimisation (fig.8) leads to the pattern (P2) presented in the first part (upper bound)

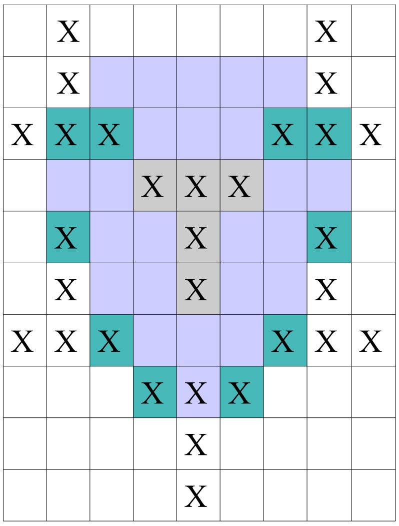


FIGURE 9 : reconstitution of the placement of pentominoes from the positions of N-cells leading to the pattern (P2). The cells marked with an X are surrounded by a pentomino.

•<u>Number optimisation</u>

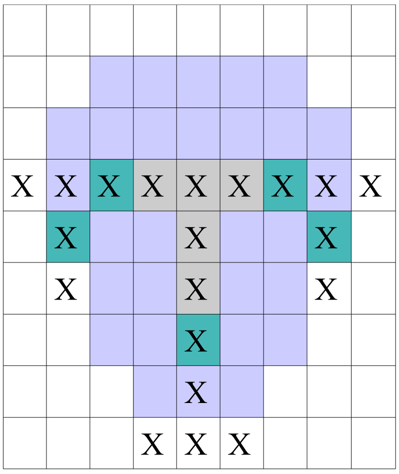


FIGURE 10 : N-cells of a number optimisation, and reconstitution of the placement of pentominoes, leading to the pattern (P1)

2)
There is always a winning strategy for one of the two players. We will design by J1 the first player, and by J2 the other one.

Indeed, this play has three characteristics which insure the existence of a winning strategy for one of the two players:

1 - There is no chance or uncertainty (each player knows all the data, there is not a “hidden card” or another uncertainty of the kind)

2 - The two only exits of the game are “J1 victory” and “J2 victory”

3 - The game can’t be prolonged indefinitely (for a tray of N boxes, the part is inevitably finished before n/5 placements)

There is thus a winning strategy: since certain positions are losing at once for the player who receives them (when one can't place pentamino any more) the positions which lead (it/there) are winning for the one who receives them.The positions forcing to give a winning position to the other one are losing, etc. In the end, we denote all the positions: J1 has a winning strategy if the starting position is winning and J2 otherwise.

•If n = 3 or n = 4 and m is odd : J1 has a winning strategy. We can write $m = 2k - 1$. J1 places a pentomino on the k-th column, folding the board into 2 symetrical parts. Then, J1 answers to each placement of J2 symetrically. When J2 plays, the number of pentominoes that can still be placed on the board will never be of 1 (because of the symetry). Thus, J1 has a winning strategy.

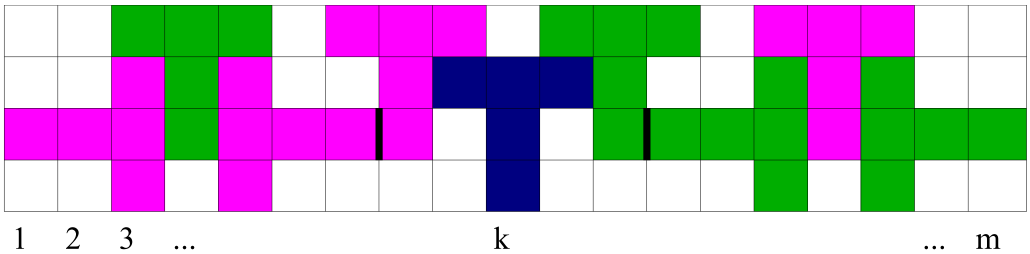


FIGURE 11 : A winning strategy in the case (n=4, m odd). The blue pentomin is placed by J1 in the first turn.

•If n = 3, We will define the two sets S_1 and S_2 such that :
for all $m \in \mathbb{N}$, $m \in S_1 \Leftrightarrow$ J1 has a winning strategy on the 3*m board (resp. S_2)
We have $S_1 \cup S_2 = \mathbb{N}$
If J2 has a winning strategy on the 3 times m board, then, on the 3 times (m+3) board, J1 can play at the first turn in order to reduce the length of the board of 3 columns (see fig.11). He is the second player on this new boardof length m, and has a winning strategy. This works with m+4 too.

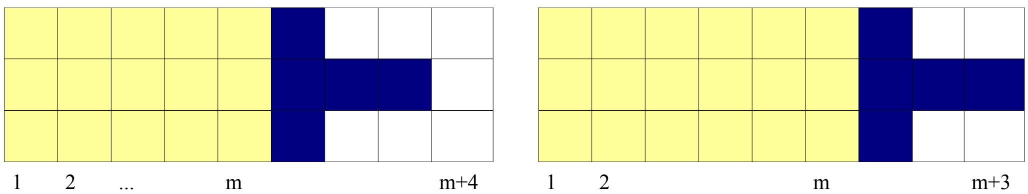
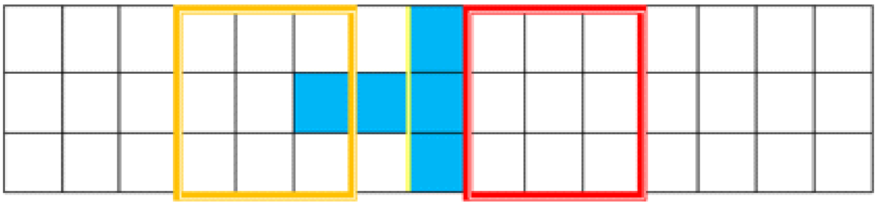


FIGURE 12 (the first pentomino placed by J1 is in blue)

Hence, the two implications :
$m \in S_2 \Rightarrow m + 3 \in S_1$ (1)
$m \in S_2 \Rightarrow m + 4 \in S_1$ (2)
In particular, we deduce from (2) that, for an even number m :
$m \in S_2$, $m = 2k \Rightarrow m + 4 \in S_1$ and m + 4 is even.
J1 has then a winning strategy in more than one 3/4 of the cases (always when m is odd, and more than one time in two when m is even)

Variant: *We suggest seeing the even case, the demonstration isn't complete. We want to show that the first player wins.*

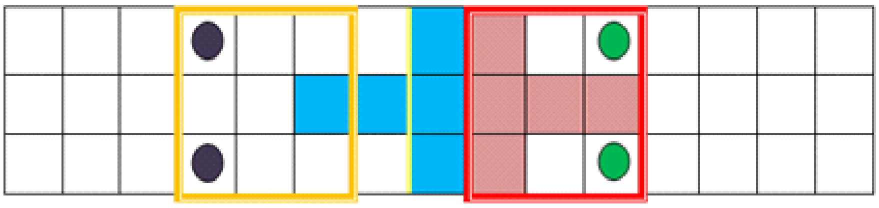
FIGURE 13:



The yellow line indicates the environment of the field 3x2k. The first one begins by putting his pentamino as it is shown above. Then he is going to play by a vertical symmetry (i.e.to put all these pentaminos **symmetrically with regard to the vertical line** which passes in the middle of the field) or a **central symmetry** (i.e to put all these pentaminos by a central symmetry with regard to the center of the field; two strategies differ by orientation of pentamino vertical).

We have an only *small* problem, squares by-passed by red and orange are not completely symmetric.

FIGURE 14:



More exactly, if the second player puts his pentamino in " pink position ", we can't put symmetrically. Nevertheless, fact that the second player can put the pink pentamino implies that at least one of the positions marked by the green circle is free, thus it doesn't much matter which strategy of a symmetry use the second player, at least one of the stone floors marked by a blue circle still free.

FIGURE 15:

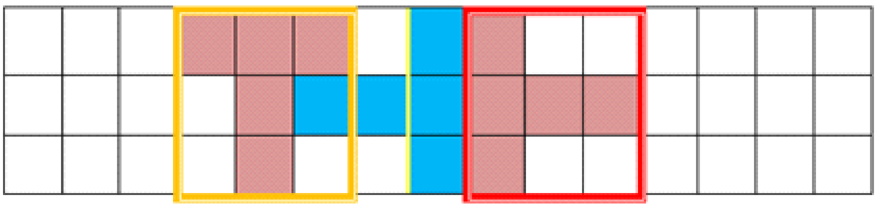
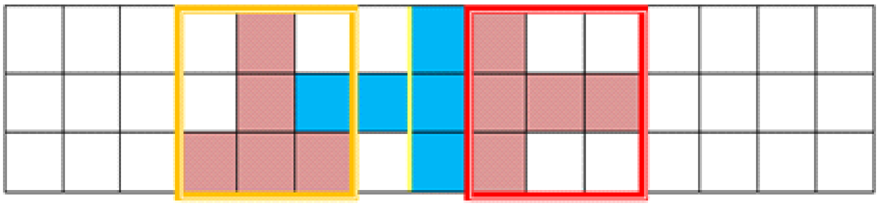
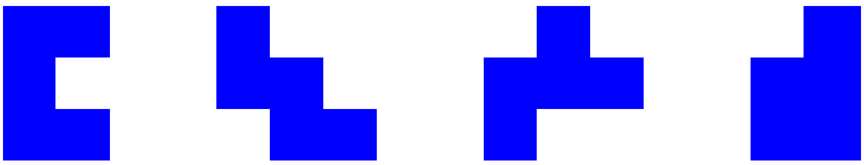


FIGURE 16:



If two " stone floors with green circles " are still free and can be to use by the second player, after our combination in the center, the second player has the freedom to choose the orientation of pentamino to block us. Besides, the first one can choose either to play by a vertical symmetry or by a central symmetry, which gives exactly the same effect of choice of orientation. But it is not evident that the second can't " make the fork ", that is by playing pentaminos far from center to provoke the first one to choose a symmetry in such a way as after this doesn't allow the first player to pose the pentamino close of the center.

3)
We have studied the 4 following pentominoes :
The methods used in 2 still apply with them.



C(m,n) W(m,n) R(m,n) D(m,n)

For the Pentominoes R and D, that are not symmetric, two questions are possible :
1- How many (R or D) pentominoes must be placed <u>with the right to return them</u> so that there is no place on the free cells for an other one ?
2 -How many (R or D) pentominoes must be placed <u>without the right to return them</u> so that there is no place on the free cells for an other one ?

In those cases, we have chosen here the first question.

All the uppers and lower bounds given are only asymptotic.

•<u>Pentominoes W</u>

Lower bound

	2	3	1	1	3	2		
	1	6	10	10	6	4	2	
		8	20	20	13	6	3	
	1	6	13	20	20	10	1	
	2	4	6	13	20	10	1	
		2	4	6	8	6	3	
			2	1		1	2	

FIGURE 17 : $S_0 = 282$

The figure 13 is an analog of the fig.5 in the case of the W pentomino. The pentomino appears in the middle, its neighborhood in blue. This time, we have $S_0 = 282$, and the lower bound (asymptotical) is :

$$W(m,n) \geqslant \frac{m \cdot n}{14.1}$$

Upper bound

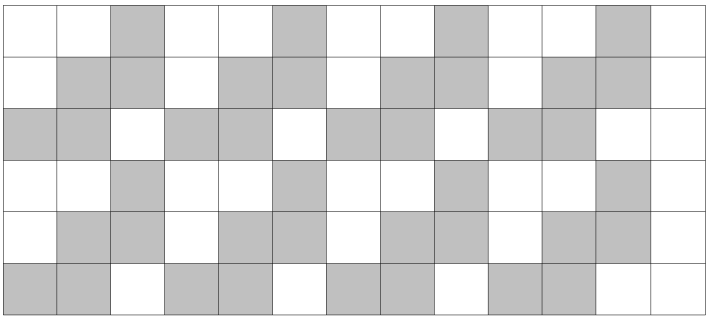


FIGURE 18 : $W(13{,}6) \leqslant 8$

This pattern provides the upper bound : $W(m,n) \leqslant \frac{m \cdot n}{9}$

•<u>Pentominoes C :</u>

Lower bound

		2	4	4	2		
	2	6	11	10	5	2	
	4	11	20	20	10	4	
	4	11	20	18	9	4	
	4	11	20	20	10	4	
	2	6	11	10	5	2	
		2	4	4	2		

FIGURE 19 : $S_0 = 306$

Upper bound

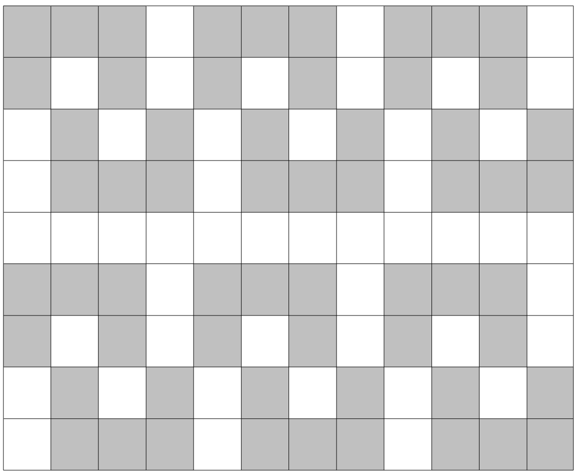


FIGURE 20 : $C(9,12) \leqslant 12$

Conclusion : $\frac{m \cdot n}{15.3} \leqslant C(m,n) \leqslant \frac{m \cdot n}{12}$

•Pentominoes D

Lower bound

0	2	5	5	2	0
2	8	16	16	8	2
5	13	40	40	16	5
6	13	40	40	16	5
5	13	40	32	8	2
2	9	10	9	3	0
0	2	4	2	0	0

FIGURE 21 : $S_0 = 446$

The figure 13 is an analog of the fig.5 in the case of the W pentomino. The pentomino appears in the middle, its neighborhood in blue. This time, we have $S_0 = 446$, and the lower bound (asymptotical) is :

$$W(m,n) \geqslant \frac{m \cdot n}{11.3}$$

Upper bound

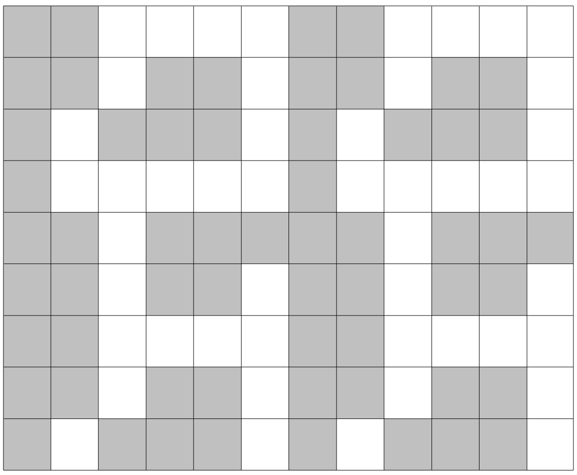


FIGURE 22 : $W(9,12) \leqslant 12$

This pattern provides the upper bound : $D(m,n) \leqslant \frac{m \cdot n}{9}$

•Pentominoes R

Lower bound

0	0	0	3	6	6	3
0	3	11	14	14	7	3
3	11	18	40	40	12	4
4	12	40	40	26	11	2
3	11	18	40	19	3	0
0	3	11	12	5	2	0
0	0	3	4	3	0	0

FIGURE 23 : $S_0 = 498$

The figure 13 is an analog of the fig.5 in the case of the W pentomino. The pentomino appears in the middle, its neighborhood in blue. This time, we have $S_0 = 498$, and the lower bound (asymptotical) is :

$$R(m,n) \geqslant \frac{m \cdot n}{12.9}$$

Upper bound

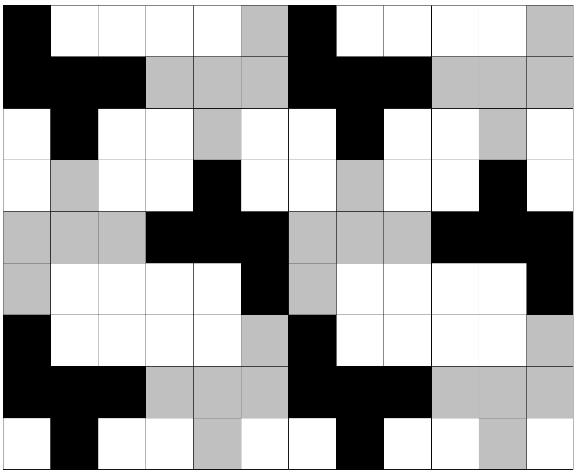


FIGURE 24 : $R(9,12) \leqslant 12$

This pattern provides the upper bound : $R(m,n) \leqslant \dfrac{m \cdot n}{9}$