Problem 6: PATTERN GRAPHS

June 25, 2009

Some important notations [First part]:

- We use all the notations of the statement
- We denote by u, v, w … the patterns

Let u be a pattern of length n, we also write:

$$u = \boxed{\begin{matrix}\sigma_u \\ \sigma'_u\end{matrix}} = \boxed{\begin{matrix} u_1 & u_2 & u_3 & \dots & u_n \\ u'_1 & u'_1 & u'_3 & \dots & u'_n \end{matrix}}$$

With σ_u and σ'_u which are permutations of the set $[\![1, n]\!]$

- A connected component of the graph G_n is generally denoted by K, L, M …
  We write k = Card K

Question 1)

A. We denote : $[a, b] = a^{-1}b^{-1}ab$

$(a, b) \xrightarrow{A} [ba, b]$

$a^{-1}b^{-1}ab \qquad [(ba)^{-1}b^{-1}(ba)b] = \quad a^{-1}b^{-1}ab$

And

$(a, b) \xrightarrow{B} [a, ab]$

$a^{-1}b^{-1}ab \qquad [a^{-1}(ab)^{-1}(ab)a] = \quad a^{-1}b^{-1}ab$

[a,b] is an invariant for the two applications A and B on the patterns.
We show that the permutation [a,b] associated to the first pattern is different from the second one , that provides us the proof that it is impossible to obtain the first pattern from the second. We denote u the permutation associated to the first pattern and v the permutation for the second pattern.
$u_1 = a^{-1}b^{-1}ab^1 = a^{-1}b^{-1}a_2 = a^{-1}b^{-1}{}_1 = a^{-1}{}_n = n$
$v_1 = a^{-1}b^{-1}ab_1 = a^{-1}b^{-1}a_2 = a^{-1}b^{-1}{}_3 = a^{-1}{}_2 = 1$
$u_1 \neq v_1$ therefore the answer is no.

B.

Theorem: Any permutation of S_n may be written as products of transpositions[1]

We denote sgn(s) the number $(-1)^A$, with A a number of transpositions of decomposition in produces of transpositions. (we agree $\text{sgn}(id) = 1$)

sgn $(S_n, 0) \to (\{-1,1\})$ is a morphism of groups[2]

s sgn(s)

$$* \ \text{sgn}(ss') = \text{sgn}(s)\ \text{sgn}(s')$$

sgn(s) is called sign of σ

We have for any p-cycle σ, $\text{sgn(s)} = (-1)^{p-1}$

Indeed, $s = [a_p, a_{p-1,} \dots, a_2, a_1] = [a_p\ a_{p-1}][a_{p-1}\ a_{p-2}] \dots [a_2\ a_1]$

Let v= $\boxed{\begin{matrix} s_u \\ s'_u \end{matrix}}$ be a pattern

The operation A consists in considerer a= $s(i)$, then in taking the number of the second line which is in the place.

$\sigma(i)$ that is $s'(s(i))$, ie $s's(i)$

Thus A(v) = $\boxed{\begin{matrix} s's \\ s' \end{matrix}}$

also B(v) =$\boxed{\begin{matrix} s \\ ss' \end{matrix}}$

From there, $A^{-1}(v) = \boxed{\begin{matrix} s'^{-1}s \\ s' \end{matrix}}$ $et\ B^{-1}(v) = \boxed{\begin{matrix} s \\ s^{-1}s' \end{matrix}}$

We notice that if sgn(s) and sgn(s') are equal in 1, so sgn(s's) and sgn(s′) are equal in 1, because $\text{sgn}(s's) = \text{sgn}(s)\ \text{sgn}(s')$.

By calling an even permutation a permutation of signature 1 (and odd otherwise), we can't have in the same connected component $\boxed{\begin{matrix} even \\ even \end{matrix}}$ and $\boxed{\begin{matrix} odd \\ even \end{matrix}}$ or $\boxed{\begin{matrix} odd \\ odd \end{matrix}}$ or $\boxed{\begin{matrix} even \\ odd \end{matrix}}$.

If n is odd, then $\text{sgn}([1,2,\dots,n]) = (-1)^{n-1} = 1$

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[1] Refer to the appendix D[1]
[2] Refer to the appendix D[2]

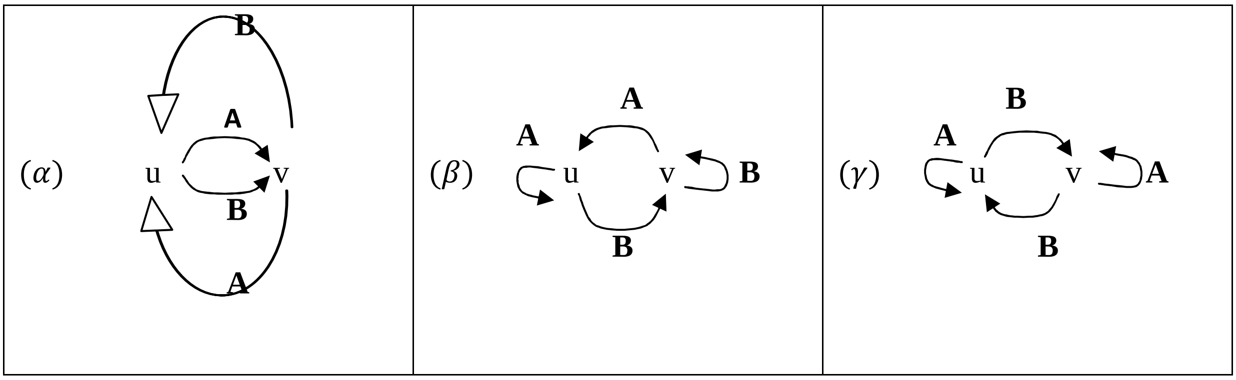
Then, n= $\boxed{\begin{matrix} even \\ even \end{matrix}}$ and u'= $\boxed{\begin{matrix} odd \\ even \end{matrix}}$ u et u' are not in the same connected component of G_n

2) (a)

Let K be a connected component of G_n such as k=2.

We denote u_1 and v_7 both motives of K.

Three arrangements are possible for K :



*) Let us suppose (α) or (β)

$$\begin{cases} u \xrightarrow{A} v \Rightarrow \sigma'_u\sigma'_v \\ u \xrightarrow{B} v \Rightarrow \sigma_u\sigma_v \end{cases}, so\ (\alpha) \text{ or } (\beta) \Rightarrow\ u = v\ and\ k = 1\ \ which\ absurd$$

*) Let us suppose (γ)

$$A \circlearrowleft u \ \Leftrightarrow\ u = \boxed{\begin{matrix} u_1 & u_1 & u_1 & \cdots & u_n \\ 1 & 2 & 3 & \ldots & n \end{matrix}}$$

$$A \circlearrowleft v \ \Leftrightarrow\ v = \boxed{\begin{matrix} v_1 & v_1 & v_1 & \cdots & v_n \\ 1 & 2 & 3 & \ldots & n \end{matrix}}$$

Furthermore,

$u \xrightarrow{B} v \Rightarrow \sigma_u = \sigma_v$ so u=v and k=1 which absurd

We deduce that from it: no connected component of the graph G_n has exactly 2 patterns. Furthermore, the only connected component containing a single pattern is the one formed by

$u_0 = \boxed{\begin{matrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \ldots & n \end{matrix}}$ (We have indeed $A \circlearrowleft u_0 \circlearrowright B$)

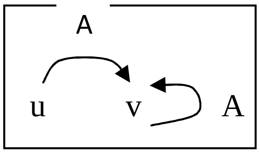
We know that G_n consists of $(n!)^2$ patterns, so we obtain a first rise of g_n:

$\forall n \geq 2, g_n \leq \left\lfloor \frac{(n!)^2 - 1}{3} \right\rfloor$, où $\lfloor x \rfloor = E(x)$ integer part

(b) *b.1*

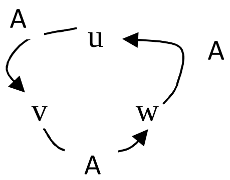
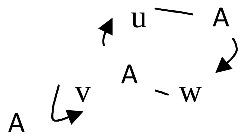
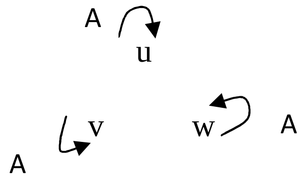
Let K be a connected component of G_n such as k=3. We denote u,v and w the three patterns of K (Different). Let us enumerate possible dispositions for K.

The applications A and B being reversal, it is impossible to have



(for example)

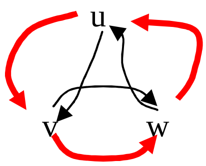
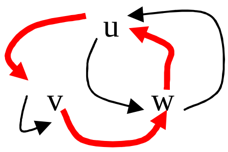
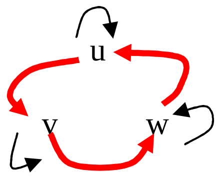
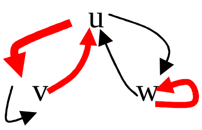
By creating only a single application (A for example), the dispositions of K are all of the shape:

(α) 	*Or* (β) 	*Or* (γ) 

(one more or less a pattern)

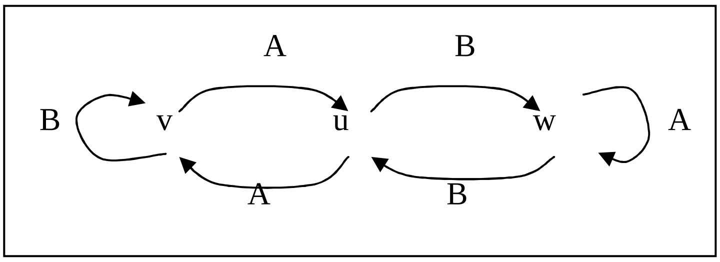
By creating both applications A and B, the dispositions thus appear as 3 ² = 9 possible forms.

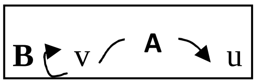
Really, some are absurd (we argue as in the question 2 (a)). The table lists the cases:

Form of B[3] / Form of A	α	β	γ
α	 *Absurd* u=v=w	 *Absurd* u=v=w	 *Absurd* u=v=w
β		 *Possible*	*Absurd* *No connected*
γ			*Absurd* *No connected*

[3] Remark :  means A, and  means B. We so improve the legibility

All the connected components of size 3 are built as the following arrangement:



As we have 

(for example), the pattern u is such as $u = \bar{u}$. Furthermore, we have $w = \bar{v}$

We must have $\forall k \in [\![1, n]\!], u_{u_k} = k$

➢ What are the permutations which agree?

☞All the patterns $u = \begin{bmatrix} a-1 & a-2 & \cdots 2 & 1 & n & n-1 & \cdots & a \\ a-1 & a-2 & \cdots 2 & 1 & n & n-1 & \cdots & a \end{bmatrix}$ agree (with $a \in [\![1, n]\!]$)

☞If n is even, $u = \begin{bmatrix} 1+\frac{n}{2} & 2+\frac{n}{2} & \cdots & n & \cdots & \frac{n}{2} \\ 1+\frac{n}{2} & 2+\frac{n}{2} & \cdots & n & \cdots & \frac{n}{2} \end{bmatrix}$ agrees

b.2. On the sequence a_n

Let S_n be the set of permutations of $[\![1, n]\!]$.

Any permutation of S_n can be written as a composition of disjointed cycles.

Let $\sigma \in S_n$ be an involution, ie a permutation such that $\sigma^2 = Id_{[\![1,n]\!]}$.

We write $\sigma = c_1 \circ c_2 \circ c_3 \circ \ldots \circ c_r$, where c-i are the cycles. The cycles c_i being disjointed, their product is commutative: $c_i \circ c_j = c_j \circ c_i$.

Then: $\sigma^2 = {c^2}_1 \circ {c^2}_2 \circ {c^2}_3 \circ \ldots \circ {c^2}_r$

And: $\sigma^2 = Id_{[\![1,n]\!]} \Leftrightarrow \forall i \in [\![1, n]\!], {c^2}_i = Id_{[\![1,n]\!]}$

Moreover, ${c^2}_i = Id \Leftrightarrow c_i$ is a 2-cycle, ie a transposition.

The choice of an involution in a set of cardinal n corresponds then to the choice of p subsets of cardinal 2 that will be the supports of p transpositions, where $p \leq \left[\frac{n}{2}\right]$ ([x] designing the integer part of x)

Which gives the sum :

$$a_n = \sum_{p=0}^{\left[\frac{n}{2}\right]} \left[\binom{n}{2p} \prod_{i=1}^{p} (2 \cdot i - 1) \right]$$

but we have also:

$$\prod_{i=1}^{p} (2 \cdot i - 1) = \frac{(2p)!}{\prod_{i=1}^{p} (2 \cdot i)} = \frac{(2p)!}{2^p \cdot p!}$$

We can then simplify the expression of :

$$a_n = \sum_{p=0}^{\left[\frac{n}{2}\right]} \left[\binom{n}{2p} \frac{(2p)!}{2^p \cdot p!} \right]$$

$$a_n = \sum_{p=0}^{\left[\frac{n}{2}\right]} \left[\frac{n!}{(2p)! \cdot (n-2p)!} \cdot \frac{(2p)!}{2^p \cdot p!} \right]$$

$$a_n = \sum_{p=0}^{\left[\frac{n}{2}\right]} \left[\frac{n!}{2^p \cdot p! \cdot (n-2p)!} \right]$$

3)

Some important notations [Second part]:

- Let u = $\boxed{\begin{matrix}\sigma_u \\ \sigma'_u\end{matrix}}$ be a pattern, we denote by $\bar{u}$ the pattern $\boxed{\begin{matrix}\sigma'_u \\ \sigma_u\end{matrix}}$ and call it *the conjugate of* u.

  By extension, if K is a set of patterns, we denote by $\bar{K}$ the set formed by the conjugates of all the patterns of K.

- We have u = $\begin{array}{|ccccc|}\hline u_1 & u_2 & u_3 & \dots & u_n \\ u'_1 & u'_1 & u'_3 & \dots & u'_n \\ \hline\end{array}$

  We denote u* = $\begin{array}{|ccccc|}\hline n+1-u_n & n+1-u_{n-1} & n+1-u_{n-2} & \dots & n+1-u_1 \\ n+1-u'_n & n+1-u'_{n-1} & n+1-u'_{n-2} & \dots & n+1-u'_1 \\ \hline\end{array}$

  u* is called « studed » of u. By extension, if K is a set of patterns, we denote by K^* the set of studed of all the patterns of K.

- We define *conjugation* to be the application which associates to each pattern u the pattern $\bar{u}$. Similarly, we define *studding* to be the application which associates to each pattern u the pattern $\bar{u}$.

Properties of the congugation and the studding:

- The conjugation and studding are involutions, that is

$$\forall u \in G_n, \bar{\bar{u}} = u^{**} = u$$

- We have:

$$\begin{cases} u \xrightarrow{A} v & \Leftrightarrow \bar{u} \xrightarrow{B} \bar{v} \\ u \xrightarrow[A]{} v & \Leftrightarrow u^* \xrightarrow[A]{} v^* \quad \text{(Resp. with } \xrightarrow{B} \text{)} \end{cases}$$

Remark: when we drew a connected component K of a graph G_n, it is interesting to see if there are patterns K which have not their conjugate (resp. studed) in K. We would have then K≠ $\bar{K}$ (resp. K≠K^*), and 2 connected components instead of one would be studied.

Fixed point :

Let u be a pattern of length n.
We say that u admits a number f ($1 \le f \le n$) as a fixed point if and only if :

$$u_f = u'_f = f$$

We denote $F(u) = \{f_1, f_2, f_3 \dots f_a\}$, all the fixed points of u
Remark : Let (u,v) ∈ G_n^2 be a couple of patterns
If F(u)≠ $F(v)$, then u and v are in different connected components of graph.

Under bond of the suite $(g_n)_{n\geq 3}$

A. Let $(u,v) \in G_n^2$ be , if F(u)≠F(v) , u and v are in different connected component.

g_n is thus superior among possible sets F(u). F(u) may be any subset of $[\![1,n]\!]$ having n-1 no element.

We thus have

$$g_n \geq \sum_{k=0}^{n-2} C_n^k + C_n^n$$

$$g_n \geq \sum_{k=0}^{n} C_n^k - n$$

Now we have,

$$\sum_{k=0}^{n} C_n^k = (1+1)^n = 2^n$$

Thus

$$g_n \geq 2^n - n$$

For example, for n=3, $g_n \geq 8 - 3 = 5 \ \ (really\ g_3 = 7)$

Upper bound :

no components have exactly 4 or 5 patterns. We have got the upper bound

$$g_n \leq \frac{n!^2 - 1 - a_n}{6} + a_n$$

For example, with n = 4 :

$$25 \leq g_4 \leq 104$$

Other method :

We will first show by induction that : $\forall n \in IN,\ 2^{\left[\frac{n}{2}\right]} \cdot \left(\left[\frac{n}{2}\right]+1\right)! \leq a_n$ (1)

The relation (1) is true for and n=1 and n=2

Let us suppose that, for $n \in IN$ given, (1) is verified for n and n+1

We have $a_{n+2} = a_{n+1} + (n+1) \cdot a_n$

Then,

- *if n is even :*

$$a_{n+2} = a_{n+1} + (n+1)a_n$$

$$a_{n+2} \leq 2^{\frac{n}{2}} \cdot \left(\frac{n}{2}+1\right)! + 2^{\frac{n}{2}} \cdot \left(\frac{n}{2}\right)!$$

$$a_{n+2} \leq 2^{\frac{n}{2}} \cdot \left(\frac{n}{2}\right)! \cdot [\frac{n}{2}+1+n+1]$$

$$a_{n+2} \leq 2^{\frac{n}{2}} \cdot \left(\frac{n}{2}\right)! \cdot \frac{3 \cdot n + 4}{2}$$

$$a_{n+2} \leq 2^{\frac{n+2}{2}} \cdot \left(\frac{n+2}{2}\right)! \cdot \frac{3 \cdot n + 4}{2 \cdot n + 2}$$

$$a_{n+2} \leq 2^{\frac{n+2}{2}} \cdot \left(\frac{n+2}{2}+1\right)!$$

- *If n is odd :*

$$a_{n+2} = a_{n+1} + (n+1)a_n$$

$$a_{n+2} \leq 2^{\frac{n+1}{2}} \cdot \left(\frac{n+1}{2}\right)! + (n+1) \cdot 2^{\frac{n-1}{2}} \cdot \left(\frac{n+1}{2}\right)!$$

$$a_{n+2} \leq 2^{\frac{n+1}{2}} \cdot \left(\frac{n+1}{2}\right)! \cdot [2+n+1]$$

$$a_{n+2} \leq 2^{\frac{n+1}{2}} \cdot \left(\frac{n+1}{2}\right)! \cdot \left[\frac{n+3}{2}\right]$$

$$a_{n+2} \leq 2^{\frac{n+1}{2}} \cdot \left(\frac{n+3}{2}\right)!$$

We can also prove that $\forall n \in IN, a_n \leq 3^{\left[\frac{n}{2}\right]} \cdot \left(\left[\frac{n}{2}\right] + 1\right)!$ (2)

Furthermore, we have the estimation $(2^n \cdot n!)^2 \underset{n \to \infty}{\sim} C\,(2 \cdot n + 1)!$, where $C \in IR$ (by induction)

Using Stirling's formula $n! \underset{n \to \infty}{\sim} \sqrt{2\pi} \cdot \sqrt{n} \cdot n^n \cdot e^{-n}$, we get the value $C = \sqrt{\pi}$

Notice : $u_n \underset{n \to \infty}{\sim} v_n \Leftrightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = 1$

We use Stirling's result on the approximation of factorials :

$$n! \underset{n \to \infty}{\sim} \sqrt{2\pi} \cdot \sqrt{n} \cdot n^n \cdot e^{-n}$$

$$(2 \cdot n)! \underset{n \to \infty}{\sim} \sqrt{4\pi n} \cdot n^{2n} \cdot e^{2n(ln2-1)}$$

While on the other side :

$$2^n \cdot n! \underset{n \to \infty}{\sim} \sqrt{2\pi n} \cdot n^n \cdot e^{n(ln2-1)}$$

$$(2^n \cdot n!)^2 \underset{n \to \infty}{\sim} 2\pi \cdot n \cdot n^{2n} \cdot e^{2n(ln2-1)}$$

We then have :

$$(2^n \cdot n!)^2 \underset{n \to \infty}{\sim} \sqrt{\pi\, n} \cdot (2n)!$$

Which leads in our case to :

$$2^{\left[\frac{n}{2}\right]} \cdot \left[\frac{n}{2}\right]! \underset{n \to \infty}{\sim} \sqrt{n!} \cdot \sqrt[4]{\pi \cdot n}$$

Thus, we obtain : $\exists\, N \in IN, \forall n \geq N, a_n \geq \sqrt{n!}$

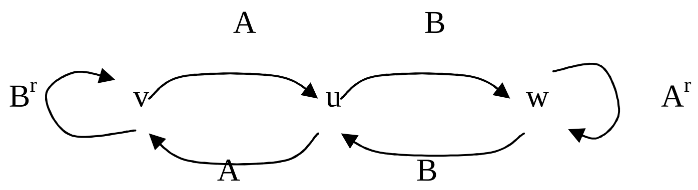
Hence, we have $a_n \geq \sqrt{n!}$ (3)

Let u be the pattern composed with the two different involutions $\sigma_1$and σ_2 .

u is invariant by the applications A² and B²

We introduce the number r, which is the order of the permutation $\sigma_1 \circ \sigma_2$ (ie, the smaller number of times it has to be composed with itself to form the identity)

The connected component associated to u is under the reducted form of a part of the component :



<u>Prismatic components :</u>

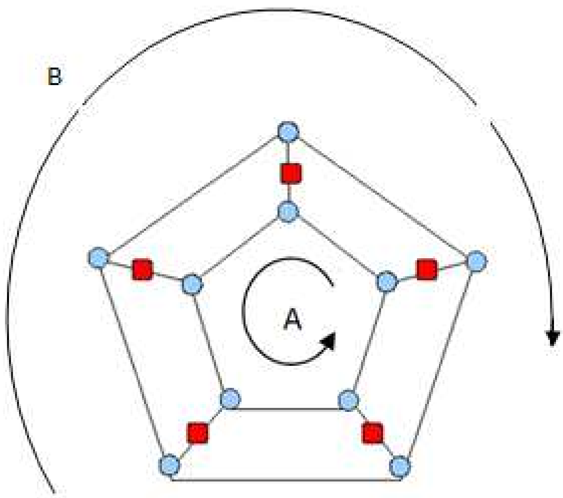
Furthermore, we have : $B°A°B(u)$ of the u type :

We have : $\boxed{\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}}\ \vec{B}\ \boxed{\begin{matrix}\sigma_1\\ \sigma_1\sigma_2\end{matrix}}\ \vec{A}\ \boxed{\begin{matrix}\sigma_1\sigma_2\sigma_1\\ \sigma_1\sigma_2\end{matrix}}\ \vec{B}\ \boxed{\begin{matrix}\sigma_1\sigma_2\sigma_1\\ \sigma_1\end{matrix}}$

Because $\sigma^2{}_1 = \sigma^2{}_2 = id$, we have $(\sigma_1\sigma_2\sigma_1)^2$: , and $v = \begin{matrix}\sigma_1\sigma_2\sigma_1\\ \sigma_1\end{matrix}$ is of the u type (formed with two involutions).

The connected component generated by u has then *3xk* patterns (in subsets of 3 patterns, with a pattern of the u type in each one of them).

Here is an artist's view (anonymous) :



(r=5, pentagonal prism. The patterns are the squares)

Study of the order r

Let σ_1 andσ_2 be two involutions of S_n.

Denote by r the order of $\sigma_1\sigma_2$, ie the smallest number such that $(\sigma_1\sigma_2)^r = id$. r is also the order of $\sigma_2\sigma_1$, (its reciprocal permutation).

σ_1 and σ_2 are two involutions so they can be written respectively as products of disjointed transpositions.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

FIGURE: Two involutions of S_{18} represented with the supports of their constitutive transpositions in colour.

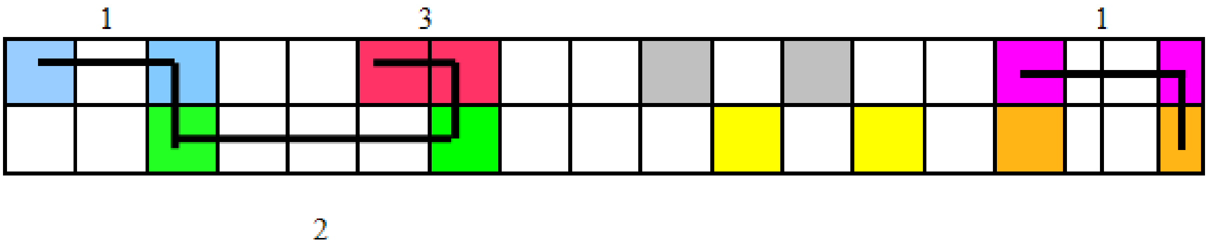


FIGURE : Cycles of $\sigma_2\sigma_1$ one of length 4 (blue-green red) and the other of length 1 (pink-orange)

The order r of $\sigma_2\sigma_1$ is then equal to LCM (a_k) , where the a_k are the length of its cycles. In our case, $\sigma_2\sigma_1$ is equal to LCM (2,4)=4.

We have .

$$\sum_{k=1}^{m} a_k \leq n$$

And then

$$r \leq max\left(\prod_{\substack{p\in[\![2,n]\!] \\ \sum p\leq n}} p\right)$$

For example, with n=15, $r \leq 3 \times 5 \times 7$ =105 , (with 3+5+7=15)

We deduce from there that : All the prismatic components of G_5 contains less than 105 patterns of the $\begin{matrix}\sigma_1\\\sigma_2\end{matrix}$ type.

There are $a_{15}^2 \approx 1{,}07.10^{14}$ patterns of $\begin{matrix}\sigma_1\\\sigma_2\end{matrix}$ the type in G_{15} , so :

$$g_n \geq \frac{a_{15}^2}{105} \approx 1{,}02 \cdot 10^{12}$$

This lower bound is quite close to : $15! \approx 1.02 \cdot 10^{12}$

Here we can make a reference to the works of ***Jean Pierre Massias :*** **"Majoration explicite de l'ordre maximum d'un élément du groupe symétrique"**[4]

"Let $g(n) = \sup_{a \in S_n}$(order of σ) (S_n being the group of permutations of n elements). Landau proved in 1909 that $log\ g(n) \sim \sqrt{n\ log\ n}$.

The aim of this article is to give the following upper bound: $log\ g(n) \leq 1{,}05313 \ldots \sqrt{n\ log\ n}\ \ (\forall n \geq 1)(with\ equality\ at\ n = 1319766)$.

Our techniques are close to those used by Ramanujan in the study of the function d(n) (number of divisors of n); in particular, we introduce certain numbers (analogous to highly composite numbers) that play here an important role."

[4] http://archive.numdam.org/ARCHIVE/AFST/AFST_1984_5_6_3-4/AFST_1984_5_6_3-4_269_0/AFST_1984_5_6_3-4_269_0.pdf

4)

A. Symmetry

We recall the properties of conjugation and studding :

$$A(\bar{u}) = \overline{B(u)}$$

$$B(\bar{u}) = \overline{A(u)}$$

$$A(u^*) = A(u)^*$$

$$B(u^*) = B(u)^*$$

Let u be a pattern belonging to a connected component K. If $\bar{u}$ (or u^*, or any composition of conjugation and studding) is in K, then K can be drawn with with respect to a central or orthogonal symmetry[5].

B. Can they always be drawn one the plane without intersection of the edges ?

We will exhibit a counter-example.

Let σ_0 be a non-involutive permutation of order 4 (ie such that $\sigma^4 = id$ but$\sigma^2 \neq id$)

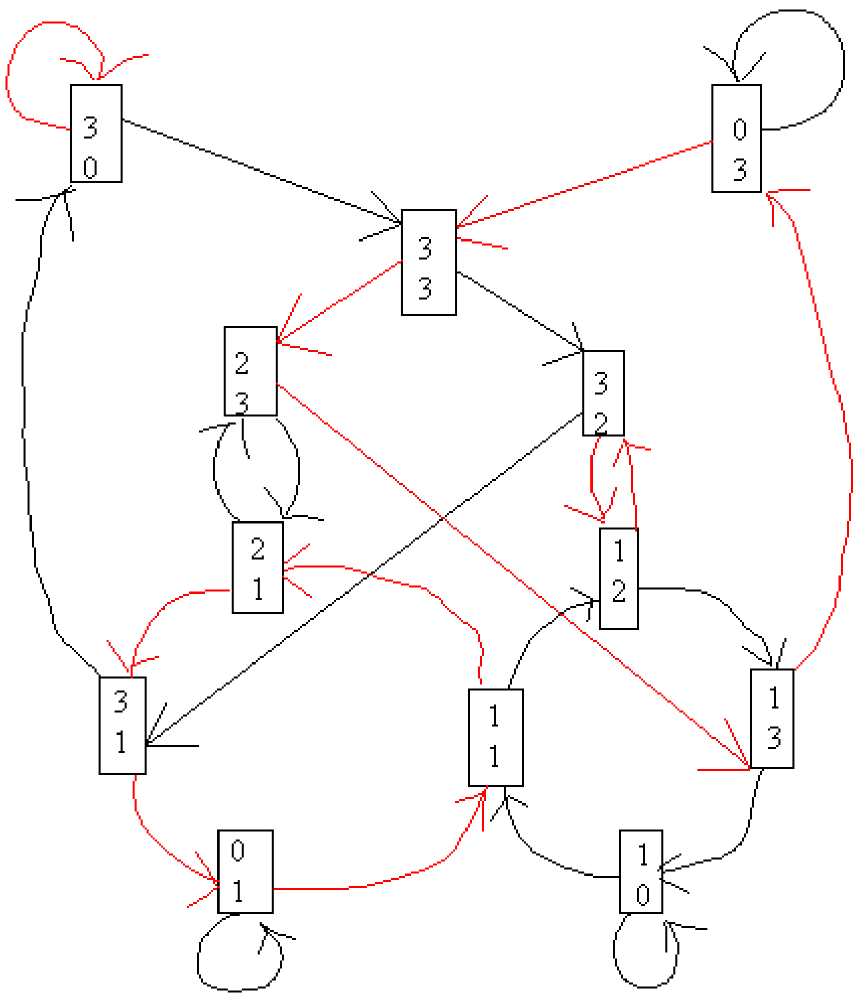
There exists permutations of the σ_0 type in S_4:

For example, the one defined by : $k + 1 \equiv \sigma_0\ (k)[4]$ or $2k \equiv \sigma_0\ [5]$ (the order of 2 being 4 modulus 5)

We can consider the connected component K of the graph $G_n, n \geq 4$ generated by u such that $\sigma_u = \sigma'_u = \sigma_0$. Any patterns w of K have its two permutations under the form $\sigma_0^a \leq a \leq 3$. We can then write them under the reducted form $w = \frac{a_1}{a_2}$, where a_1 and a_2 are the corresponding numbers. K contains all the patterns w under this form (there are 4^2=16), excepted the identity ($a_1 = a_2 = 0$) and the three patterns of the 3-component generated by the involutive permutation (${\sigma^2}_0$). K has then 12 patterns. We can draw it.

[5] Refer to the appendix for demonstration D[4]

FIGURE of K



Appendix

Continuing the method with invariant [a,b]. One can estimate the number of connected components of G_n. First we note that if for a in S_n we have [a,b]= [a,c] for some b, c in S_n then $b.c^{(-1)}$ lies in centralizer C(a) of a. If f is a fixed point of a (we consider a as an application from {1,2, ..., (n-1), n} to itself) then b(f) is also fixed point of a.
We say that f is a fixed point of a of order k if is the smallest number such that a^k (f)=f. If f is a fixed point of order k of a then there are k numbers f(1), .. f(k) such that a acts as k-cycle on them. If b is in C(a) then (b(f(1), b(f(2)), ..., b(f(k)) is also a k-cycle for a. From another side if b sends some f(i) to the set {f(1),...f(k))} then b also acs as k-cycle on this set. Lets apply this to estimate the number of connected components for some n. For $n = \frac{p(p+1)}{2}$ one can consider permutation w_n = (1)(2,3)(4,5,6)(...) (..) (for example w_{10} = (1)(2,3)(4,5,6)(7,8,9,10)) There is not more then one cycle for every k in w_n. Thus one can easily calculate C_{w_n} and obtain that $Card\ C_{w_n}$ = p! (rem. factoriel de p). Thus since the commutator [w_n, *] is an invariant for a connected component, we see that there must be at least $\frac{n!}{p!} = \frac{\frac{p(p+1)}{2}!}{p!}$ connected components in this case.

Demonstrations:

***) D[1]** *Any permutation of S_n can be written as transpositions or as the identity :*

Reminder: we denote S_n all the permutations.

We can this statement by induction on the cardinality of the set (a_1, ..., a_n).

For the case n= 1 or n= 2, it is true because for n= 1, there is only the identity and for n= 2 we have the identity and the permutation s such that : $s(a_1) = a_2$ and $s(a_2) = a_1$, which is the definition of the permutation (a_1, a_2). Assuming the statement for a integer n $\geq$ 3, we have few possibilites for a set of n + 1 elements considering the image and the inverse image by s of a_{n+1} :

- If $s(a_{n+1}) = a_i$ and $s(a_i) = a_{n+1}$. Considering the restriction s' of s to

A' = $A_{\backslash(a(n+1) \cup a(i))}$ with card A' = card A - 2 $\geq$ 2 ,which is a permutation , the induction hypothesis provides us a decomposition of s' as a product of transposition $\prod$, and we have : s = $(a_{n+1}, a_i).\prod = \prod.(a_{n+1},a_i)$ and thus s is a product of transposition.

- The inverse image of a_{n+1} is different from a_i, inverse image that we denote a_j and $a_{j'}$ the inverse image of a_j by s. There is here two subcases :

➢ if j' = i.

Then $s(a_{n+1}) = a_i$, $s(a_i) = a_j$, $s(a_j) = a_{n+1}$. As we do above, we get a decomposition of the restriction of s to $A' = A\backslash(a_{n+1} \cup a_i \cup a_j)$ (card A' $\geq$ 1) as a product of transposition or the identity (we denote this decomposition $\prod$) by the induction hyptothesis, and then s is the product of a 3-cycle ((a,b,c) = (a,b)(b,c)) and $\prod$, so that s is a product of transposition.

➢ if j' $\neq$ i, we consider the application s' from $A' = A_{\backslash(a(n+1) \cup a(i))}$ defined as :

for all k of A', k $\neq$ j', $s'(a_k) = s(a_k)$

$s(a_{j'}) = a_i$

s' is a permutation of A' and we can write s' as a product of transpositions by the induction hypothesi, that we denote $\prod$. Then $s = (a_{n+1},a_i).(a_i, a_j).\prod$ is a product of transposition.

<u>Indeed, we have :</u>

☞ for k $\neq$ j, j' and n+1, $(a_{n+1},a_i).(a_i, a_j).\prod (a_k) = (a_{n+1},a_i).(a_i, a_j). (s(a(_k)) = s(a_k)$ because $s(a_k) \neq a_j,a_{n+1}$ and a_i.

☞ for k= j',$(a_{n+1},a_i).(a_i, a_j).\prod (a_j) = (a_{n+1},a_i).(a_i, a_j).s'(a_{j'}) = (a_{n+1},a_i).(a_i, a_j) (a_i) = (a_{n+1}, a_i) (a_j) = a_j$

☞ for k= j, $(a_{n+1},a_i).(a_i, a_j).\prod (a_j) = (a_{n+1},a_i).(a_i, a_j) (a_j = a_{n+1})$

☞ for k = n + 1, $(a_{n+1},a_i).(a_i, a_j).\prod(a_{n+1}) = (a_{n+1},a_i).(a_i, a_j). (a_{n+1}) = a_i$

Thus $s = (a_{n+1}, a_i).(a_i, a_j).\prod$ and s is a product of transposition. ■

***) D[2]** <u>*For the signature*</u>

We use the following notation :

u and v are two elements of S_n

C_n is the set of the couples (i,j) with i< j, $1\leq i\leq n$, $1\leq j\leq n$

I_u is the set of elements (i,j) of C_n such that $u_i > u_j$

N_u is the complement of I_u in C_n

the signature of a permutation u is the number $(-1)^{\text{card } I(u)}$

The signature is a group homomorphism from S_n to {1,-1, x } :

We want to show that :
sgn(vu) =sgn(v)* sgn(u)
$\Leftrightarrow (-1)^{\text{card } I(vu)} = (-1)^{\text{card } I(v)} * (-1)^{\text{card } I(u)}$
<=> Card I(vu) =Card I(u) + Card I(v) [2]

For a couple (i,j) of C_n we have four possibilities :

- $u_i < u_j$, ie (i,j) E N(u)
  - $v(u_i) < v(u_j)$, ie (u_i,u_j) E N(v)
  - $v(u_i) > v(u_j)$, ie (u_i,u_j) E I(v)
- u(i) > u(j), ie (i,j) E I(u)
  - $v(u_i) < v(u_j)$, ie (u_i,u_j) E I(v)
  - $v(u_i) > v(u_j)$, ie (u_i,u_j) E N(v)

These four sets are disjoint two by two and each couple (i,j) belongs to one of these four sets. Consequently, I(vu) = ((i,j) E N(u) and (u_i,u_j) E I(v)) U ((i,j) E I(u) and (u_i, u_j) E N(v)) and:

card I(vu) = card (((i,j) E N(u) and (u_i, u_j) E I(v)) U ((i,j) E I(u) and (u_i, u_j) E N(v)))
= card ((i,j) E N(u) and (u_i, u_j) E I(v)) + card ((i,j) E I(u) and (u_i, u_j) E N(v)

= card ((u_i, u_j) E I(v)) - card ((i,j) E I(u) and (u_i, u_j) E I(v)) + card ((i,j) E I(u)) -

card ((i,j) E I(u) and (u_i, u_j) E I(v))
= card ((u_i, u_j) E I(v)) + card ((i,j) E I(u)) - 2*card ((i,j) E I(u) and (u_i, u_j) E I(v))

u is a permutation so card ((u_i, u_j) E I(v))= card ((i, j) E I(v)) and we get :
card I(vu) = card ((i, j) E I(v)) + card ((i,j) E I(u)) - 2*card ((i,j) E I(u) and (u_i, u_j) E I(v))
⇒ card I(vu) = card (I(v)) + card (I(u)) [2]■

***) D[3] :** *Any involution of S_n can be written as a product of disjoint transpositions or as the identity :*

We denote by n the cardinality of the considered set.
For n = 2, it's true. For a set with its cardinality n superior to 3, we have just two possibilities for $s(a_{n+1})$:

- $s(a_{n+1}) = a_{n+1}$, the restriction s' of s to $A_{\backslash(a(n+1))}$ is a permutation, that we can write as a product of disjoint transpositions by the induction hypothesis, this product can be extended to A by defining s'(a_{n+1}) = a_{n+1}, then s' = s (s' denotes here the extended permutation to A).

- s(a_{n+1}) = a_i. s is a involution, then s(a_i) = a_{n+1}. Hence, s = (a_{n+1},a_i) $\prod$, with $\prod$ the product of disjoint transpositions of the restriction of s to A $\backslash(a(n+1) \cup a(i))$, is a product of transpositions. The transpositions that appear in the product $\prod$ are disjoint and have their *support* in A((a_{n+1} ∪ a_i)^2, and therefore they are disjoint two by two with (a_{n+1}, a_i). s is a product of disjoint transpositions. ■

***) D[4]** (with the conjugate $\bar{u}$ as an example, but it applies with any composition of conjugation and studding)

u being a pattern of K , any pattern of K can be obtain from u using the applications A and B , A^{-1} and B^{-1} . But we have $A^{-1} = A^{r-1}$, where r is the order of the permutation u'. Any pattern of K can then be obtain from u using only the applications A and B.

If $\bar{u} \in K$, then $A(\bar{u}) = \overline{B(u)} \in K$ and all pattern of K have its conjugate in K. We have then $K = \bar{K}$.

- ***If k = card K is an odd number***, there exists necessarily in the component K an odd number d of patterns v such that $v = \bar{v}$. If d=1 then K can be drawn with central symmetry (with the unique pattern v as center). If $d \geq 3$ then K can be drawn with othogonal symmetry (placing the patterns of the v type on an axis, with respect to which any couple of patterns u and $\bar{u}$ are symmetric)
- ***If k= card K is an even number***, there exists necessarily in the component K an even number d of patterns v such that $v = \bar{v}$. If d=0 then K can be drawn with central symmetry (with no pattern in the center). If $d \geq 2$ then K can be drawn with orthogonal symmetry.

There are *n!* patterns v such that $v = \bar{v}$ (choice of a permutation in S_n) There are $\left[\frac{n}{2}\right]$! patterns v such that v=v* (choice of the $\left[\frac{n}{2}\right]$ first images of two permutations of S_n , the following obeing to the symmetry $\sigma(k) = n + 1 - \sigma(n - k + 1))$)

Also : Given a pattern u, if $[\sigma_u, \sigma'_u]$ is not an involution, then u and $\bar{u}$ are not in the same component. Indeed, $[a, b] = a^{-1}b^{-1}ab = [b, a]^{-1}$. ■