A functionnal equation

June 26, 2009

Let be k a constant number

1. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + kx) = xf(x) \quad (E_k)$
We will suppose that f is a continuous fucntion.
- if $k \notin \{1, 2, 3\}$
We can notice that the constant function 0 is a solution of the equation E_k for all the real number k.
we denote R_f the set of the zeros of f.
Using the equality for x = 0, we get $f(f(0)) = 0$ thus $f(0) \in R_f$. Moreover, using directly the equality E_k we have for x a zero of f $f(kx) = 0$. So all the numbers the set $\{f(0)k^n | n \in \mathbb{N}\}$ is a subset of R_f. Particularly, f has infinitely many zero so f is not a polynomial different from the constant polynomial zero.
If $|k| < 0$ the sequence $(f(0)k^n)_{n\in\mathbb{N}}$ converges to 0 so :

$$f(0) = \lim_{n\to\infty} f((f(0)k^n) = 0$$

We will now prove the implication :

$$f(x) + kx = m \Rightarrow kx^2 - mx + f(m) = 0$$

Indeed, if x is such that

$$f(x) + kx = m$$

then,

$$f(f(x) + kx) = xf(x) = x(m - kx)$$

ie

$$f(m) = xm - kx^2$$

ie

$$kx^2 - xm + f(m) = 0$$

We call this quadratic equation the quadratic equation associated to m or the m-quadratic equation.
We define the function g by :

$$\forall x \in \mathbb{R}, g(x) = f(x) + kx$$

Because we suppose f a continuous function, g is a continuous function too. we have :

$$\begin{aligned} f(g(x)) &= f(f(x)+kx) \\ &= g(f(x)+kx) - k(f(x)+kx) \\ &= g(g(x)) - kg(x) \end{aligned}$$

$$x(f(x)) = xg(x) - kx^2$$

The two previous properties satisfied by f becomes for g :

$$\forall x \in \mathbb{R}, g(g(x)) = (x+k)g(x) - kx^2$$

$$\forall x \in \mathbb{R}, \forall m \in \mathbb{R}, g(x) = m \Rightarrow kx^2 - xm + g(m) - km = 0$$

If k_1 and k_2 are two real numbers that satisfies the m-quadratic equation, then :

$$k_1 + k_2 = \frac{m}{k}$$

Moreover, we have $\lim_{x\to\infty} g(x) \notin \mathbb{R}$.
Indeed, if $\lim_{x\to\infty} g(x) = l \in \mathbb{R}$ then :

$$g(l) = \lim_{x\to\infty} g(g(x)) = \lim_{x\to\infty} ((x+k)g(x) - kx^2)$$

By the definition of the convergence of g in $+\infty$, for $x > -k$ there exists a real number ϵ such that $l - \epsilon \leq g(x) \leq l + \epsilon$ then :

$$(x+k)(l-\epsilon) - kx^2 \leq ((x+k)g(x) - kx^2) \leq (x+k)(l+\epsilon) - kx^2$$

In $+\infty$, the two polynomials $(x+k)(l-\epsilon) - kx^2$ and $(x+k)(l+\epsilon) - kx^2$ are equivalent to kx^2 and because kx^2 diverges to $+\infty$ or $-\infty$ we would have $|g(l)| = +\infty$ which is absurd.

The quadratic equation shows that g takes the same value at the most two times. For these reasons, we can say we have four possilities for g :

$\lim_{x\to-\infty} g(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = -\infty$

$\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$

$\lim_{x\to-\infty} g(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$

$\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{x\to+\infty} g(x) = -\infty$

We suppose now that $f(0) = 0$
We have also $g(0) = 0$.If there exists an other real number k such that $f(k) = 0$ then we must have $k + 0 = 0$ ie $k = 0$. Therefore, g is monotonic on $\mathbb{R}_+$ and $\mathbb{R}_-$.

- If k= 0

$$f(f(x)) = xf(x)(E_0)$$

$f(f(0)) = 0$ and if we use the equality E_0 to y = f(0) :

$$f(f(f(0))) = f(0)f(f(0))$$

ie

$$f(0) = 0$$

2. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that :

$$\forall (x, y) \in \mathbb{R}^2, f(f(x) + f(y) + kxy) = xf(y) + yf(x) \quad (E'_k)$$

The zero function is a solution for every real number k.
We have $f(2f(0)) = 0$ and for $y = 0$ we get $f(f(x) + f(0)) = xf(0)$ which implies that f is an injective application if we suppose that $f(0) \neq 0$.
Indeed,

$$f(x) = f(y) \Rightarrow yf(0) = xf(0) \Rightarrow x = y$$

Moreover, if x_0 is such that $f(x_0) = 0$ then, using the equality for $x = y = x_0$, we have $f(kx_0^2) = 0$ ie kx_0^2 is also a zero of f. If $f(0) \neq 0$, $2f(0)$ and $k(2f(0))^2$ are two zeros and consequently are equal.

$$2f(0) = 4kf(0)^2 \Rightarrow f(0) = 0 \text{ or } f(0) = \frac{1}{2k}$$

Therefore, we have two possible values for $f(0)$: $f(0) = 0$ or $f(0) = \frac{1}{2k}$.

-If $f(0) = 0$:
Then f is such that, for any real number x, $f(f(x)) = 0$, which is equivalent to $f(Imf) = 0$. But the set Imf has to satisfy :

$$\forall (x, y) \in (Imf)^2, kxy \in Imf$$

-If $f(0) = \frac{1}{2k}$, k $\notin \{0, \frac{1}{2}\}$:
f must satisfy the equality:

$$f(f(x) + \frac{1}{2k}) = \frac{x}{2k}$$

we introduce the function g such that $g(x) = f(x) + \frac{1}{2k}$.The previous equality becomes :

$$g(g(x)) = \frac{x+1}{2k}$$

We suppose now that g is differentiable at every real number and that its derivative is continuous. By differentiating the previous equality we get the relation between $g'(x)$ and $g'(g(x))$:

$$\forall x \in \mathbb{R}, g'(x)g'(g(x)) = \frac{1}{2k}$$

Using the equality for $g(x)$, we get :

$$g'(g(x))g'(g(g(x))) = \frac{1}{2k}$$

ie

$$g'(g(x))g'(\frac{x+1}{2k}) = \frac{1}{2k}$$

These two equalities give together :

$$g'(x) = g'(\frac{x+1}{2k}) \quad (1)$$

that we write $g'(x) = g'(t(x))$ with $t(x) = \frac{x+1}{2k}$.
We define now the relation $\sim$ by :

$$x \sim y \Leftrightarrow \exists n \in \mathbb{Z}, x = t^n(y)$$

The relation $\sim$ is an equivalence relation and because of the equality (1), the derivative of g at two points of the same equivalence class is the same. The only finite equivalence class is the singleton $\{\frac{1}{2k-1}\}$.Moreover, each equivalence class converges to $\frac{1}{2k-1}$.
Let a and b two real numbers from different equivalence classes, we denote [a] the equivalence class of a and the same for [b]. Then, because we suppose that g' is continuous :

$$\begin{aligned} g'(\frac{1}{2k-1}) &= \lim_{x\in[a]} g'(x) = g(a) \\ &= \lim_{x\in[b]} g'(x) = g(b) \end{aligned}$$

Therefore, g' is a constant function, :

$$\forall x \in \mathbb{R}, g'(x) = \frac{1}{\sqrt{2k}} \Rightarrow g(x) = \frac{1}{\sqrt{2k}}x + b \text{ with } b = g(0) = f(0) + \frac{1}{2k} = \frac{1}{k}$$

Thus, we have,

$$\forall x \in \mathbb{R}, g(x) = \frac{1}{\sqrt{2k}}x + \frac{1}{k}$$

ie

$$\forall x \in \mathbb{R}, f(x) = \frac{1}{\sqrt{2k}}x + \frac{1}{2k}$$

However, f is not a solution to the general equation E'_k.
This method applies to the general case too.