ITYM : Problem 1 - Specular colorings

June 25, 2009

Notations :

We keep the notations of the terms.

We say a line (row or column) is satisfied if and only if two coloured squares are symmetric with respect to this line, and these squares are satisfying the line. It may be more than two squares satisfying the same line.

The squares are numerated according to their place : the left low square is (1,1) and the top right one is (m,n).

*Let be a m*n grid with coloured squares, the associated graph G of this grid is the simple graph in which vertices are the the coloured squares, two vertices being linked by an edge if and only if they are satisfying a line.*

Question 1 : What is the maximal number M of squares of a m*n grid which can be coloured such that no line is satisfied ?

On first, notice that if two adjacent squares are coloured, the line between them is satisfied.

Draw a « snake » like drawn below :

« tail »-1 --->	------ 2------>	----- 3------->	------- 4----->	------- 5----->	------- 6----->	7↓
↓			…	<-- 10-------	<------ 9-----	<-----8
…						
----------- --->	----------- --->	----------- --->	----------- --->	----------- --->	----------- --->	----- ↓
« head » mn	← mn-1 -----				…	<--------- ---

According to the parity of m and n the head and the tail of the snake may be or not in the same column but it does not matter.

What matters is that the snake crosses each square once and that two consecutive squares in the snake are adjacent and so can not be both coloured. Considering that each coloured square is preventing you from colouring the next one in the snake, the best colouring is obtained by colouring half the squares if they are an even number, and the first and the last if they are an odd number.

This is equivalent to colour the grid like a chessboard, colouring the four corners if m and n are odd numbers. The number of coloured squares is then $mn/2$ is m or n is even and (mn+1)/2 in the opposite case. In all cases, M = [(mn+1) / 2] where [x] is the integer part of x.

Question 2 : Let S(m,n) be the minimal number of squares you need to colour to satisfy all the lines. We call a colouring specular if this all the lines are satisfied. Estimate S(m,n).

Lower bound :

Let G be the associated graph of a specular colouring The graph G has at least m+n-2 edges, one for each line satisfied, and has the minimal number of vertices among all the associated graphs specularly coloured.

Two squares satisfying a line have different parities about the sum of their coordinates; if it was not the case, the arithmetic mean of these coordinates, one of them being the same as the squares are on the same row or column, would be a integer numbers and the squares would be symmetrical by respect to a square and not to a line.

So, if there are three squares two by two satisfying a line, we could find 3 integer numbers two by two of different parity, which is impossible. So the graph G does not include any triangle, i.e. among three vertices at least two are not linked by an edge.

Then the Mantel-Turan theorem asserts that a simple graph with no orientation,with n vertices, which does not contain any triangle, contain less than $n^2/4$ edges, or exactly $n^2/4$ in he case where n is even and the graph is the bipartite complete graph $K_{n/2\,,\,n/2}$.

Here, as the graph has at least m+n-2 edges, stating $k^2/4$ = m+n-2 , the number of vertices is at least $2\sqrt{(m+n-2)}$.

As G ≠ $K_{n/2\,,\,n/2}$ (except maybe cases m or n =1) because if it was the case, all the squares would be on the same line and some lines would not be satisfied, or squares will be symmetric with squares not on the same line, which is absurd too. The number of vertexes is then at least

$\ulcorner 2\sqrt{(m+n-2)} \urcorner$, which is the smallest integer strictly greater than $2\sqrt{(m+n-2)}$.

Upper bound for S(1,n)

We are still considering the rectangle (segment) $1\times n$
We denote each cell of the segment by $0, 1, 2, \ldots, n-1$.
We design each interior line of the segment by the sum of the two cells that it separates. Thus, the $n-1$ are labelled by all the odd numbers between 1 and $2n-3$.
We are looking for the smallest subset P of $[\![0 ; n-1]\!]$ such that any odd number between 1 and $2n-3$ can be written as the sum of two elements of P. (If an odd cell c_1 and an even cell c_2 are coloured, they are satisfying the line $c_1 + c_2$).
We remark that any odd number of $[\![1 ; 2n-3]\!]$ is congruent to $1, 3, 5, \ldots$ or $2k-1 \,(mod\, 2k)$, where k is an integer smaller than n. An idea is then to colour the cells $0, 2k, 4k, \ldots$ plus the cells $1, 3, 5 \ldots 2k-1$ on the left (and the same number of odd cells on the right) . Each interior line would then be satisfyied by exactly two cells, which is in our interest.
We will show on the figure this algorithm of construction (here with a well chosen case, but it always works roughly...)

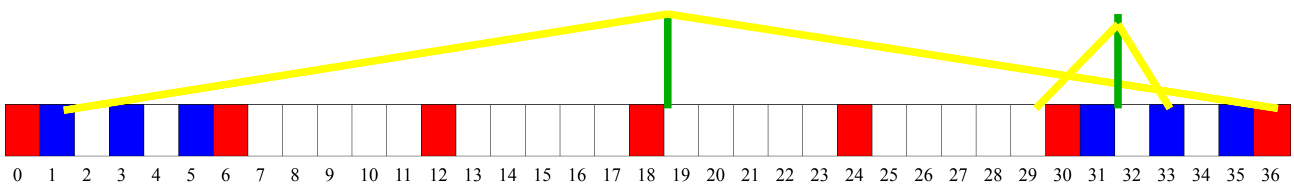

FIGURE 1 : $S(1{,}37) \leqslant 13$. Specular colouring of the 1×37 rectangle, with k=3 (two examples of symmetry are shown with the line 37 and 63)

But the number of coloured cells will vary according to the value of k. We will then choose k in order to minimise this number of coloured cells, that we call f(k).

$$f(k) = \left[\frac{n}{2k}\right] + 1 \;+\; 2k$$

The difficulty in looking for the exact value realising the minimum comes from the fact that f is not a continuous fonction (because of the integer part).

We can use an approximation of f that we will call $\tilde{f}$.

$$\tilde{f}(k) = \frac{n}{2k} + 1 + 2k$$

This time, we can determine easily the minimum of $\tilde{f}$ by calculating its derivated $\tilde{f}\,'$:

$$\forall\, k \in \mathbb{R}_+ ,\ \tilde{f}\,'(k) = 2 - \frac{n}{2 \cdot k^2}$$

$$\tilde{f}\,'(k) = 0 \Leftrightarrow k = \frac{\sqrt{n}}{2}$$

Furthermore, we have $|\tilde{f}(k) - f(k)| < 1$, so the differnce between the minima of these two functions is at most of 1. Calling m the minimum of f and $\tilde{m}$ the one of $\tilde{f}$, we have necessarily $m > \tilde{m} - 1$.

We have :

$$\tilde{m} = \tilde{f}\left(\frac{\sqrt{n}}{2}\right) = 2\sqrt{n} + 1$$

$$2\sqrt{n} < m \leqslant 2\sqrt{n} + 1$$

$$m = [2\sqrt{n}] + 1$$

Associating lower and upper bound, we obtain the following estimation :

$$[2\sqrt{n-1}] + 1 \leqslant S(1,n) \leqslant [2\sqrt{n}] + 1$$

Indeed, the graph associated to a specular colouring constructed with the algorithm is almost or exactly the complete bipartite graph $K_{S(m,n)/2, S(m,n)/2}$ (the odd and even coloured cells are constituing the two parts of the graph)

Upper bound of S(m,n)

We are now considering the $n \times m$ grid
Then knowledge of $S(1,n)$ enables us to give an upper bound of $S(n,m)$: We can obtain a specular colouring of the $m \times n$ rectangle using only one row and one column for the coloured cells. If we place the intersection of this row and this column such that it corresponds to a coloured cell, the final colouring has $S(1,n)+S(1,m)-1$ coloured cells (see fig.2):

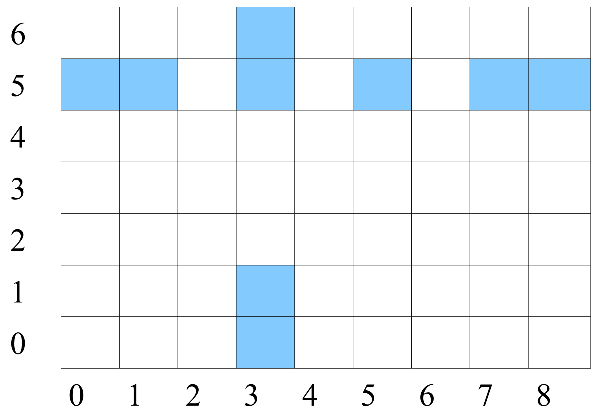

FIGURE 2 : Specular colouring of the 9×7 rectangle using the row 3 and the column 5

We can choose arbitrarily the Row 0 and the Column 0. We are sure that the cell (0,0) is coloured. The colouring is then entirely contanied in a « L area » (formed by the row and the column). We introduce the value:

$$S_L(m,n) = S(1,n) + S(1,m) - 1$$

Where $S_L(m,n)$ is the minimal specular colouring contained in the L area.
The L area being a subset of the entire grid, we can write :

$$S(m,n) \leqslant S_L(m,n)$$

$$S(m,n) \leqslant [2\sqrt{n}] + [2\sqrt{m}] + 2$$

Lower bound of S(m,n)

We are going to show that :

$$S(m,n) \geqslant S_L(m,n) - 6$$

Additional notation : C being a coloured cell in a (specular) colouring, we mean by *neighbours* of C the cells that are symmetric to C with respect to any line of grid.
In the proof, we sometime mix the coloured cells and their associated vertexes in the graph, in order to simplify the sentences (for instance we can use the expression « the cells of a cycle »)

Let us consider a minimal specular colouring of the $m \times n$ rectangle, where m and n even numbers. We will call G its associated graph.

We add the 4 corners of the rectangle to the colouring. The colouring contains now at most $S(n,m)+4$ cells. We call its associated graph G'. The vertexes associated to the four corners are in all in a same connected component of G' that we will denote by K. Any coloured cell C belonging to a border of the rectangle has its associated vertex in K (because m and n are even, C is necessarily linked to one or the other of the corners, and the corners have all their associated vertex in K).

Remark : if the graph G' contains a k-cycle, then k is necessarily an even number : Considering a coloured cell C of a k-cycle of coordinates (i_C, j_C) , we introduce the value $V(C) = i_C + j_C$
All the neighbours X of the cell C are such that $V(X)$ is odd if $V(C)$ is even, $V(X)$ is even if $V(C)$ is odd. . This principle is already used in the first question (When we are in fact colouring all the cells of the same parity with respect to the V function). Then, if G contains a k-cycle, k is even (If not, $V(C)$ is an odd and even number at the time,

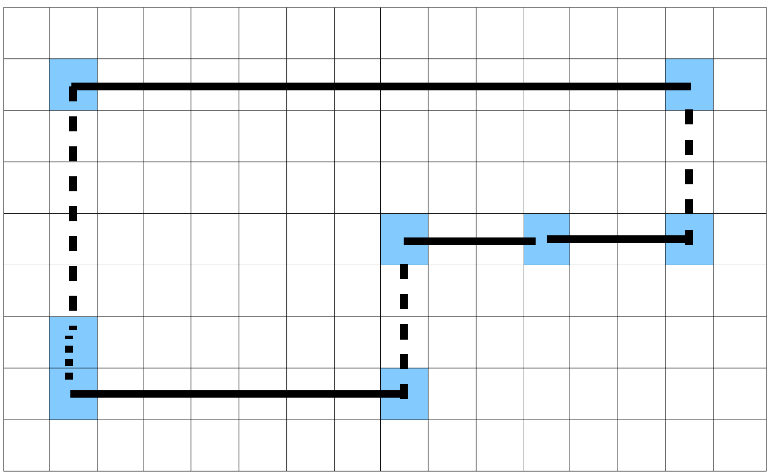

FIGURE 3 : An 8-cycle of the graph. The continuous straits are representing the vertical symmetries, while the interupted lines represent the horizontal ones.

We can always colour the cells of the cycle with 2 colours such that any couple of cells bounded by a continuous line are of the same colour, and every couple of cells bounded by an interrupted line are of different colours.

Indeed, we can project the cells of the cycle on a single column, and then use the fact that the cycle obtained is a 2k-cycle :

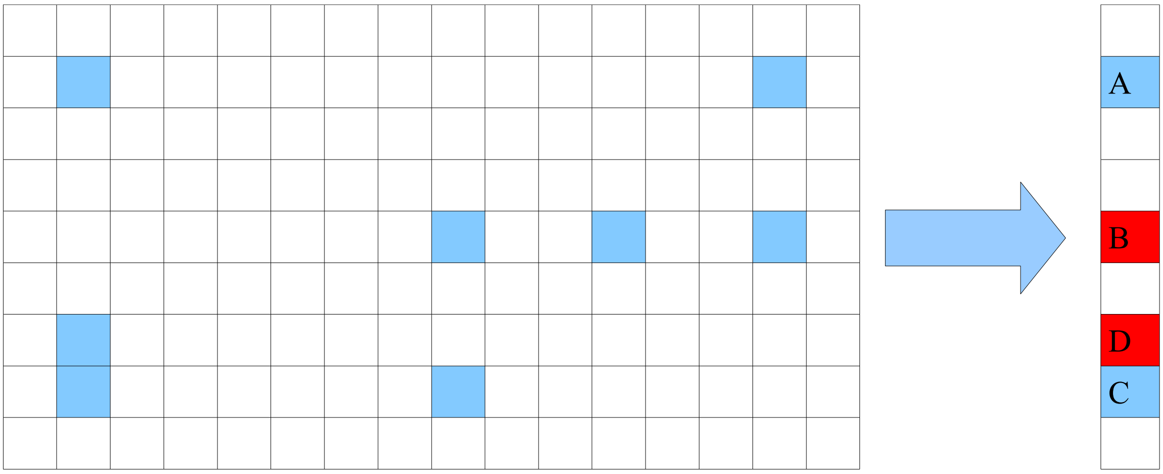

FIGURE 4 : Projection of an 8-cycle on a single column

Like shown on an example on the fig.8, we obtain a 2k-cycle (here, k=4). We can then colour the cells of this new cycle with only 2 colours, such that if two cells are bounded, their colours are different.

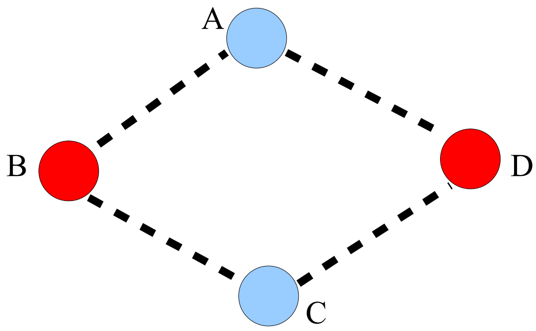

FIGURE 5 : Colouring of a 2k-cycle

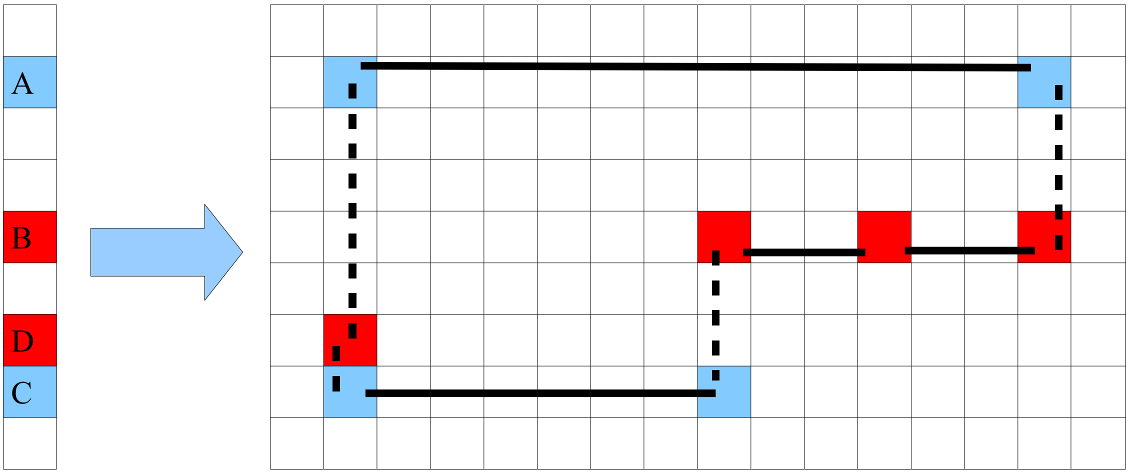

FIGURE 6

2) We consider successively all the connected component K_i of the graph G'. For each component we denote by P the associated cell that is the closest one to a border of the rectangle (there exists at least one), and we design by p its distance to the border (p = 0 if P belongs to a border of the rectangle).
If p = 0, K_i is connected to K
If $p \geqslant 1$, We can move the cells corresponding to K of a distance of p, following the process :

–We move the cell P so that it joins its nearest border.

–For each cell moved, we move its neighbours so that the symmetry with respect to the lines are kept.

If a cell is moved of a distance p, its neighbours will be moved of a distance p too (orthogonal symmetry is an isometry of the plane), so that none of the cell of the component will « get out » of the rectangle (because of the minimality of p). All the cells of the component move in the same direction (even if not always in the same sense). There are two possibles senses for every coloured cell, like the two colours used in the fig.6, and they obey to the same « rules ».

It is possible to move all the cells without contradiction (we can get to a cell by several ways, but because of the property « all k-cycles are 2k-cycles », the indication of movement cannot be controversial.
Once the process ended, the graph G" obtained is connected, and the colouring is specular with at most $S(n,m)+4$ cells.

If n and m are even, there is a specular colouring whose associated graph is connex, with less than $S(n,m)+4$ cells.
If n is odd and m is even, there is a specular colouring whose associated graph is connex, with less than $S(n,m)+5$ cells (one more cell on the new row/column is sufficient to keep th specularity)
If n and m are odd, there is a specular colouring whose associated graph is connex, with less than $S(n,m)+6$ cells.

We are now considering the general case. We have proved the following statement :
There always exists one specular colouring with less than $S(m,n)+6$ cells coloured, whose associated graph G" is connected.

The graph G" being connected, it is possible to travel all over it following the lines (passing several times through each vertex if necessary). We can then choose a starting coloured cell C_1 , that correponds to the strating vertex in the graph. We colour its projection(s) on the L area. At each vertex encountered for the first time, we project the corresponding cell on the L area (one projection is actually sufficient, because the other has already been done with the cell from which we are coming, see fig.7).

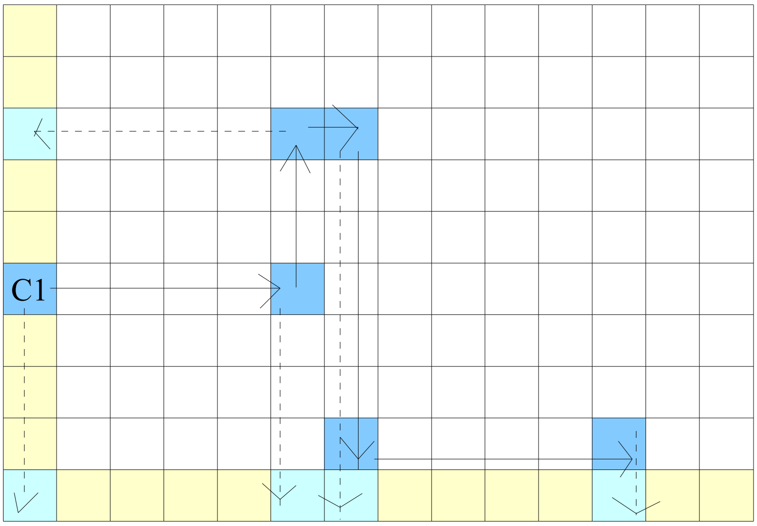

FIGURE 7 : Projection of a specular colouring into the L area. Only a few coloured cells have been represented.

Finally, we obtain : $S(m,n)+6+1 \geqslant S_L(m,n)$ (each cell has been projected one time, and the starting cell two times)

Because of its specularity, the colouring contains necessarily one cell coloured in the first row and one cell coloured in the first column (if not, there wouldn't be any couple of cells that are symmetric with respect to the first vertical line, neither with respect to the first horizontal line).

- if the corner at the intersection of the first line and the first column is coloured, then it will be our starting cell. It is projected two times at the same place, and one coloured cell in the L area has was counted twice. Then we can write : $S(m,n)+6 \geqslant S_L(m,n)$
- If not, there are two distinct coloured cells in the L area, that will be both projected in the corner. The corner has been counted twice, and we can write $S(m,n)+6 \geqslant S_L(m,n)$

We can deduce the estimation : $S_L(m,n)-6 \leqslant S(m,n) \leqslant S_L(m,n)$

Using the estimation of $S_L(m,n)$ established before and the relation :

$$[2\sqrt{n-1}]+1 \leqslant S(1,n) \leqslant [2\sqrt{n}]+1$$

We get : $[2\sqrt{n-1}]+[2\sqrt{m-1}]+1 \leqslant S_L(m,n) \leqslant [2\sqrt{n}]+[2\sqrt{m}]+2$

And, finally : $[2\sqrt{n-1}]+[2\sqrt{m-1}]-5 \leqslant S(m,n) \leqslant [2\sqrt{n}]+[2\sqrt{m}]+2$.

The difference betwween the lower bound and the upper bound is never larger than 9.

In our researches, we have first conjectured that there is always a minimal specular colouring contained in the L area. Then, we remarked that, if there exists one minimal specular colouring whose associated graph is entirely connected, the first conjecture is true. We have not been able to *prove any of these two statements. However, further ideas showed that their importance on the final result – the estimation of S(m,n) – was little. But we are still strongly believing that they are true ; and, then, that* $S_L(m,n)=S(m,n)$ *holds for any couple of integers m and n.*

QUESTION 3) Study three-dimensional analogs of the problem.

One could consider several possibilities for a 3-dimensional generalization of the problem :

A) Some cells of an $m\times n\times p$ cubic honeycomb are coloured blue. We call such a colouring plane-specular if for any interior plane of the grid there are two blue cells that are symmetric with respect to this plane. Denote by $S_2(m,n,p)$ the minimal number of blue cells in a plane-specular colouring of an $m\times n$ cubic grid.

B) Some cells of an $m\times n\times p$ cubic honeycomb are coloured blue. We call such a colouring line-specular if for any interior plane of the cubic grid there are two blue cells that are symmetric with respect to this line. Denote by $S_1(m,n,p)$ the minimal number of blue cells in a line-specular colouring of an $m\times n\times p$ cubic grid.

Upper bounds

A)The method used in 2 still apply. The L will only become an Y (there are now three axis)
We recall briefly the results obtained in the question 2, adapted to the new case :

$$[2\sqrt{n-1}]+1 \leqslant S_2(1,1,n) \leqslant [2\sqrt{n}]+1$$

$$S_Y(m,n,p) = S_2(m,1,1)+S_2(1,n,1)+S_2(1,1,p)-1$$

$$[2\sqrt{n-1}]+[2\sqrt{m-1}]+[2\sqrt{p-1}]+2 \leqslant S_Y(m,n,p) \leqslant [2\sqrt{n}]+[2\sqrt{m}]+[2\sqrt{p}]+3$$

We have then the upper bound : $S_2(m,n,p) \leqslant [2\sqrt{n}]+[2\sqrt{m}]+[2\sqrt{p}]+3$

B)

We will extend the algorithm of colouring presented in 2, and give an upper bound based on a colouring contained only in the cells of three faces.
First, we have to consider We are considering the 2-dimensional analog of the problem, remplacing the interior lines by the interior vertexes of the grid.
The algorithm used in the last question gives the following result :

Denote any vertex by $\begin{matrix}a\\b\end{matrix}$, where a is the number associated to its vertical line and b is the number associated to its horizontal line.
We are looking for the smallest subset $P_0\subset[\![0;24]\!]\times[\![0;16]\!]$ such that, for any couple (a,b) of even numbers in $[\![1;47]\!]\times[\![1;31]\!]$, there are two elements of P_0 (p_1,p_2) and (q_1,q_2)

such that $a = p_1 + q_1$ and $b = p_2 + q_2$
a is congruent to $1, 3, 5, \ldots \text{ or } 2k_1 - 1 \ (mod \ 2k_1)$
b is congruent to $1, 3, 5, \ldots \text{ or } 2k_2 - 1 \ (mod \ 2k_2)$

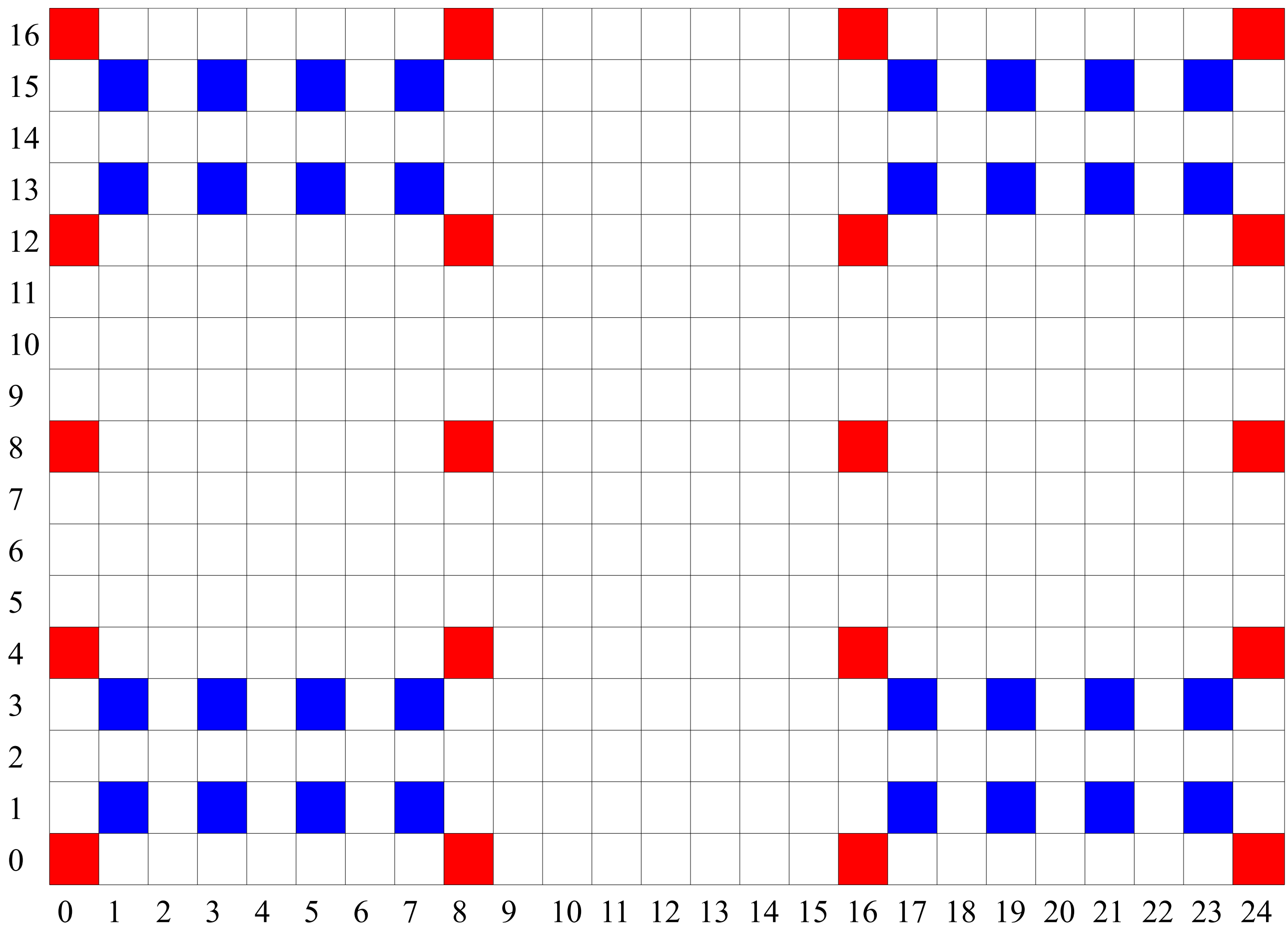

FIGURE 9 : A (non minimal) vertex-specular colouring of the 17 times 25 rectangle, with $k_1 = 4$ and $k_2 = 2$

This time the approximate calculus give :

$$f(k_1, k_2) = \left[\frac{m}{2k_1}\right] \cdot \left[\frac{n}{2k_2}\right] + 4 \cdot k_1 k_2$$

$$\tilde{f}(k_1, k_2) = \frac{m}{2k_1} \cdot \frac{n}{2k_2} + 4 \cdot k_1 k_2$$

One variable is sufficient : $k_1 k_2$

$$\tilde{f}_0(k_1 k_2) = \frac{mn}{4 k_1 k_2} + 4 \cdot k_1 k_2$$

$$\tilde{f}'_0(k_1 k_2) = 4 - \frac{mn}{4 k_1^2 k_2^2}$$

$$\tilde{f}'_0(k_1 k_2) = 0 \Leftrightarrow k_1 k_2 = \frac{1}{4} \cdot \sqrt{mn}$$

$$\tilde{m}_0 = \tilde{f}(\frac{1}{4} \cdot \sqrt{mn}) = 2\sqrt{mn}$$

We have then $S_0(m,n) \leqslant [2\sqrt{mn}] + C(m,n)$, where C(m,n) is little.

In our 3-dimensional case, this algorithm of colouring can be applied on three adjacent faces of the rectangular parallelepiped.

We have then the upper bound : $S_1(m,n,p) \leqslant S_0(m,n) + S_0(m,p) + S_0(n,p)$

$S_1(m,n,p) \leqslant [2\sqrt{mn}] + [2\sqrt{np}] + [2\sqrt{pm}] + C(m,n,p)$, where C(m,n,p) is little.

We still can do a little better by modifying the values of k_1 or k_2 , keeping on each rectangle the same product $k_1 k_2$: The rectangles are sharing three segments $m \times 1 \times 1$, $1 \times n \times 1$ and $1 \times 1 \times p$. In two of them, we can make the coefficients k be equals.

<u>Lower Bounds.</u>

In both cases A and B, the method used in 2 breaks down because by this way, it is sometimes impossible to connect the graph associated to a spectular colouring : if we try to move the coloured cell C that is the nearest of an edge of the rectangle parallelepiped, we take the risk of getting an other coloured cell C' out of it. Indeed, considering the cell C', its distance to the nearest edge can be greater than its distance to the nearest face.
We must then use other methods.

First, we can use the method presented in 2 for the lower bound of S(1,n) :
We count the number of elements that will require an edge in the graph

A) In the parallepiped, the number of interior planes is equal to :

$$N_2 = m + n + p - 3$$

(there are $m-1$, $n-1$ and $p-1$ in each direction respectively)

B) In the parallepiped, the number of interior line is equal to :

$$N_1 = (m-1)(n-1) + (n-1)(p-1) + (p-1)(m-1)$$

The number of edges of a graph counting k vertexes and no triangle is less than $\frac{k^2}{4}$.

A) $\frac{k^2}{4} \geqslant N_2 \Leftrightarrow k \geqslant 2\sqrt{m+n+p-3}$

B) $\frac{k^2}{4} \geqslant N_1 \Leftrightarrow k \geqslant 2\sqrt{(m-1)(n-1)+(n-1)(p-1)+(p-1)(m-1)}$

(we remark that $(m-1)(n-1)+(n-1)(p-1)+(p-1)(m-1) \approx mn+np+pm$)

Thus, $2\sqrt{m+n+p-3} \leqslant S_2(m,n,p) \leqslant [2\sqrt{n}]+[2\sqrt{m}]+[2\sqrt{p}]+3$

and : $2\sqrt{(m-1)(n-1)+(n-1)(p-1)+(p-1)(m-1)} \leqslant S_1(m,n,p)$
$\leqslant [2\sqrt{mn}]+[2\sqrt{np}]+[2\sqrt{pm}]+C(m,n,p)$

We have, for all strictly positive numbers a,b and c, the inequality :

$$1 \leqslant \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a+b+c}} \leqslant \sqrt{3}$$

Then, in both cases A and B, upper and lower bound are such that their ratio does not exceed $\sqrt{3}$.

In the A case, we can add little improvements, considering the associated graph of a minimal plane-specular colouring :
If a connected component contains exactly 2 coloured cells, then we can translate them in a movement that is parallel to the element they are symmetric to, and connect them to the rest of the coloured cells.
If a connected component contains exactly 3 coloured cells, then they are contained in a plane, and we can move them perpendiculary to the plane to connect them to the rest of the coloured cells.
Etc.

Then, there exists necessarily a minimal specular colouring of the m times n times p rectangle with all its connected components containing more than 4 coloured cells (we can go further than 4 with the study of addditional cases).

Let K be its number of connected components.
We have necessarily $K \leqslant \frac{S_2(m,n)}{4}$.
We can project them independantly on the Y : Each time, we begin with a cell C_1 that we project three times, and then we project one time all the other coloured cells in the component.

Finally, we have : $S_Y(m,n) \leqslant S_2(m,n)+2\text{K}$

$$S_Y(m,n) \leqslant \frac{3}{2} \cdot S_2(m,n)$$

And the lower bound is this time $\frac{2}{3}$ of the upper bound.

$$max\left(2\sqrt{m+n+p-3}\,,\ \frac{2}{3}\cdot M(m,n,p)\right) \leqslant S_2(m,n,p) \leqslant M(m,n,p)$$

Where $M(m,n,p)$ is the upper bound : $M(m,n,p) = [2\sqrt{n}]+[2\sqrt{m}]+[2\sqrt{p}]+3$