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PROBLEM NINE

GOOD NUMBERS

Any rational number x may be expressed as a continued fraction

$$x = a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_m}}}$$

where a_0 is the integer part of *x*, and the numbers a_1, a_2, . . . , a_m are positive integers called partial quotients of *x*. We will also write x = [a_0 ; a_1 , a_2 , . . . , a_m].

1. Find all numbers $n > 2$ that can be expressed as the sum of two positive integers $n = a + b$ so that $a < b$ and the continued fraction for $\frac{a}{b}$ has all its partial quotients equal to 1. For example, 13 = 5 + 8 and $\frac{5}{8}$ = [0; 1, 1, 1, 1, 1].

2. A number $n > 2$ is called 2-good if for some positive integers $a < b$ we have $n = a + b$ and the partial quotients of $\frac{a}{b}$ are equal to 1 or 2. For example, 11 = 4+7 and $\frac{4}{7}$ = [0; 1, 1, 2, 1].

(a) Are there infinitely many odd numbers that are not 2-good?

(b) Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.

(c) Describe the set of all 2-good numbers.

3. In general, a number *n* is called *k*-good if it can be expressed as the sum of two positive integers $n = a + b$ so that $a < b$ and the continued fraction for $\frac{a}{b}$ has all its partial quotients not greater than *k*.

Does there exist a positive integer *k* such that all positive integers n > 2 are *k*-good?

4. Describe the set of numbers *n* with the property: there exist two positive integers $b > a$ such that $n = a + b$ or $n = b - a$ and the partial quotients of $\frac{a}{b}$ are equal to 1 or 2.

5. Suggest and study additional directions of research.

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THEORETICAL PART

In our investigation , for compactness, we will denote the continued fraction [a_0; a_1, a_2, . . . , a_m], $a_0 = 0$, $a_i = 1,2$ with $[0;X]$. In contrast to the convention, we will consider that the last element of the fraction is 1.

1) The expression $a_0 + \frac{1/}{/a_1} + \frac{1/}{/a_2} + ... + \frac{1/}{/a_n} + ...$ (*), where $a_0 \in Z_o^+$, $a_i \in N$, $i = 1,...,n, ...$ is called simple continued fraction.

2) The finite simple continued fractions

$$k_0 = a_0,\ k_1 = a_0 + \frac{1}{a_1},\ k_2 = \frac{1/}{/a_1} + \frac{1/}{/a_2},\ k_n = a_0 + \frac{1/}{/a_1} + \frac{1/}{/a_2} + ... + \frac{1/}{/a_n}$$

are called terms of the fraction (*).

3) The quantity obtained by including *n* terms of continued fraction $c_n = \frac{p_n}{q_n} = a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ... + \cfrac{1}{a_n}}}$ is called *n*th convergent.

4) Define $p_{-1} = 1,\ q_{-1} = 0;\ p_0 = a_0,\ q_0 = 1$. The subsequent terms can be calculated from the recurrence relations:

a. $p_{k+1} = p_k a_{k+1} + p_{k-1}$

b. $q_{k+1} = q_k a_{k+1} + q_{k-1}$

c. $p_{k+1} q_k - p_k q_{k+1} = (-1)^k$

5) The convergents of infinite simple continued fraction satisfy

$c_0 < c_2 < c_4 < ....c_{2n-2} < c_{2n} < ... < x$

$x < ... < c_{2n+1} < c_{2n-1} < .... < c_5 < c_3 < c_1$

6) Euler's Theorem. Any $a \in R$ corresponds to exactly one continued fraction. Finite continued fractions correspond to rational numbers, and the infinite – to irrational numbers.

7) The fraction $K\left(\frac{1/}{/a_n}\right)$ is called periodic (with period *k*), if it has the form $a_1 + \frac{1/}{/a_2} + ... + \frac{1/}{/a_k} + \frac{1/}{/a_1} + \frac{1/}{/a_2} + ... + \frac{1/}{/a_k} + a_1 + \frac{1/}{/a_1} + ...$

For compactness, we will write $\left(a_1 + \frac{1/}{/a_2} + ... + \frac{1/}{/a_k}\right)$.

8) The fraction $k\left(\frac{1}{a_n}\right)$ is mixed-periodic, if it has the form $b_0+\frac{1/}{/b_1}+\frac{1/}{/b_2}+\ldots+\frac{1/}{/b_i}+\left(\frac{1/}{/a_1}+\frac{1/}{/a_2}+\ldots+\frac{1/}{/a_k}\right)$.

9) Any quadratic irrational has a continued fraction which is periodic after some point.

10) Lagrange's theorem. The real roots of quadratic expressions with integral coefficients have periodic continued fractions

PART ONE

Problem 1. Find all numbers $n>2$ that can be expressed as the sum of two positive integers $n=a+b$ so that $a<b$ and the continued fraction for $\frac{a}{b}$ has all its partial quotients equal to 1.

Solution. By induction it follows that the numbers $\frac{a}{b}$ are of the form $\frac{a}{b}=\frac{F_{k-1}}{F_k}$ (F_i is a Fibonacci number). The basis is obvious ([0;1]=1) and if [0;1,1,…,1] = $\frac{F_{n-1}}{F_n}$, where we have n units, then [0;1,1,…,1] with n+1 ones is $\frac{1}{1+\frac{F_{n-1}}{F_n}}=\frac{1}{\frac{F_{n+1}}{F_n}}=\frac{F_n}{F_{n+1}}$. Consequently all the numbers $n>2$ from the Fibonacci sequence and their multiples can be represented in the wanted form.

Consequently the Fibonacci numbers and their multiples, describe the following families of 1-good numbers:

$S_1=$3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 ...

$S_2=$6, 10, 16, 26, 42, 68, 110, 178, 288, 466, 754, 1220, ...

$S_3=$9, 15, 24, 39, 63, 102, 165, 267, 432, 699, 1131, 1830, ...

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$S_k=$3*k*, 5*k*, 8*k*, 13*k*, 21*k*, 34*k*, 55*k*, 89*k*, 144*k*, 233*k*, 377*k*, 610*k*...

PART TWO

A number $n > 2$ is called 2-good if for some positive integers $a < b$ we have $n = a + b$ and the partial quotients of $\frac{a}{b}$ are equal to 1 or 2. For example, 11 = 4+7 and $\frac{4}{7}$ = [0; 1, 1, 2, 1].

Analysis. Suppose that all 2-good numbers $\frac{a}{b}$ with $a + b \le n$ are found and we are looking for these with $a + b = n + 1$. If $\frac{a}{b}$ is 2-good, then $\frac{a}{b} = \frac{1}{\frac{b}{a}}$ and we have 2 variants:

– $\left[\frac{b}{a}\right] = 1$, then $\frac{a}{b} = \frac{1}{\frac{b}{a}} = \frac{1}{1 + \frac{b-a}{a}}$ and $\frac{b-a}{a}$ has sum of the numerator and the denominator $b < b + a = n + 1$, i.e. we know whether $\frac{b-a}{a}$ is 2-good number or not and then we can say whether $\frac{a}{b}$ will be 2-good.

– $\left[\frac{b}{a}\right] = 2$, then $\frac{a}{b} = \frac{1}{\frac{b}{a}} = \frac{1}{2 + \frac{b-2a}{a}}$ and $\frac{b-2a}{a}$ have an sum of the numerator and the denominator $b - a < b + a = n + 1$, i.e. we know whether $\frac{b-2a}{a}$ is 2-good, or not and then we can say whether $\frac{a}{b}$ will be also good.

Problem 2. Describe the set of all 2-good numbers.
Solution.

Let $\frac{a}{b} = [0; X]$, where $X = a_1, a_2, ..., a_m$ and $a_i = \{1,2\}$, consequently $\frac{b}{a+b} = [0;1, X]$, $\frac{a+b}{a+2b} = [0;1,1, X]$ and so forth. By this way, we construct a sequence $\{s_n\}$, for which $s_1 = a$, $s_2 = b$ and $s_{n+2} = s_{n+1} + s_n$. All of its terms are 2-good numbers. Consequently it is enough to find only the first two terms of this kind of row and they will determine a whole family of 2-good numbers (the first two terms of the sequence, i.e. the numbers *a* and *b* is not necessary to be 2-good).

The numbers *a* and *b* are defined from the sequence *X*. For definiteness, we will consider that $a_1 = 2$ (in opposite case we will choose a_2 as a beginning of *X*).

Case 1. $X=(2)=(1,1)$ i.e. $\frac{a}{b}=[0;2]=[0;1,1]$

$\frac{a}{b}=\frac{1}{1+\frac{1}{1}}=\frac{1}{2}$. Consequently $a=1$, $b=2$.

$\{s_n\}=$ 1, 2, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... (the Fibonacci numbers)

Case 2. $X=(2,1)$ i.e. $\frac{a}{b}=[0;2,1]$

$\frac{a}{b}=\frac{1}{2+\frac{1}{1}}=\frac{1}{3}$. Consequently $a=1$, $b=3$.

$\{s_n\}=$ 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... (Lucas numbers)

Case 3. $X=(2,1,1)$ i.e. $\frac{a}{b}=[0;2,1,1]$

$\frac{a}{b}=\frac{1}{2+\frac{1}{1+\frac{1}{1}}}=\frac{2}{5}$. Consequently $a=2$, $b=5$.

$\{s_n\}=$ 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, ...

Case 4. $X=(2,2,1)$ i.e. $\frac{a}{b}=[0;2,2,1]$

$\frac{a}{b}=\frac{1}{2+\frac{1}{2+\frac{1}{1}}}=\frac{3}{7}$. Consequently $a=3$, $b=7$.

$\{s_n\}=$ 3, 7, 10, 17, 27, 44, 71, 115, 176, ...

Case 5. $X=(2,1,1,1)$ i.e. $\frac{a}{b}=[0;2,1,1,1]$

$\frac{a}{b}=\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}=\frac{3}{8}$. Consequently $a=3$, $b=8$.

$\{s_n\}=$ 3, 8, 11, 19, 30, 49, 79, 128, 207, ...

Case 6. $X=(2,2,1,1)$ i.e. $\frac{a}{b}=[0;2,2,1,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}}}=\frac{5}{12}$$. Consequently $a=5$, $b=12$.

$\{s_n\}$ = 5, 12, 17, 29, 46, 75, 121, 196, 317, ...

Case 7. $X=(2,1,2,1)$ i.e. $\frac{a}{b}=[0;2,1,2,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}}}=\frac{4}{11}$$. Consequently $a=4$, $b=11$.

$\{s_n\}$ = 4, 11, 15, 26, 41, 67, 108, 175, 283, ...

Case 8. $X=(2,2,2,1)$ i.e. $\frac{a}{b}=[0;2,2,2,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1}}}}=\frac{7}{17}$$. Consequently $a=7$, $b=17$.

$\{s_n\}$ = 7, 17, 24, 41, 65, 106, 171, 177, ...

Case 9. $X=(2,1,1,1,1)$ i.e. $\frac{a}{b}=[0;2,,1,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}=\frac{5}{13}$$. Consequently $a=5$, $b=13$.

$\{s_n\}$ = 5, 13, 18, 31, 49, 80, 129, 209, ...

Case 10. $X=(2,1,1,2,1)$ i.e. $\frac{a}{b}=[0;2,1,1,2,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}}}}=\frac{7}{18}$$ Hence $a=7$, $b=18$.

$\{s_n\}$ = 7, 18, 25, 43, 68, 111, 179, 290, ...

Case 11. $X=(2,1,2,1,1)$ i.e. $\frac{a}{b}=[0;2,1,2,1,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}}}}=\frac{9}{23}$$. Consequently $a=9$, $b=23$.

$\{s_n\}=$ 9, 23, 32, 55, 87, 142, 229, 371, ...

Case 12. $X=(2,2,1,1,1)$ i.e. $\frac{a}{b}=[0;2,2,1,1,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}=\frac{8}{13}$$. Consequently $a=8$, $b=13$.

$\{s_n\}=$ 8, 13, 21, 34, 55, 89, ...

Case 13. $X=(2,1,2,2,1)$ i.e. $\frac{a}{b}=[0;2,1,2,2,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1}}}}}=\frac{10}{27}$$. Consequently $a=10$, $b=27$.

$\{s_n\}=$ 10, 27, 37, 64, 101, 165, 266, 431, ...

Case 14. $X=(2,2,1,2,1)$ i.e. $\frac{a}{b}=[0;2,2,1,2,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}}}}=\frac{11}{25}$$. Consequently $a=11$, $b=25$.

$\{s_n\}=$ 11, 25, 36, 61, 97, 158, 255, 413, ...

Case 15. $X=(2,2,2,1,1)$ i.e. $\frac{a}{b}=[0;2,2,2,1,1]$

$$\frac{a}{b}=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}}}}=\frac{12}{29}$$. Consequently $a=12$, $b=29$.

$\{s_n\}=$ 12, 29, 41, 70, 111, 181, 292, 473, ...

Case 16. $X = (2,2,2,2,1)$ i.e. $\frac{a}{b} = [0;2,2,2,2,1]$

$$\frac{a}{b} = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1}}}}} = \frac{17}{41}$$. Consequently $a = 17$, $b = 41$.

$\{s_n\}$ = 17, 41, 58, 99, 157, 256, 413, ...

Conjecture. Since the previous sequences were constructed on the base of Fibonacci sequence, we assume that all 2-good numbers can be obtained through Fibonacci sequence.

Analysis

1. Lucas sequence: $L_n = L_{n-1} + L_{n-2}$, $L_1 = 1, L_2 = 3$; $L_n = F_{n-1} + F_{n+1}$.
2. In the same way we obtain the sequences 3, 5 and 9.

a. In case 3 for s_1 and s_2 we take the second and the fourth term of the Fibonacci sequence.

b. In case 5 for s_1 and s_2 we take the third and the fifth term of the Fibonacci row.

c. In case 9 for s_1 and s_2 we take the fourth and the sixth term of the Fibonacci sequence.

In this way, if the first terms of the new family are two consecutive even/odd terms of the Fibonacci row, then this family consists of 2-good numbers. The ratio of every two consecutive numbers of the sequence, which is formed in this way, is of the form $[0;1,1,...,1,2,1,1,...,1]$.

3. Analogically from every new sequence we can make another.

a. For example the sequence from case 4 is obtained when for s_1 and s_2 we take the second and the fourth term of the Lucas sequence

b. In case 7 the sequence is obtained when for s_1 and s_2 we take the third and the fifth term of the Lucas sequence.

c. And so on and so forth for the all sequences, which are obtained in this way from the Lucas sequence it is realized that the ratio between every two consecutive numbers is of the form $[0;1,1,...,1,21,...,1,2,1]$.

4. Analogically from the sequence, described in case 3, we can get new families of 2-good numbers, for which the ratio of every two consecutive numbers is of the form $[0;1,1,...,1,21,...,1,2,1,1]$ (Example is the sequence in case 6).

5. Applying this principle for composing new families to every next new obtained sequence, we will get all the families, for which the ratio of every two consecutive numbers is of the form $[0;1,1,...,1,21,...,1,2,1,1,...,1]$.

a. The left over 2-good numbers, which have three or more partial quotients equal to 2, can be constructed on the same principle like the previous rows. For example:

b. The sequence in case 15 is obtained when for s_1 and s_2 are taken the second and the fourth term of the sequence, which is described in case 6.

c. The sequence in case 16 is obtained when for s_1 and s_2 are taken the second and the fourth term of the sequence, which is described in case 8.

PART THREE

In this part it is shown another algorithm that could find 2-good numbers.

According to the conditions of the problem $a_0 = 0,\ a_i \in \{1;2\}$, $i = 1,...,n,\ ...$, and again we will use the symbol $[0;X]$. Then the sum of *n*-th numerator and *n*-th denominator of (*) is $S_n(K_n) = A_n + B_n$ и $\frac{A_n}{B_n} = [0;X]$, i.e. $S_n(K_n)$ are 2-good numbers.

For convenience, we will call 2*i*-good number the value of periodic continued fractions with elements $a_i \in \{1,...\ n,...\}$.

We could find some of them with the following

Algorithm 2.

1. Given a periodic continued fraction.
2. Find out its respective square equation.
3. Determine quadratic irrational (the value of the fraction).
4. Consecutively separating the integer part a_i on every step obtain quadratic irrational giving continued fraction periodical to the initial.
5. It the fraction is mixed-periodic we construct an algorithm to find its value placing the *n*th convergent instead of the period.
6. Describe S_n.

Problem 3. Find the value a of $1+\frac{1/}{/1}+\frac{1/}{/1}+...+=\left(1+\frac{1/}{/1}\right)$.

Solution. In part one we find out that $1(=2^0)$-good numbers are described with Fibonacci numbers (and their multiples). Presented solution will enable us to find 2-good numbers using continued fractions with periodic part $\left(1+\frac{1/}{/1}\right)$. Since the continued fraction is periodical, for a is true that $a=\left(1+\frac{1/}{/1}\right)=\frac{1}{1+a} \Leftrightarrow a^2+a-1=0$.

The roots of the equation are $a_{1,2}=\frac{-1\pm\sqrt{5}}{2}$, but only one of them is positive - $\frac{\sqrt{5}-1}{2}>0$. Since $\left[\frac{\sqrt{5}-1}{2}\right]=0$, consequently $K_1=\frac{1}{\frac{(\sqrt{5}+1)}{2}}$.

From $\frac{\sqrt{5}+1}{2}=1+\frac{1\sqrt{5}-1}{2}=1+\frac{1}{\frac{\sqrt{5}+1}{2}}$ follows $a_1=1$ and so on. $a_i=1$.

Problem 4. Find a when $K_i^1=\frac{1/}{/2}+\left(1+\frac{1/}{/1}\right)$.

Solution. $\frac{1}{2+\frac{\sqrt{5}-1}{2}}=\frac{2}{\sqrt{5}+3}$, hence $a=\frac{2}{\sqrt{5}+3}=\frac{2(3-\sqrt{5})}{4}=\frac{3-\sqrt{5}}{2}$.

Note that a is a root of the equation $a^2-3a+1=0$.

So convergents $\frac{A_i}{B_i}$ of $K_i^1=\frac{1/}{/2}+\left(1+\frac{1/}{/1}\right)$ are $K_1^1=\frac{1}{2}; K_2^1=\frac{1}{3}; K_3^1=\frac{2}{5}; K_4^1=\frac{3}{8},...,$ From them follows the family of 2-good numbers $S_1=3; S_2=4; S_3=7; S_4=11,...$. (It is well-known that these are Lucas number that define the number $\frac{3-\sqrt{5}}{2}$ which is 2-good.

We could obtain the same result if we replace a periodic part in K_i with the respective convergents from Fibonacci numbers: $F_1=\frac{1}{1}, F_2=\frac{1}{2}, F_3=\frac{2}{3},...(F_0)=0$. Consequently:

$$K_1=\frac{1}{2+F_0}, K_2=\frac{1}{2+F_1}=\frac{1}{2+\frac{A_{F_1}}{B_{F_1}}}=\frac{B_{F_1}}{2B_{F_1}+A_{F_1}}, K_3=\frac{1}{2+F_2}=\frac{1}{2+\frac{A_{F_2}}{B_{F_{21}}}}=\frac{B_{F_2}}{2B_{F_2}+A_{F_2}},...$$

By induction on *i* we find out the formula $K_{i+1}^1=\frac{B_{F_1}}{2B_{F_i}+A_{F_i}}$, in which the numbers $S_i=A_i+B_i$ are 2-good (A_{F_i}, B_{F_i} are the i-th numerator and denominator of $\left(1+\frac{1/}{/1}\right)$.

Problem 5. Find a when $K_i^2=\frac{1/}{/2}+\frac{1/}{/2}+\left(1+\frac{1/}{/1}\right)$.

Solution. From $\frac{1}{2+\frac{3-\sqrt{5}}{2}}=\frac{2}{7-\sqrt{5}}$, follows $a=\frac{7+\sqrt{5}}{22}$, which is a root of the equation $11a^2-7a+1=0$. Convergents $\frac{A_i}{B_i}$ of K_i^2 are $K_1^2=\frac{1}{2}; K_2^2=\frac{2}{5}; K_3^2=\frac{3}{7}; K_4^2=\frac{5}{9};...$.

From here follows the family of 2-good numbers $S_i^2: 3,7,10,14,24,...$

It is easy to find the formula $K_{i+1}^2=\frac{1}{2+\frac{1}{2+F_i}}=\frac{2B_{F_i}+A_{F_i}}{5B_{F_i}+2A_{F_i}}$, where $S_i^2=A_i+B_i$ are 2-good numbers, such that $A_i/B_i=[0;2,2,1,...,1,1]=[0;2,2,1,...,1,2]$.

It follows from problem 4 and 5 that for mixed-periodic continued fractions $[0;\underbrace{2,\dots,2}_{P},1,\dots,1,\dots]$ exist the following recurrence relations:

$K_i^P = \dfrac{1}{2+K_i^{(P-1)}}$, where p is the number of pairs.

Analogically we will describe a method for finding another periodic fraction.

Problem 6. Describe all 2-good numbers $S_i^P = A_i^P + B_i^P$, such that $K_i^P = A_i^P / B_i^P = [0;\underbrace{1,\dots,1}_{P},2,2,\dots,2]$.

Solution. A) Let $p=0$. The continued fraction $\left(2+\dfrac{1/}{/2}\right)$ is periodic, so for $a=\dfrac{1}{2+2}$ we have the equation $a^2+2a-1=0$ with positive root $a=-1+\sqrt{2}.\ (a_0=[-1+\sqrt{2}]=0)$.

So $K_1^1=\dfrac{1}{2};\ K_2^1=\dfrac{2}{5};\ K_3^1=\dfrac{5}{12};\ K_4^1=\dfrac{12}{29};\ K_5^1=\dfrac{29}{70};\ K_6^1=\dfrac{70}{169};\dots$ give the family $S_i^1: 3,\ 7,17,41,99,239,\dots$ of 2-good numbers.

Comment. Note that from $\sqrt{2}-1=\dfrac{1}{\sqrt{2}+1}=\dfrac{1}{2+(\sqrt{2}-1)}$ It follows that the number $\sqrt{2}+1$ describes the fraction $[2;2,2,\dots,2]$. It means that $\{\sqrt{2}+1\}$ is fraction which is "subfraction" of the initial.

Б) When $p=1$ from $a=\dfrac{1}{1+\sqrt{2}-1}=\dfrac{\sqrt{2}}{2}$ we find the equation $2a^2-10a-1=0$. In this case the family S_i^2 is defined from the convergents $K_1^2=\dfrac{1}{1+K_1^1}=\dfrac{2}{3};\ K_2^2=\dfrac{1}{1+K_2^1}=\dfrac{5}{7},\dots$ and it is: $5,12,29,70,169,408\dots$

В) When $p\geq 1$ by induction on p we could easy find the relation $K_i^P = A_i^P / B_i^P = \dfrac{A_i+pB_i}{A_i+(p+1)B_i}$, where A_i and B_i are the i-th numerator and denominator of the number $(-1+\sqrt{2})$.

Problem 7. Describe all 2-good numbers $S_i = A_i + B_i$, where $K_i = A_i / B_i = [0;1,2,1,2,\dots,1,2,\dots]$ or $[0;2,1,2,1\dots,2,1,\dots]$.

Solution. Consider the continued periodic fraction $\left(1+\frac{1/}{/2}\right)$. From $a=\frac{1}{1+\frac{1}{2+a}}$ we get the equation $a^2+2a-2=0$ with positive root $(-1+\sqrt{3})$.

Since $a=1+\frac{1/}{/2}=1+\left(\frac{1/}{/2}+\frac{1/}{/1}\right)$ and $-1+\sqrt{3}=\frac{1}{\frac{\sqrt{3}+1}{2}}$, consequently $b=\frac{\sqrt{3}+1}{2}-2=\frac{\sqrt{3}-1}{2}.(2b^2+2b-1=0)$ is the value of $\left(2+\frac{1/}{/1}\right)$.

Consequently the 2-good numbers $S_i=A_i+B_i$, for which A_i/B_i are written with a period $(1,2)$, are defined with the sums of i-th numeration and denominator of the 2i-good numbers $\sqrt{3}-1$ or $\frac{\sqrt{3}-1}{2}$.

The families 2-good numbers in this case are $5,7,19,26,...$ and $3,4,11,15,41,...$, which are sum of the numerators and denominators respectively of the convergents in the presented above:

$$\frac{2}{3},\frac{3}{4},\frac{8}{11},\frac{11}{15},... \text{ and } \frac{1}{2},\frac{1}{3},\frac{3}{8},\frac{4}{11},\frac{11}{30},....$$

Comment. It follows from problem 7 that we could find convergents of mixed-periodic fractions $(b_i=1,2)$, that include $\left(1+\frac{1/}{/2}\right)/\left(2+\frac{1/}{/1}\right)$. The sums of numerators and denominators of the new convergents describe new families 2-good numbers.

Problem 8. Describe all 2-good numbers $S_i=A_i+B_i$, where $K_i=A_i/B_i=[0;1,1,2,1,1,2,...,1,1,2,...]$, $[0;1,2,1,1,2,1...,1,2,1,...]$ or $[0;2,1,1,2,1,1,...,2,1,1,...]$.

Solution. From $a=\frac{1}{1+\frac{1}{1+\frac{1}{2+a}}}$ we could find the equation $2a^2+4a-3=0$ with positive root $a=\frac{-2+\sqrt{10}}{2}$.

Analogically, the second periodic continued fraction has value $b=\frac{\sqrt{10}-1}{3}(3b^2+2b-3=0)$, and the third one - $d=\frac{-2+\sqrt{10}}{3}(3d^2+4d-2=0)$.

Obviously the 2i-good numbers are connected each to other

$$a=\frac{1}{1+b},b=\frac{1}{1+d},d=\frac{1}{1+a}.$$

We find the following family for s_i:

(a) $3\left(\frac{1}{2}\right); 8\left(\frac{3}{5}\right); 11\left(\frac{4}{7}\right); 19\left(\frac{7}{12}\right); 49\left(\frac{18}{31}\right); ...$

(б) $5\left(\frac{2}{3}\right); 7\left(\frac{3}{4}\right); 12\left(\frac{5}{7}\right); 31\left(\frac{13}{18}\right); 43\left(\frac{18}{25}\right); ...$

(в) $3\left(\frac{1}{2}\right); 4\left(\frac{1}{3}\right); 7\left(\frac{2}{5}\right); 18\left(\frac{5}{13}\right); 25\left(\frac{7}{18}\right); 43\left(\frac{12}{31}\right); ...$

Comment. We will notice that the numbers $\frac{\sqrt{10}+1}{3}(=1+\frac{\sqrt{10}-2}{3})$, $\frac{\sqrt{10}+2}{2}\left(=2+\frac{\sqrt{10}-2}{3}\right)$ and $\frac{\sqrt{10}+2}{3}\left(=1+\frac{\sqrt{10}-1}{3}\right)$ are expanded respectively $[1;1,2,1,1,1,2,...,1,2,1,...]$, $[2;1,1,2,1,1,...,2,1,1]$ and $[1;2,1,1,2,1,...,2,1,1,...]$. In more common meaning $(a_0=1;2)$ they are also $2i$-good numbers.

Problem 9. Find the 2i- good numbers which could be expressed as $\left(1+\frac{1/}{/2}+\frac{1/}{/2}\right)$, $\left(2+\frac{1/}{/2}+\frac{1/}{/1}\right)$ or $\left(2+\frac{1/}{/1}+\frac{1/}{/2}\right)$.

Solution. From $a=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+a}}}=\frac{2a+5}{3a+7}$ follows $a=\frac{-5+\sqrt{85}}{6}$,

$(3a^2+5a-5=0)$. Analogically $b=\frac{-5+\sqrt{85}}{10}$ and $b=\frac{-7+\sqrt{85}}{6}$. From these 2i-good number we could find the following family of 2-good numbers:

(a) $5\left(\frac{2}{3}\right); 12\left(\frac{5}{7}\right); 17\left(\frac{7}{10}\right); 46\left(\frac{19}{27}\right);...$

(б) $3\left(\frac{1}{2}\right); 7\left(\frac{2}{5}\right); 10\left(\frac{3}{7}\right); 27\left(\frac{8}{19}\right); 37\left(\frac{11}{26}\right);...$

(в) $3\left(\frac{1}{2}\right); 4\left(\frac{1}{3}\right); 11\left(\frac{3}{8}\right); 15\left(\frac{4}{11}\right); 41\left(\frac{11}{30}\right); ...$

Comment.

1. Our purpose in this part was to consider the 2i-good numbers, because of their relation with other math fields – number theory, (number

properties), algebra (equations, Diophantine equations, periods and so on), coding, set theory, combinatory geometry and so on.

In [1.] there are a lot of examples for the worth of some families of 2-good numbers.

2. On the other hand we realize that even finding those with period 4 and 5 is a big work.

For example: (1, 1, 2, 2), (1, 2, 2, 1), (2, 2, 1, 1) and (2, 1, 1, 2) have values $\frac{-9+\sqrt{221}}{10}, \frac{-5+\sqrt{221}}{10}, \frac{-9+\sqrt{221}}{14} u \frac{-11+\sqrt{221}}{10}$. (The first is root of the equation $5a^2+9a-7=0$).

As we have already proved from each of the families could be get new families “sticking” the respective periods to continued fraction with $b_i = 1, 2$.

3. New families could be obtained by multiplying the previous families with any natural number.

PART FOUR

Problem 10. Find all pairs natural numbers $(k,k+1)$, such that $\frac{k}{k+1}=[0;a_1,...a_n], a_i=1,2.$

Solution. Since $\frac{k}{k+1}=\frac{1}{1+\frac{1}{k}}$, when $k\geq 4$ the fraction is not 2-good. Finally, the numbers are (1; 2)→[0; 2]; (2; 3)→[0; 1, 2] and (3;4)→[0;1,2,1].

(We recall that the fraction [0, a_1,…,2,1] is regarded as 2-good).

Problem 11. Find all pairs natural numbers $(k-1,k+2)$ such that $\frac{k-1}{k+2}=[0;a_1,...,a_n], a_i=1,2$.

Solution. From $\frac{k-1}{k+2}=\frac{1}{1+\frac{3}{k-1}}$ follows two cases:

I. $\left[\frac{k-1}{3}\right]=1$. Then $k=4,5,6$ and the respectively pairs are (3,6); (4,7);(5,8). Their continued fractions respectively are: [0;2];[0;1,1,2,1];[0;1,1,1,2]=[0;1,1,1,1,1].

II. $\left[\frac{k-1}{3}\right]=2$. it is true when $k=7,8,9$. Then $(6,9)=[0;2,1]$, $(7,10)=[0;1,2,2,1]$, $(8,11)=[0;1,2,1,2]$.

Problem 12. Find all pairs natural numbers such that $\frac{k-2}{k+3}=[0;a_1,...,a_n], a_i=1,2.$

Solution. Since $\frac{k-2}{k+3}=\frac{1}{1+\frac{5}{k-2}}$ we have two cases:

I. $\left[\frac{k-2}{5}\right]=1$ when $k=7,8,9,10,11$. We could directly check that $(5,10)=[0,2],(6,11)=[0;1,1,5]$ - does not satisfy the condition, $(7,12)=[0;1,1,2,2],(8,13)=[0;1,1,1,1,1,1], (9,14)=[0;1,1,1,4]$ - does not satisfy the condition.

II. $\left[\frac{k-2}{5}\right]=2$ when $k=12,13,14,15,16$. The pairs that satisfy the condition are: $(10,15);(12,17);(13,18)$.

If $\frac{k-2}{k+3}=\frac{1}{2+\frac{7-k}{k-2}}$ then $3\le k\le 7$. We could directly check and find out that the condition is true when $k=5\to(3;8)$ and $k=7\to(5;10)$.

Problem 13. Find the number of representations of natural numbers up to 60, that are 2-good.

If $n=a+b$ is 2-good number in the interval [2;60] find B(*n*) – (the number of representations) and all possible values of *a* and *b*.

Solution.

S(n)	B(n)	(a+b), a<b	S(n)	B(n)	(a+b), a<b	S(n)	B(n)	(a+b), a<b
1	0	-	21	4	6+15,7+14, 8+13, 9+12	41	3	11+30,12+29, 17+24
2	0	-	22	2	6+16,8+14	42	4	12+30,14+28, 16+26,18+24
3	1	1+2	23	0	-	43	1	18+25
4	1	1+3	24	5	6+18,7+17, 8+16,9+15, 10+14	44	5	11+33, 12+32, 13+31, 16+28, 17+27
5	1	2+3	25	2	7+18,10+15	45	5	12+33, 13+32, 15+30, 18+27, 19+26
6	1	2+4	26	3	7+19,10+16, 11+15	46	2	17+29, 19+27
7	2	2+3,3+4	27	3	8+19,9+18, 10+17	47	2	13+34,18+29
8	2	2+6,3+5	28	3	7+21,8+20, 12+16	48	5	12+36, 14+34, 16+32, 18+30, 20+28
9	1	3+6	29	3	8+21,11+18, 12+17	49	4	14+35,18+31, 19+30, 21+28
10	2	3+7,4+5	30	5	8+22,9+21, 10+20, 11+19,12+18	50	5	14+36, 15+35, 19+31, 20+30, 21+29
11	2	3+8,4+7	31	2	12+19,13+18	51	3	15+36, 17+34, 21+30
12	3	3+9,4+8, 5+7	32	2	8+24,12+20	52	2	14+38, 20+32
13	1	5+8	33	3	9+24,11+22, 12+21	53	0	-
14	2	6+8,4+10	34	3	10+24,13+21,	54	5	15+39, 16+38,

					14+20			18+6, 20+34, 21+33
15	3	4+11, 5+10, 6+9	35	3	10+25,14+21, 15+20	55	5	16+39, 20+35, 21+34, 22+33, 24+31
16	2	4+12, 6+10	36	5	9+27,10+26, 12+24 14+22,15+21	56	4	14+42, 16+40, 21+35, 24+32,
17	2	5+12, 7+10	37	2	10+27,11+26	57	3	19+38, 21+36, 24+33
18	3	5+13, 6+12, 7+11	38	2	14+24,16+22	58	4	16+42, 17+41, 22+36, 24+34
19	2	7+12, 8+11	39	2	13+26,15+24	59	0	-
20	3	5+15, 6+14, 8+12	40	5	10+30,11+29, 12+28, 15+25, 16+24	60	7	15+45, 16+44, 18+42, 20+40, 22+38, 24+36, 25+35

Comment. Some conclusions which follow from the results above:

1. All numbers which are multiple of 2-good numbers are 2-good.
2. When $n>3$ it is true that $B(2n)\geq 2$
3. Let $n=a+b$ is 2-good number, $\frac{a}{b}=[0,X]$ and *a* and *b* are also 2-good numbers. There are even *n*, for which such representation is possible only for even *a* and *b*. For example, 14 may be expressed only as 4+10$\to\frac{4}{10}=[0;2,2]$ and 6+8$\to\frac{6}{8}=[0;1,2,1]$. In the table these numbers are coloured in red.
4. In the investigated interval there are some prime numbers that are not 2-good – 23, 53, 59. In the table they are coloured in blue.

Conjecture. If a number is not 2-good, its multiple number (include the degrees) are 2-good numbers.

It follows from the comment above that the part of the conjecture about multiple numbers is correct.

$23^2=\frac{155}{374}=[0;2,2,2,2,1,2,2,1]$. The conjecture about the degrees is based on the following computer programme which check the degrees of the numbers which are found above and which are not 2-good:

```
#include<cstdio>
#include<iostream>
#include<cstring>
#include<cstdlib>
using namespace std;
int gcd(int a,int b)
{
        return
(b==0)?a:gcd(b,a%b);
        }
        class drob
```

```
{
 public:
 int c,z;
 void simple()
 {
  int d=gcd(c,z);
  c/=d;
  z/=d;
 }
 void recip()
 {
  if(c!=0)
   swap(c,z);
 }
 drob operator+ (drob a)
 {
  drob res;
  res.c=a.c*z+c*a.z;
  res.z=a.z*z;
  res.simple();
  return res;
 }
};
int num;
void check(int x)
{
 int k;
 drob f;
 f.c=x;
 f.z=num-x;
 f.simple();
 for(k=1;k<=13;k++)
 {
  int d=1<<(k+1)-1;
  int i,j;
  for(i=0;i<=d;i++)
  {
   drob res;
   res.c=0;
   res.z=1;
   for(j=0;j<k;j++)
    if(i&1<<j)
    {
     drob t;
     t.c=2;
     t.z=1;
     res=res+t;
     res.recip();
    }
   else
    {
     drob t;
     t.c=1;
     t.z=1;
     res=res+t;
     res.recip();
    }
   res.simple();
   if(res.c==f.c&&res.z==f.z)
   {
    printf("fraction %d/%d\n",res.c,res.z);
    printf("representation 0");
    for(j=k-1;j>=0;j--)
     if(i&1<<j)
      printf("2");
     else
      printf("1");
    printf("\n");
    //exit(0);
   }
  }
 }
 //printf("checked %d\n",x);
}
int main()
{
 scanf("%d",&num);
 int i;
 for(i=1;i<=num/2;i++)
  if(i<num-i)
   check(i);

 return 0;
}
```

Conjecture. If any number is not 2-good, consequently it is prime number.

Problem 14. Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.

Solution.

Suppose that exist numbers for which the statement is not true. Let $2k$ is the minimal of them.

We will consider only the case when k is even, since if k is odd, it is analogical. Consequently all pairs odd numbers, which sum is $2k$ are:

$1 \quad + \quad 2k-1$

$3 \quad + \quad 2k-3$

$5 \quad + \quad 2k-5$

...

$k-1 \quad + \quad k+1$

There are at all $\left[\frac{k-1}{2}\right]$ groups with two odd numbers in each. We will prove that there is at least $\left[\frac{k-1}{2}\right]+1$ odd numbers that are smaller than $2k$ and that are 2-good, i.e. we will prove that there is at least one group, in which both numbers are 2-good and so $2k$ will be 2-good number.

Consider the number $2k-2$, it is smaller than $2k$, hence it could be expressed as a sum of two odd 2-good numbers. Consequently there are at least 2 odd numbers - k_1, k_2, smaller than $2k$, which are 2-добри.

$2k-2$ also could be expressed in the wanted way, consequently there is at least one more 2-good odd number - k_3, which is different from k_1, k_2. And so on because of each even number, smaller than $2k$, there is one new 2-good odd number.

$\frac{2k-2-6}{2}=k-4$ is the number of all even numbers which are smaller than $2k$ and could be expressed by the wanted way. (From 2k numbers we eliminate 2k and the first 6 numbers.) Consequently the number of odd 2-good numbers, smaller than $2k$, is at least $k-4+1=k-3$. The numbers that we want are greater than 6, so it is obvious that $k-3>\left[\frac{k-1}{2}\right]$. Consequently at least $\left[\frac{k-1}{2}\right]+1$ odd numbers, smaller than $2k$, are 2-good. We proved that $2k$ is 2-good number, but this is contradiction. Consequently all even numbers could be expressed as a sum of two odd 2-good numbers.

PART FIVE

In general, a number *n* is called *k*-good if it can be expressed as the sum of two positive integers $n = a + b$ so that $a < b$ and the continued fraction for $\frac{a}{b}$ has all its partial quotients not greater than *k*.

Problem 15. Does there exist a positive integer *k* such that all positive integers n > 2 are *k*-good?

Solution.

We will prove that every number *n* is (*n*-1)-good.

We have $n = 1 + (n-1) \rightarrow [0; n-1] = [0; n-2, 1]$.

Let $n = k + (n-k),\ k \le (n-k)$ are the representations of any (*n*–*1*)-good number. Since $\left[\frac{n-k}{k}\right] = \left[\frac{n}{k}\right] - 1 \le \left[\frac{n}{1}\right] - 1 \le \left[\frac{n-1}{1}\right]$, then $n = k + (n-k) : \frac{k}{n-k} = [0; X],\ x = a_1, ..., a_n$, are such that $a_i \le n-k$. Obviously the greatest value of $n-k$ is reached when $k = 1$. This way the statement is proved.

Problem 16. Prove that all numbers which are multiple of *k*-good numbers are *k*-good.

Solution.

Let $n > 2$ is prime *k*-good number. It follows from problem 16 that always we can choose such *k* that the number is good. For the number k^n $(n > 1)$ we have $k^n = k.k^{n-1} = (k-p).k^{n-1} + p.k^{n-1}$, where $1 \le p < k - p \le k - 1$.

It follows from part one that $(p, k-p) = 1$ and $(p, k) = 1$. Therefore $\frac{p.k^{n-1}}{(k-p).k^{n-1}} = \frac{p}{k-p} = [0; X]$ for any *p*.

PART SIX

In this part we will called $2^{\pm}$- good the numbers *n*, for which is true $n = a \pm b,\ \frac{a}{b} = [0; X]$.

Problem 17. Describe the set of numbers *n* with the property: there exist two positive integers $b > a$ such that $n = a + b$ or $n = b - a$ and the partial quotients of $\frac{a}{b}$ are equal to 1 or 2.

Solution. It is obviously that all 2-good numbers are $2^{\pm}$- good. Consequently it is enough to prove that all numbers that are not 2-good are $2^{\pm}$- good.

Let *t* is not 2-good number and let $T = (a + t) - a$. We will find the value of a (natural number), for which $\frac{a}{a+t} = [0, X]$.

Let $a = t$. Then $\frac{t}{2t} = \frac{1}{2} = [0; 2]$, and when $a = 2t$ we get $\frac{2t}{2t + t} = \frac{2}{3} = [0; 1, 1, 1]$. Consequently the number $T = t$ is $2^{\pm}$- good.

Comment.

1. We show that if 23 (53, 59) is not 2-good number, it is $2^{\pm}$- good. Something more, it is $1^{\pm}$- good. With this method we can change each non-2-good (non 1-good) in $2^{\pm}$-good ($1^{\pm}$- good). Consequently any natural number $n > 2$ is $1^{\pm}$- good and $2^{\pm}$- good.

2. Since reasoning does not dependent on the value of *k*, it follows that each non-*k*-good number is $k^{\pm}$- good.

3. It also follows that the numbers 1 and 2 are $k^{\pm}$- good numbers.

We will called the number $n = a + b,\ (n > 2)$ 2^s-good $\left(s \in Z_0^+\right)$, if $\frac{a}{b} = [0; Y],\ Y = a_1, a_2, ..., a_k,\ a_i \in \left\{2^0, 2^1, ..., 2^s\right\}$, for every *k*.

Problem 18. Find all 2^s-good numbers in the interval $[3; 20]$.

Solution.

Obviously every 2^{s-1}-good number is 2^s-good. When $s = 2, 3, 4$, 2^s-good number, for which at least one $a^i = 2^s$, are:

$4=2^2$	5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
$8=2^3$	9, 10, 17, 18, 19
$16=2^4$	17, 18

Comment.

Note that the numbers *23, 53* and *59* are not 2-good, but they are 2^2-good which follows from the fractions $\frac{9}{14}$, $\frac{22}{31}$ и $\frac{16}{43}$.

Problem 19. Find and describe the numbers $S_i = A_i + B_i$ and the value of the number $A_i / B_i = \left(4+\frac{1/}{/4}\right)$.

Solution. Since $\left(4+\frac{1/}{/4}\right)$ is a periodic continued fraction, then $a = \frac{1}{4+a} \Leftrightarrow a^2 + 4a - 1 = 0 \Leftrightarrow a = -2 \pm \sqrt{5}.$

$$A_i / B_i : \frac{1}{4}, \frac{4}{17}, \frac{17}{72}, \frac{75}{305}, \frac{305}{1272}, ...$$

$$S_i = 5, 21, 99, 379, 1577, ...$$

Problem 20. Find the value of the number $A_i / B_i = \left(1+\frac{1/}{/4}\right)$ and describe the numbers $S_i = A_i + B_i$.

Solution. From $a = \frac{1}{1+\frac{1}{4+a}} \Leftrightarrow a = -2 \pm \sqrt{6}\,(a^2 + 4a - 4 = 0)$.

$$A_i / B_i : \left(\frac{1}{1}\right), \frac{4}{5}, \frac{5}{6}, \frac{24}{29}, \frac{29}{35}, \frac{140}{169}, ...$$

$$S_i = 9, 11, 53, 64, 309, ...$$

Comment.

Having in mind that $\frac{\sqrt{ab(ab+4)} - ab}{2a}$ is the positive root of the equation $a = \frac{1}{a + \frac{1}{b+a}}$, we can find out other families 2^s-good numbers, according to the values of *a* and *b*.

SUMMARY

The following results are achieved:

PART ONE

1. All numbers $n > 2$ that can be expressed as the sum of two positive integers $n = a + b$ so that $a < b$ and the continued fraction for $\frac{a}{b}$ has all its partial quotients equal to 1 are found.

PART TWO

2. Different families of 2-good numbers are described.
3. A conjecture for finding all 2-good numbers is deduced.

PART THREE

4. It is described a second method for finding 2-good numbers
5. A second algorithm for finding all 2-good numbers through infinite periodical fractions is described.
6. Some generalizations for different kind of 2-good numbers are made.

PART FOUR

7. The number of representations of natural numbers up to 60, that are 2-good is found, as well as, some proprieties as a result.
8. There are some investigations and the following conjecture is based on them: If any number is not 2-good, consequently it is prime number.
9. It is proved that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers.

PART FIVE

10. A positive integer *k* such that all positive integers n > 2 are *k*-good is found.
11. It is proved that all numbers which are multiple of *k*-good numbers are *k*-good.

PART SIX

12. It is proved that all natural numbers are $k^{\pm}$- good and in particular $2^{\pm}$- good.
13. A definition of 2^s-good number is given and some families of such numbers are described.

REFERENCES

[1.] http://www.research.att.com/~njas/sequences
[2.] Любенова, Г., „Непрекъснати дроби”, София, 1986
[3.] Бескин Н., „Цепные дроби”, Квант, 1983/1