I^{st} International Tournament of Young Mathematicians
27th June – 3rd July 2009, Paris, France

PROBLEM EIGHT

POSITIVITY OF SYMMETRIC POLYNOMIALS

A polynomial P(x, y) with real coefficients is symmetric if the equality P(x,y)= P(y, x) holds for all $x, y \in R$.

1. Let $P(x, y) = x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a > -1$.

2. Let $P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4.

(a) Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a < -4, b > \frac{a^2+8}{4}$ or $a > -4$, $b > -2a - 2$.

(b) Prove that $P(x, y) > 0$ for all $x, y \neq 0$ if and only if $|a| > 4$, $b > \frac{a^2+8}{4}$ or $|a| \leq 4$, $b > 2|a| - 2$.

TEAM: **MATH HIGH SCHOOL *NANCHO POPOVICH*, BULGARIA**

Leaders: LOZANOV Chavdar, CHRISTOVA Madlen
Contestants: VALKOV Mladen, YORDANOVA Yoana, MARKOVA Magdalena, KOSTADINOVA Georgina, ALEKSANDROVA Polina, ENCHEV Ivaylo

PART ONE.

Problem 1. Let $P(x, y) = x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that P(x, y) > 0 for all x, y > 0 if and only if a >−1.

Solution. We have

$$P(x,y) = x^3 + ax^2y + axy^2 + y^3 = (x+y)(x^2 + (a-1)xy + y^2) = (x+y)xy\left(\frac{x}{y} + \frac{y}{x} + a - 1\right)$$

Put $t = \frac{x}{y} + \frac{y}{x}$, then $P(x, y) > 0$ iff $f(t) = t + a - 1 > 0 \Leftrightarrow a > 1 - t$. Since $t \geq 2$ for all $x, y > 0$, we have $a > -1$.

Problem 2. Let $P(x,y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4.

(a) Prove that P(x, y) > 0 for all x, y > 0 if and only if $a < -4, b > \frac{a^2+8}{4}$ or $a \geq -4$, $b > -2a - 2$.

(b) Prove that P(x, y) > 0 for all x, y≠ 0 if and only if |a| > 4, $b > \frac{a^2+8}{4}$ or |a|≤4, b > 2|a| − 2.

Solution. We have

$P(x,y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4 = x^2y^2[t^2 + at + b - 2]$, where $t = \frac{x}{y} + \frac{y}{x}$.

(a) $P(x, y) > 0$ iff, $f(t) = t^2 + at + b - 2 > 0$.

Let $D = a^2 - 4b + 8 < 0$ or $b > \frac{a^2+8}{4}$. Then $f(t) = t^2 + at + b - 2 > 0$, so $P(x, y) > 0$.

Let $D = a^2 - 4b + 8 \geq 0$. Since $t \geq 2$ for all $x, y > 0$, we have $-\frac{a}{2} < 2 \Leftrightarrow a > -4$, $f(2) = 2a + b + 2 > 0 \Leftrightarrow b > -2a - 2$.

(b) If $xy > 0$, $t \geq 2$ and if $xy < 0$, $t \leq -2$.

Let $D = a^2 - 4b + 8 < 0$ or $b > \frac{a^2+8}{4}$. Then $f(t) = t^2 + at + b - 2 > 0$, for all $x, y \neq 0$ so $P(x, y) > 0$.

Let $D = a^2 - 4b + 8 \geq 0$. Now $P(x, y) > 0 \Leftrightarrow f(t) > 0$ if and only if

$$\begin{matrix} f(2) = 2a + b + 2 > 0 \Leftrightarrow b > -2a - 2 \\ f(-2) = -2a + b + 2 > 0 \Leftrightarrow b > 2a - 2 \end{matrix} \quad \text{or } b > 2|a| - 2$$

and $-2 < -\frac{a}{2} < 2 \Leftrightarrow -4 < a < 4$ or $|a| \leq 4$.

Problem 3. Let $P(x,y)=x^5+ax^4y+bx^3y^2+bx^2y^3+axy^4+y^5$ be a symmetric polynomial of degree 5.

(a) Prove that P(x, y) > 0 for all x, y > 0 if and only if $b>\frac{(a-1)^2+4}{4}$ or $a>-3$, $b>-a-1$.

(b) Prove that P(x, y) > 0 for all x, y≠ 0 if and only if $b>\frac{(a-1)^2+4}{4}$ or $-3<a<5$, $b>-a-1$, $b>3a-5$.

Solution. We have

$P(x,y)=(x+y)x^2y^2[t^2+(a-1)t+b-a-1]$, where $t=\frac{x}{y}+\frac{y}{x}$.

(a) $P(x,\ y)>0$ iff, $f(t)=t^2+(a-1)t+b-a-1>0$.

Let $D=(a-1)^2-4b+4<0$ or $b>\frac{(a-1)^2+4}{4}$. Then $f(t)>0$, so $P(x,\ y)>0$.

Let $D=(a-1)^2-4b+4\geq 0$. Since $t\geq 2$ for all $x,y>0$, we have $-\frac{a-1}{2}<2\Leftrightarrow a>-3$, $f(2)=a+b+1>0\Leftrightarrow b>-a-1$.

(b) If $xy>0$, $t\geq 2$ and if $xy<0$, $t\leq -2$.

Let $D=(a-1)^2-4b+4<0$ or $b>\frac{(a-1)^2+4}{4}$. Then $f(t)>0$, for all $x,y\neq 0$ so $P(x,\ y)>0$.

Let $D=(a-1)^2-4b+4\geq 0$. Now $P(x,\ y)>0\Leftrightarrow f(t)>0$ if and only if

$f(2)=a+b+1>0\Leftrightarrow b>-a-1$

$f(-2)=-3a+b+5>0\Leftrightarrow b>3a-5$

and $-2<-\frac{a-1}{2}<2\Leftrightarrow -3<a<5$.

Comment. In the same way we can proceed with

$$P(x,y)=x^6+ax^5y+bx^4y^2+cx^3y^3+bx^2y^4+axy^5+y^6=x^3y^3[t^3+at^2+(b-3)t+c-2a]$$

$$P(x,y)=x^7+ax^6y+bx^5y^2+cx^4y^3+cx^3y^4+bx^2y^5+axy^6+y^7$$
$$=(x+y)x^3y^3[t^3+(a-1)t^2+(b-2)t+c-2a-b+1]$$

Where $t=\frac{x}{y}+\frac{y}{x}$ and $|t|\geq 2$, for $x,y\neq 0$