I[st] International Tournament
of Young Mathematicians
27th June – 3rd July 2009, Paris, France

PROBLEM SEVEN

PLACEMENTS OF PENTOMINOES

1. Given an *m*. *n* rectangle, denote by T (*m*, *n*) the minimum number of non-overlapping pentominoes that must be placed (along the grid lines) so that there is no place on the free cells for another pentomino? Find or estimate the number T (*m*, *n*) and give an algorithm for constructing suitable placements.

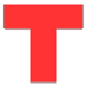
2. Two players alternately place pentominoes on the free cells of an*m. n* rectangle, along the grid lines. The loser is the one who cannot place a pentomino. Does any player have a winning strategy?

3. Study the previous questions for other polyominoes.

TEAM: MATH HIGH SCHOOL *NANCHO POPOVICH,* BULGARIA

Leaders: LOZANOV Chavdar, CHRISTOVA Madlen
Contestants: VALKOV Mladen, YORDANOVA Yoana, ALEKSANDROVA Polina, MARKOVA Magdalena, KOSTADINOVA Georgina, ENCHEV Ivaylo

PART 1.

We will denote a pentomino of the type  with T. Without loss of generality let $n \geq m$.

Problem 1.1. Find $T(3m, n)$ in a rectangle, if: A) *m*=1 B) *m*=2

Investigation.

A) *m*=1

In ***Figure 3***, ***Figure 4*** and ***Figure 5*** is shown the number and the way of placing T-minos in a strip $P(3,n)$. The placing in the other cases for $P(3m,n)$ gives an idea for estimating this number.

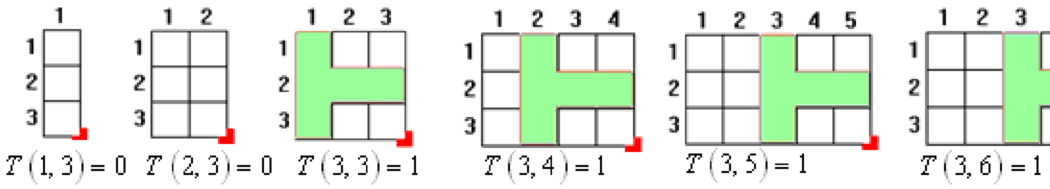


Figure 1.

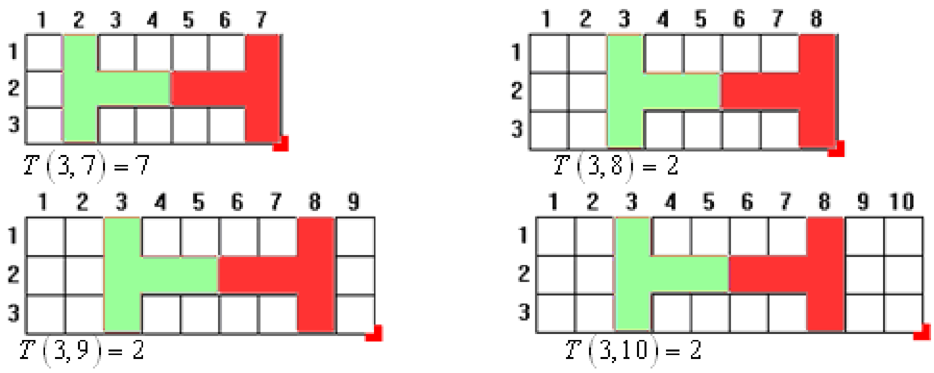


Figure 2.

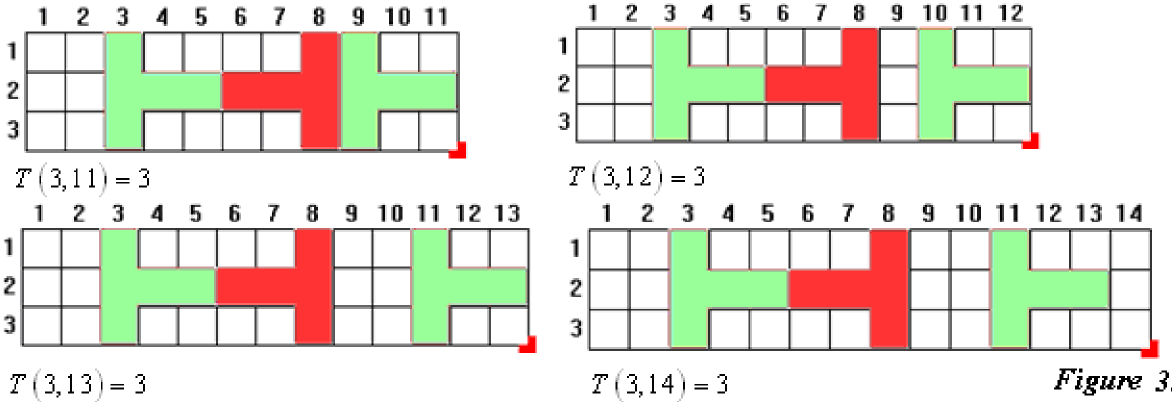


Figure 3.

B) *m*=2

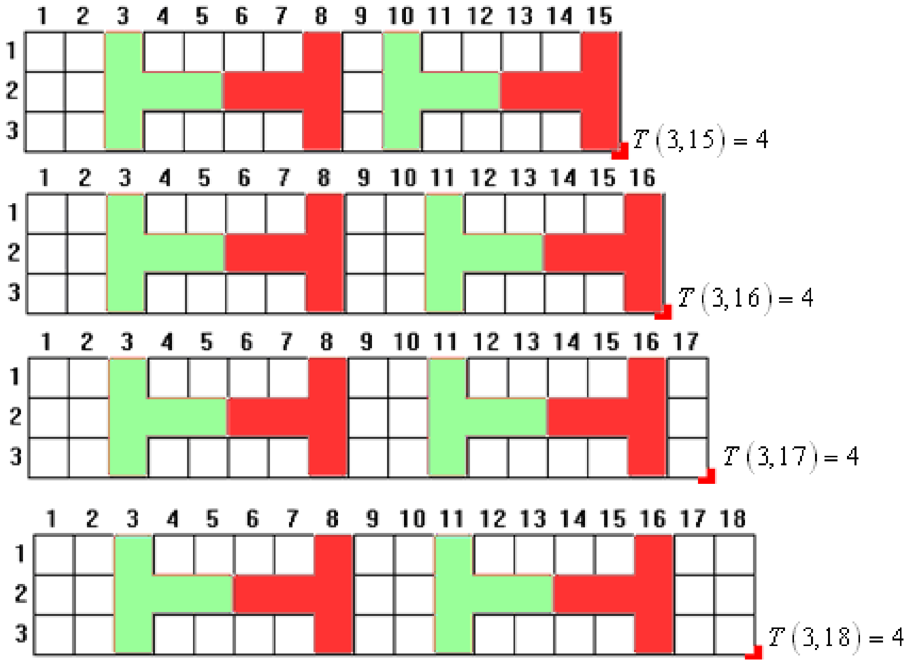


Figure 4.

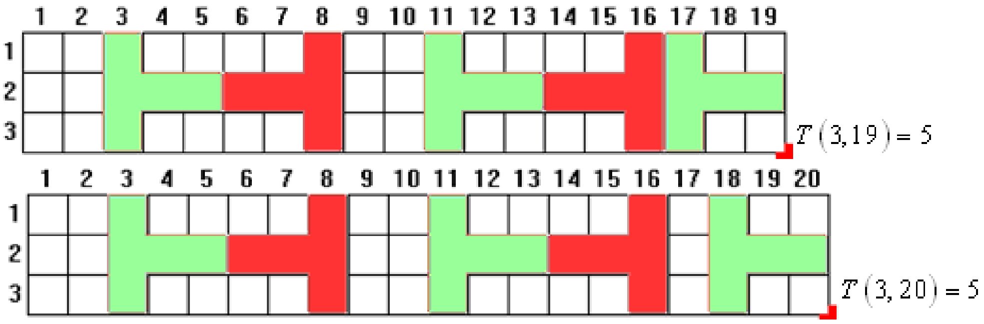


Figure 5.

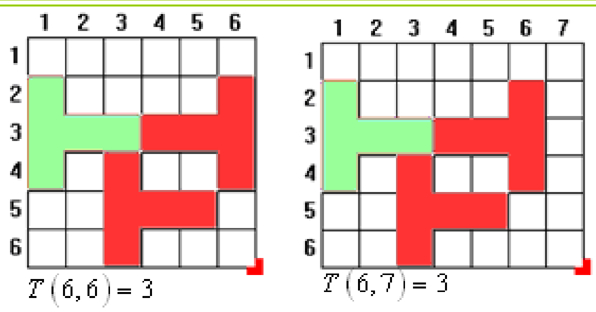


Figure 6.

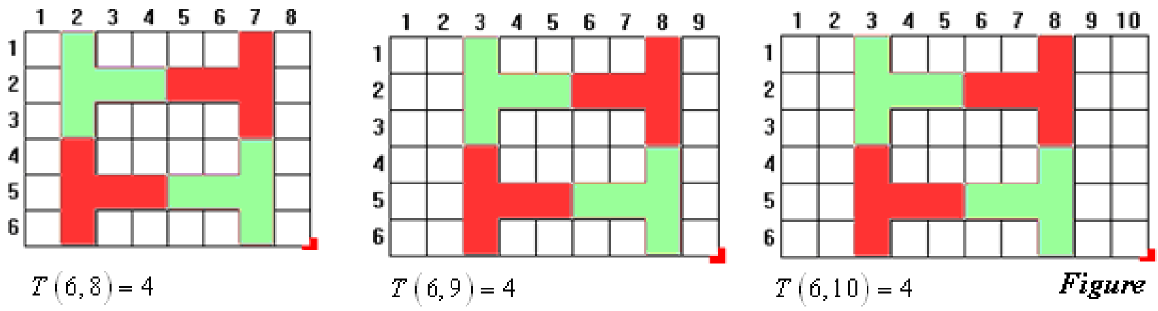


Figure 7.

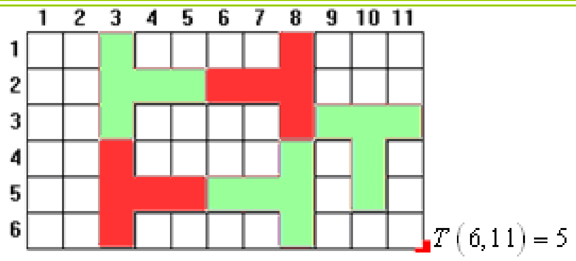


Figure 8.

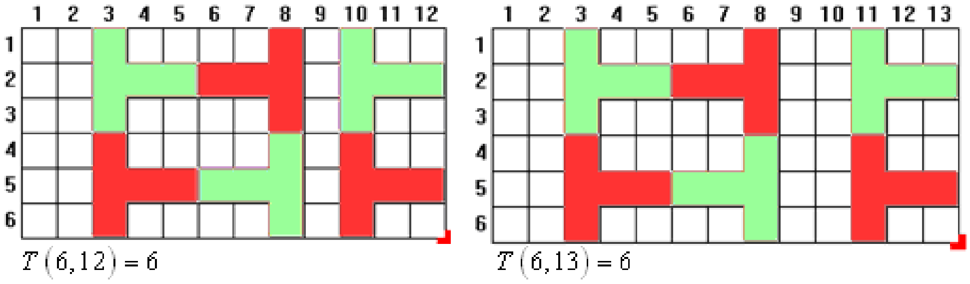


Figure 9.

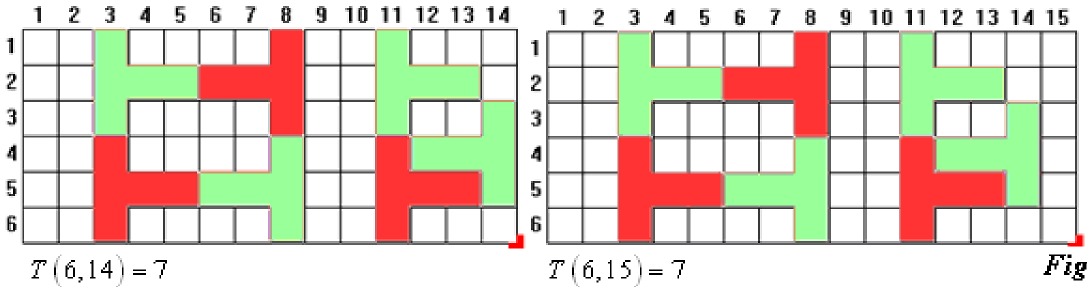


Figure 10.

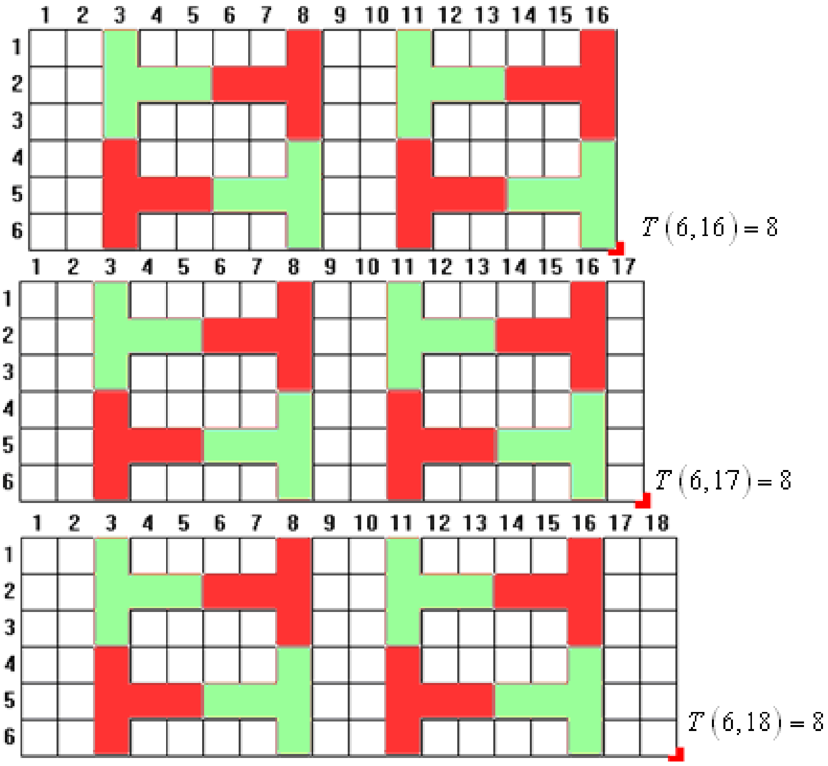
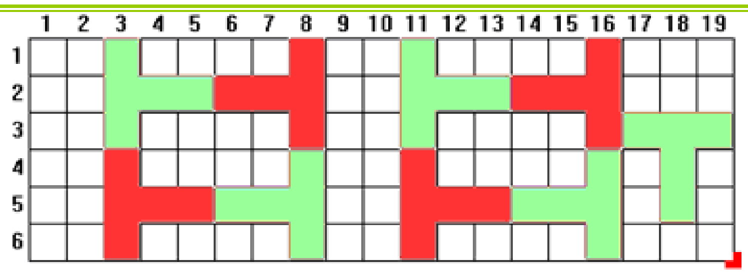
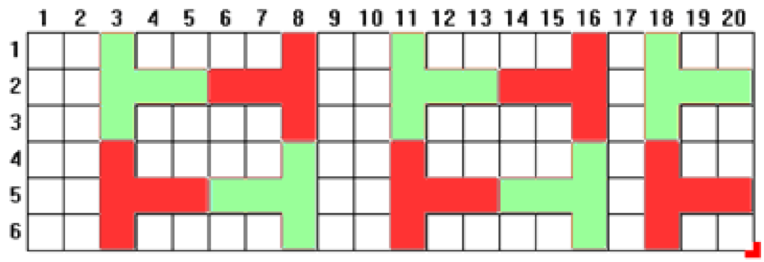


Figure 11.



$T(6,19)=9$ **Figure 12.**



$T(6,20)=10$ **Figure 13.**

Conclusion 1.1.

$$T(3,\ n)=k+1,\ 4k+3\le n\le 4k+6$$

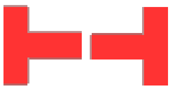
$$T(3,n)\le T(3m,n)\le m(k+1),\ 4k+3\le n\le 4k+6,\ k\in Z_0^+,\ n>3m,\ m\in \mathrm{N}$$

$$mT(3,n)-1\le T(3m,n)\le mT(3,n)+1$$

The third estimation is necessary for estimating $T(3m,3m)$.

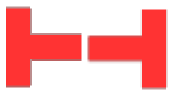
Problem $1.1'$**.** Give an algorithm for constructing suitable placements of $T(3m,n)$, where $m\in \mathrm{N}$ for the estimation made.

Algorithm 1.1.

1) Note that, since it is important how the pentomino is orientated, in the algorithm we will define explicitly its orientation  and indicate the type of the couple they make .

2) For $T(3,n)$ start placing T-minos by leaving k empty columns ($n\equiv k(\mathrm{mod}\,3),\ k\ne 0$).

3) If $k=0$, leave two empty columns before and one after placing of pentominoes. Then follow steps 4) and 5) for $n\le 6$.

4) Leave k empty columns ($n\equiv k(\mathrm{mod}\,3),\ k\ne 0$) and add a new symmetrical placed pentomino next to the one with right orientation T_1 .

5) After putting such a couple pentominoes detach the obtained rectangle $P(3,8)$. Then follow steps 4) and 5) for $6<n\le 10$.

6) Again leave k empty columns ($n-8a\equiv k(\mathrm{mod}\,3),\ k\ne 0,\ a\in \mathrm{N}$) and if $n-8a\ne 3$, go to step 8).

7) For $n-8a=3$ place compactly to the couple of pentominoes one new.

8) Repeat steps from 2) to 5). Follow steps 6), 7) and 8) if $n>10$.

9) For $T(3m,n)$ observe steps from 1) to 8) and paste m alike to the obtained strip $T(3,n)$ downwards.

10) Determine the number of pentominoes needed by the estimation: $mT(3,n)-1 \le T(3m,n) \le mT(3,n)+1$.

Commentary. Because the estimation is non-symmetrical in relation to *m* and *n*, we will consider only rectangles from the type $(3m,n)$, where $3m < n$.

Problem 1.2. Find $T(4m,n)$, if: A) *m*=1 B) *m*=2

<u>Investigation.</u>

A) *m*=1

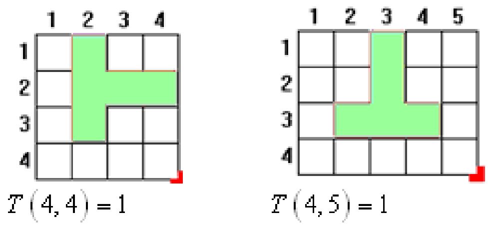


Figure 14.

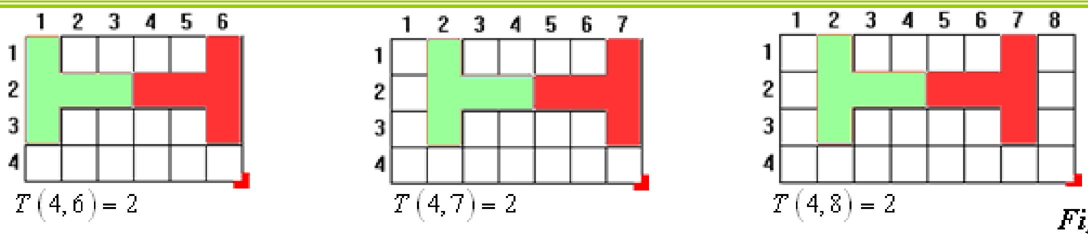


Figure 15.

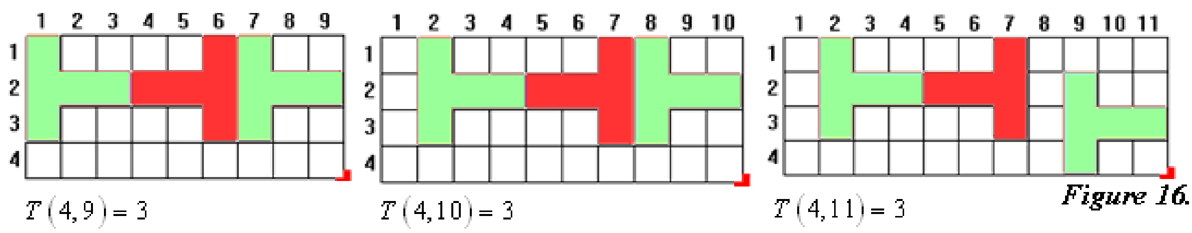


Figure 16.

B) *m*=2

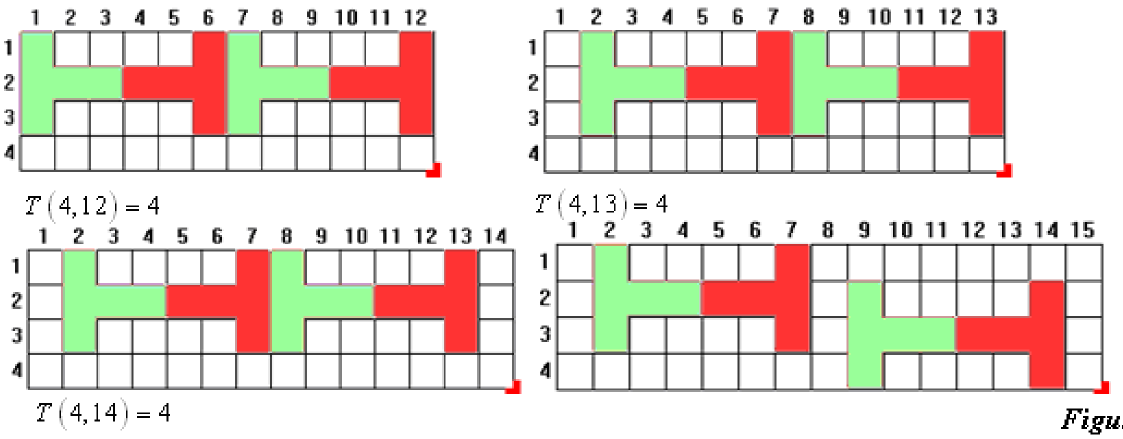


Figure 17.

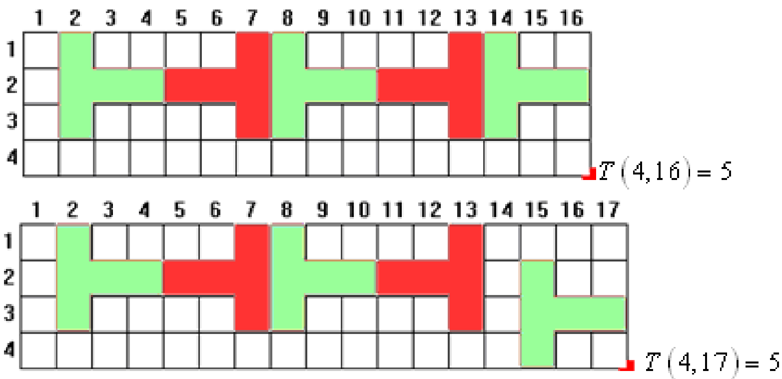


Figure 18.

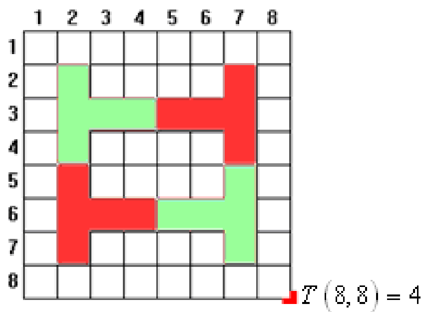


Figure 19.

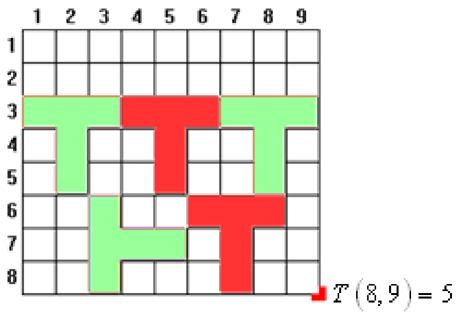


Figure 20.

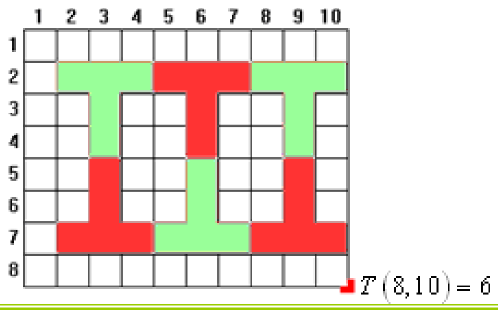


Figure 21.

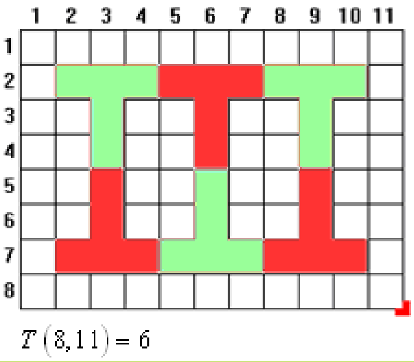


Figure 22.

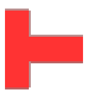
Conclusion 1.2.

$$T(4,n) \le k,\ \ 3k \le n \le 3(k+1),\ \ k \in \mathbb{N}$$
$$mT(4,n) - 1 \le T(4m,n) \le mT(4,n) + 1$$

Problem $1.2'$**.**

Give an algorithm for constructing suitable placements of $T(4m,n), m \in \mathbb{N}$

Algorithm 1.2:

1) In the algorithm we use T with orientation  (or , where is needed, to form couples).

2) For $T(4,n)$ and $n \le 5$ place a pentomino as shown on ***Figure 14.***

3) Start putting  by placing symmetrically a new one and leave k empty columns ($n \equiv k \pmod 3$, $k \ne 2$).

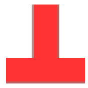
4) If $k=2$, leave a column before and after placing a pentomino couple. Follow steps 3) and 4) when $5<n\le 10$.

5) Then detach the rectangle $P(4,7)$, which we got.

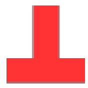
6) Let the next column be empty again and place new , so that the empty row does not coincide with the one, determined from the first couple pentominoes.

7) If $n-7a=3$, put compactly to the couple a new T-mino.

8) Repeat steps from 3) to 7). We execute 3) and 4) when $n>10$.

9) For $T(4m,n)$ place a T with orientation  (or , where is necessary such that they would form a couple).

10) Leave the first column as well as the first and last rows empty (if $n\equiv k(\mathrm{mod}\,3),\ k\ne 0$).

11) Put another pentomino  and add symmetrically other  (an exception is $T(8,9)$, whose placement is shown on ***Figure 20.***).

12) Determine the number of the pentominoes needed by the estimation: $mT(4,n)-1\le T(4m,n)\le mT(4,n)+1$.

Problem 1.3. Find the values of *m* and *n* such that in a rectangle $P(m,n)$ could be placed exactly one pentomino.

Solution. On ***Figure 1*** and ***Figure 14*** are shown the rectangles, where can be placed at least one T-mino: $P(3,3)$, $P(3,4)$, $P(3,5)$, $P(3,6)$, $P(4,4)$ and $P(4,5)$.

Problem 1.4. Find the values of *m* and *n* such that in a rectangle $P(m,n)$ could be placed *k* T-minoes, where $2\le k\le 10$.

Solution.

k* = 2** – On ***Figure 2 and ***Figure 15*** are shown the cases when *m is* 3 and 4. Two pentominoes can be placed also in a rectangle $P(5,n)$, $n=5,6,7,8$ (as can be seen in ***Figure 23***).

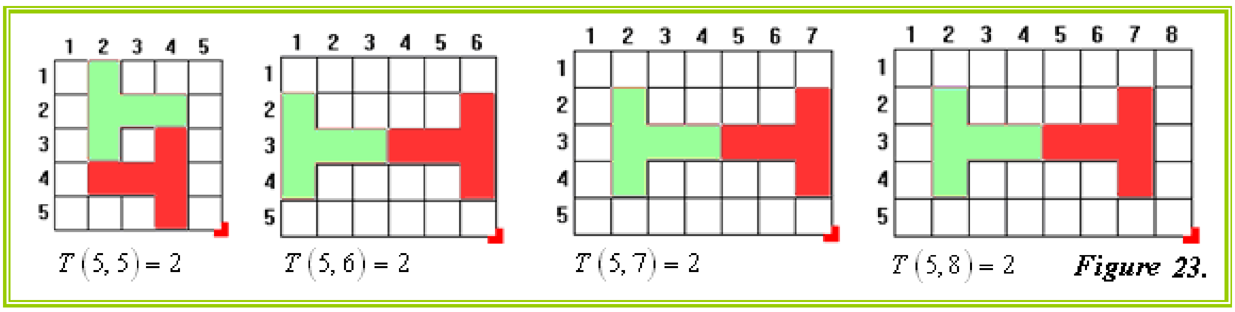


Figure 23.

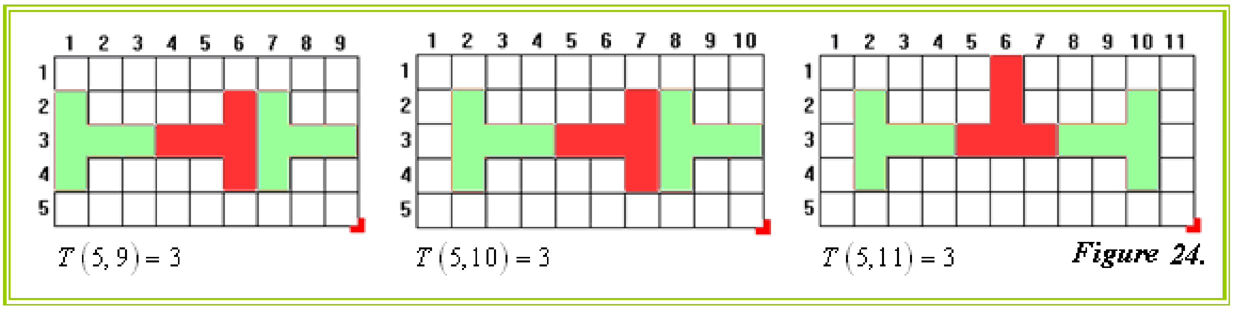


Figure 24.

K* = 3** – On ***Figure 3, Figure 6 and ***Figure 16*** are given the cases when *m* is respectively 3, 6 and 4. For $P(5,n)$ we have that $n = 9,10,11$ (***Figure 24***).

K* = 4** – On ***Figure 4, ***Figure 7***, ***Figure 17*** and ***Figure 19*** are shown the cases when m is respectively 3, 6, 4 и 8. For $P(5,n)$ we have $n = 12,13,14$ (***Figure 25***) and for $P(7,n), n = 7,8$ (***Figure 26***).

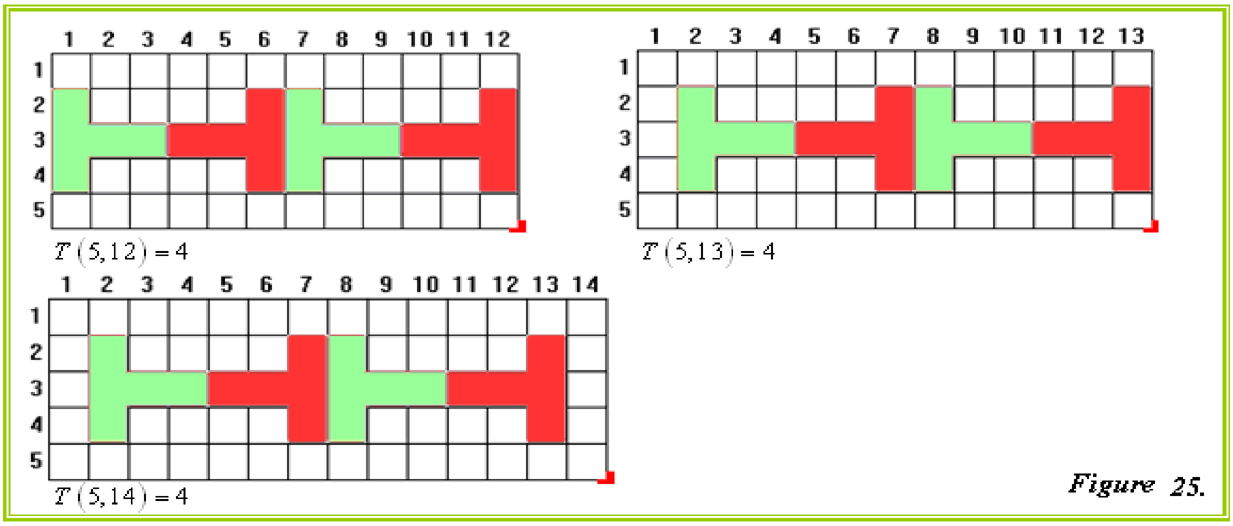


Figure 25.

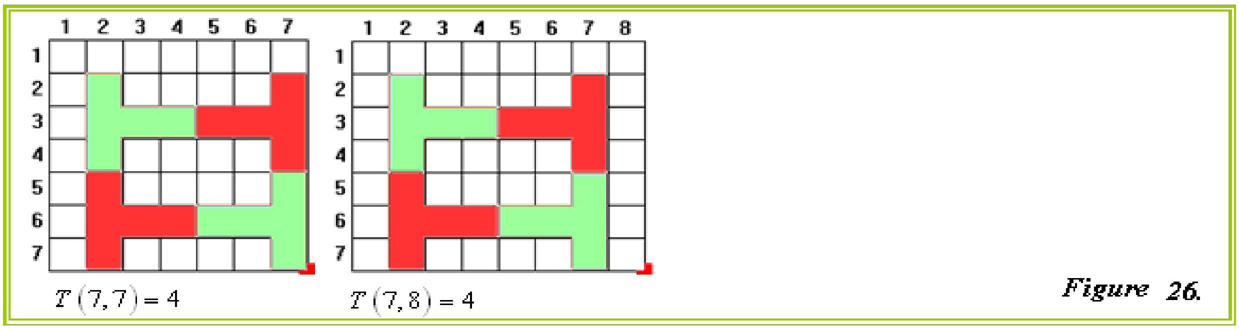


Figure 26.

K* = 5** – On ***Figure 5, ***Figure 8***, ***Figure 18*** and ***Figure 20*** are given the cases when *m is respectively* 3, 6, 4 and 8. For $P(5,n)$ we have that $n = 15,16$ (***Figure 27***) and for $P(7,n), n = 9,10$ (***Figure 28***).

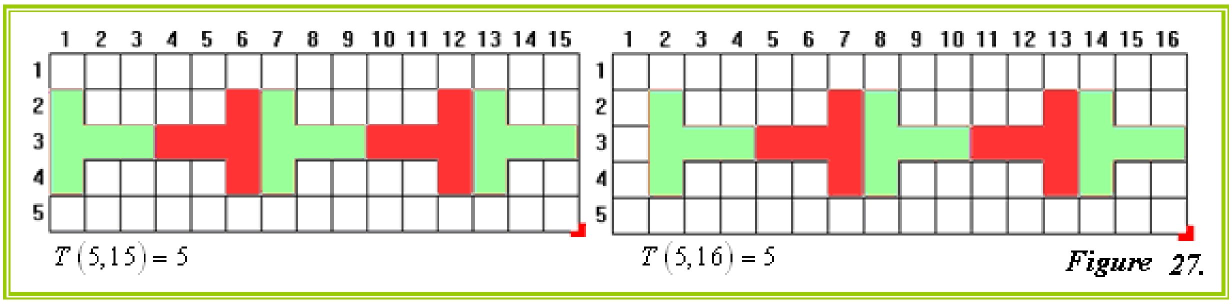


Figure 27.

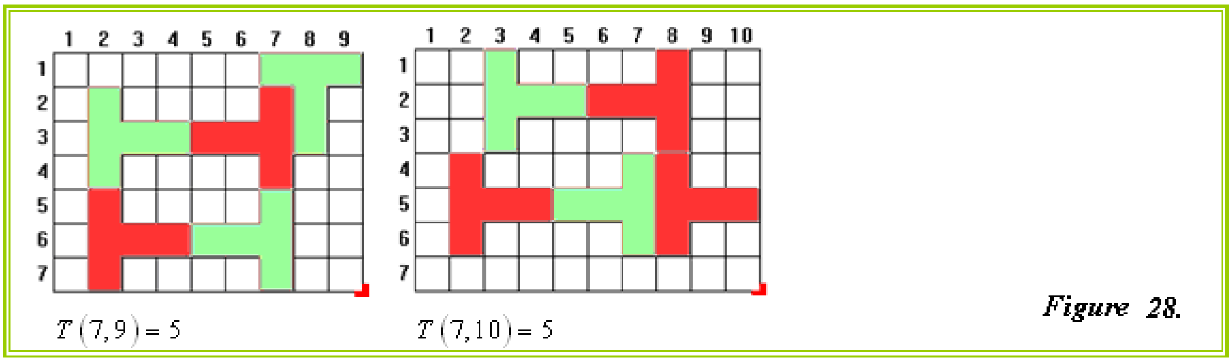


Figure 28.

K* = 6** – On ***Figure 9, ***Figure 21.*** and ***Figure 22.*** are given the cases when *m is* 6. For $P(7,n), n = 11,12$ (***Figure 29.***).

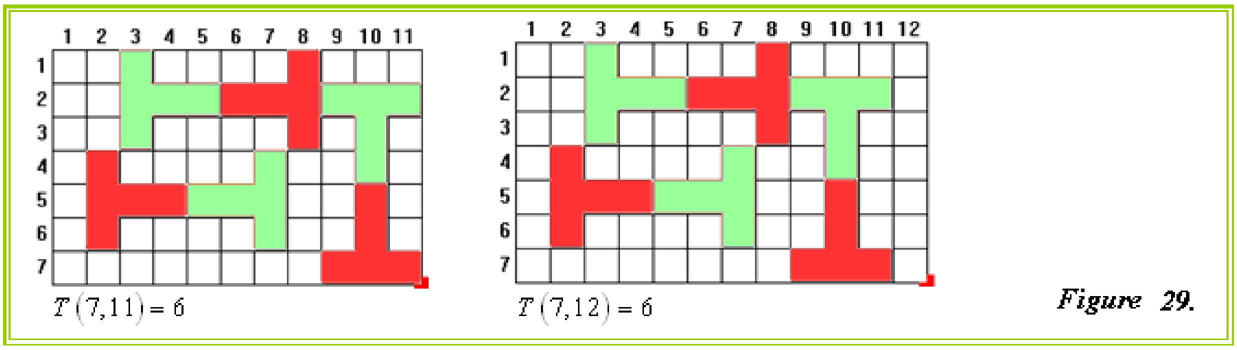


Figure 29.

K* = 7** – On ***Figure 10. are shown the cases when *m is* 6. For $P(7,n), n = 13,14$ (***Figure 30.***).

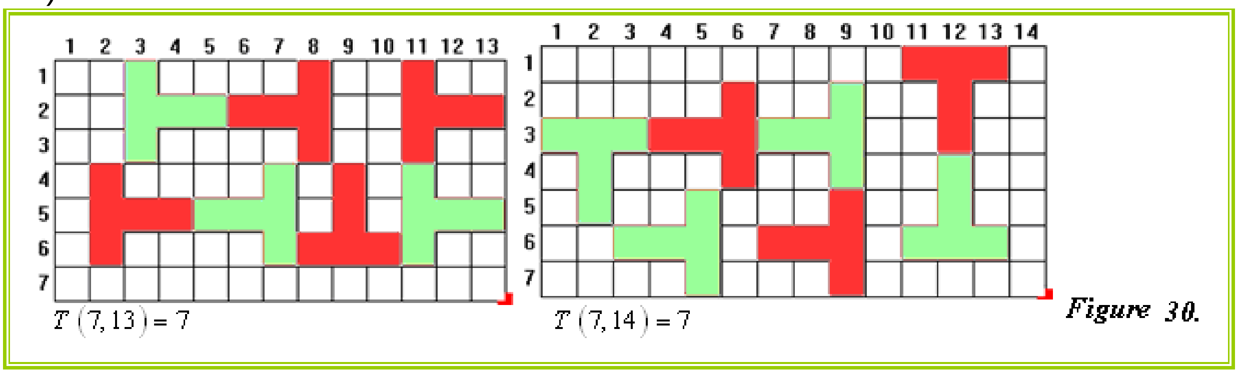


Figure 30.

K* = 8** – On ***Figure 11 and ***Figure 12*** are given the cases when *m is* 6. For $P(7,n)$ we have that $n = 15,16$ (***Figure 31.***).

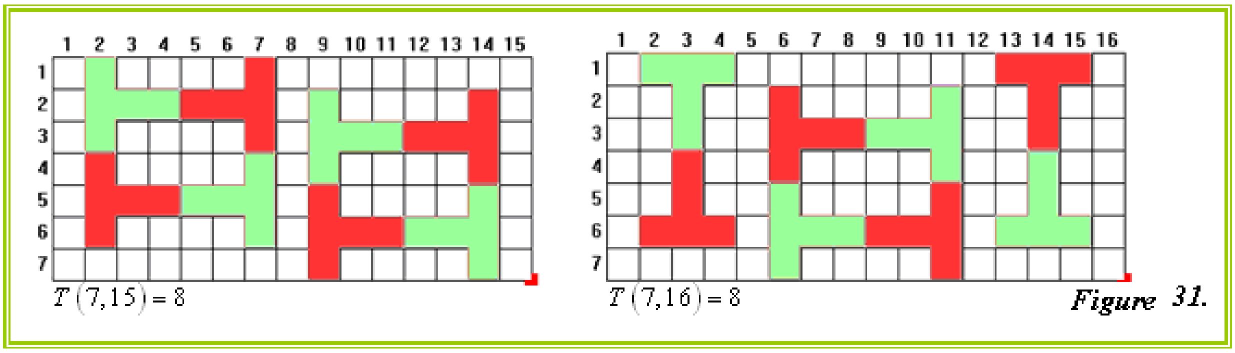


Figure 31.

K* = 9, 10** – On ***Figure 13 are shown the cases when *m is* 6. For $P(7,n)$ we have that $n=17,18$ (***Figure 32.***).

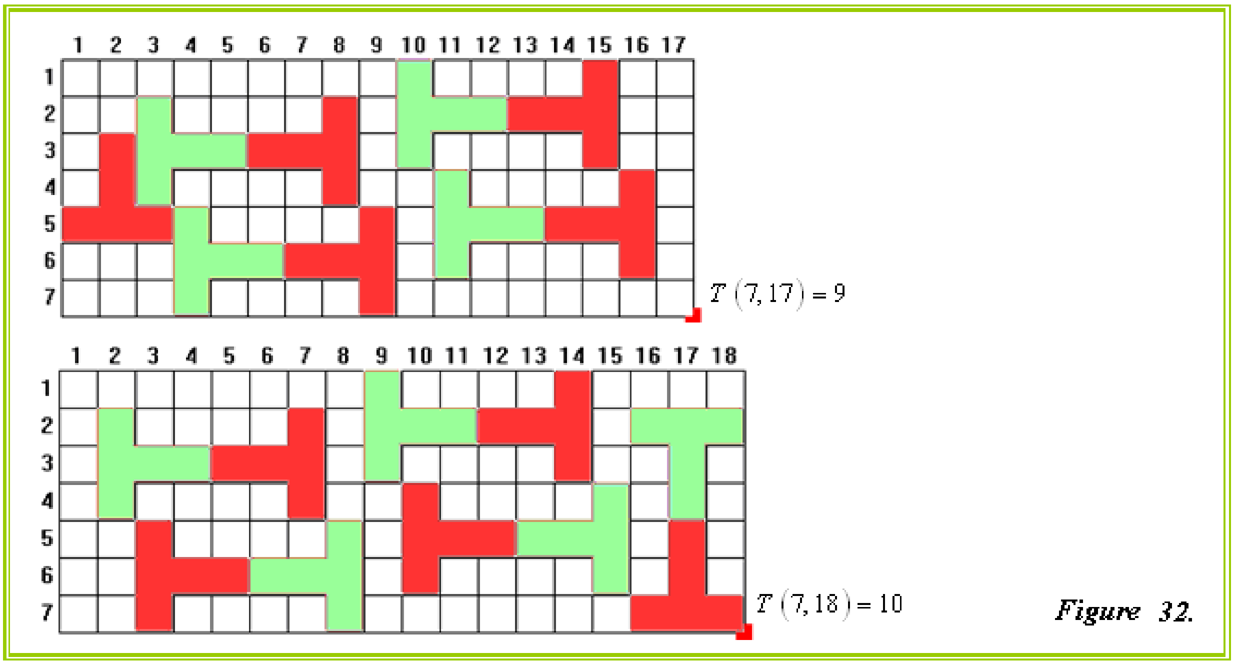
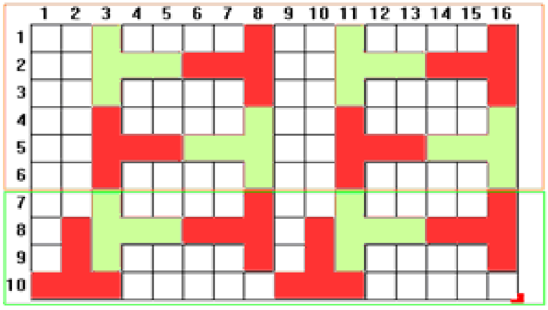


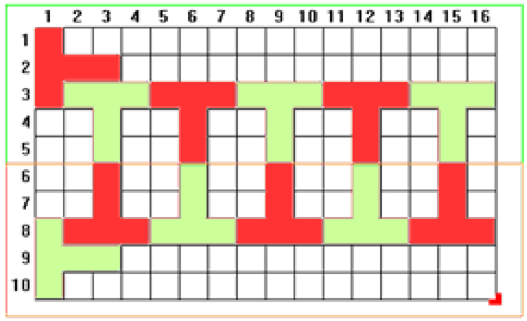
Figure 32.

Analysis.

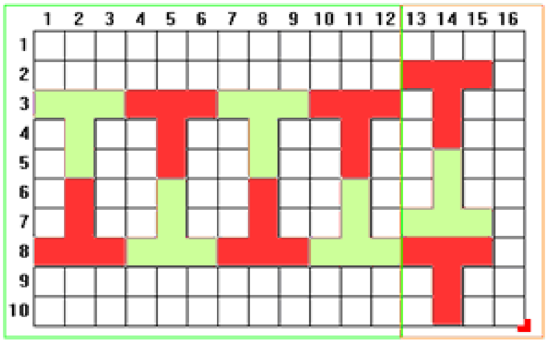
Of essential significance is the cutting up of rectangles $P(m,n)$ into suitable strips. For example, if we cut up $P(10,16)$ into $P_1(5,16)$ and $P_2(5,16)$, the number $T(10,16)=12$ (***Figure 34***). If we do the same but into strips $P_1(6,16)$ and $P_2(4,16)$, then $T(10,16)=14$ (***Figure 33***), and if $P_1(10,12)$ and $P_2(10,4)$, then $T(10,16)=11$ (***Figure 35***).



$T(10,16)=14$



$T(10,16)=12$



$T(10,16)=11$

Figure 35.

On the other hand the choice of type of k pentominoes in a given rectangle is very important. For example, two pentominoes in $P(3,6)$ - ***Figure 36.***

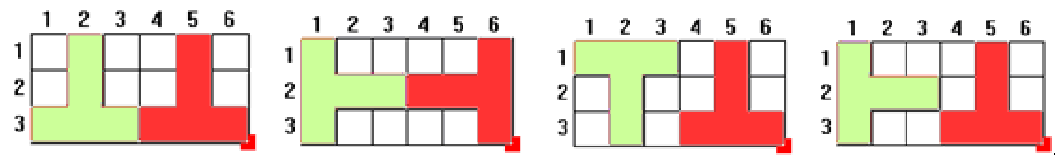


Figure 36.

Problem 1.5. Find $T(m,n)$ for $3 \le m \le 10$ and $3 \le n \le 20$

Solution.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5
4	1	1	1	2	2	2	3	3	3	4	4	4	4	5	5	5	6	6
5	1	1	2	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6
6	1	2	2	3	3	4	4	4	5	5	6	6	7	7	8	8	9	10
7	2	2	2	3	4	4	5	5	6	6	7	7	8	8	9	10	10	11
8	2	2	2	4	4	4	5	6	7	8	8	8	9	10	10	11	12	12
9	2	3	3	4	5	5	6	7	8	8	9	10	10	11	12	12	13	14
10	2	3	3	4	5	6	6	7	8	8	9	10	10	11	12	12	13	14

Hypothesis.

The optimal placements of T-minoes made until now are a basis for getting the following estimations:

$$mT(5,n)-1 \le T(5m,n) \le mT(5,n)+1,\ m \in \mathrm{N}$$

$$mT(7,n)-1 \le T(7m,n) \le mT(7,n)+1,\ m \in \mathrm{N}$$

Another hypothesis is that we can find the minimal number of pentominoes by the following estimation as pasting strips with smaller measures.

$T(m = al + bp, n) \le aT(l,n) + bT(p,n)$, where $a, b \in \mathrm{N}$ and l,n, p,n are the measures of the strips we paste, a and b are the number of strips we paste of one of these types.

We can consider that for small values of m and n, the $T(m,n)$ found and the shown instructions are a proof – all of them are got by scrutinizing all the possible situations.

PART 2.

Two players alternately place pentominoes on the free cells of an $m.n$ rectangle, along the grid lines. The loser is the one who cannot place a pentomino. Does any player have a winning strategy?

Problem 2.1. What is the minimal and maximal number of possible moves for rectangles with measures:

A) $3 \le m \le 10,\ 3 \le n \le 10$

B) $10 < m,\ 10 < n$

min/ max	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1	1	1/2	1/2	2/3	2/3	2/4	2/4	3/5	3/5	3/6	3/6	4/7	4/7	4/8	4/8	5/9	5/9
4	1	1/2	1/2	2/3	2/4	2/4	3/5	3/6	3/6	4/7	4/8	4/8	4/9	5/10	5/10	5/11	6/12	6/12
5	1/2	1/2	2/3	2/4	2/5	2/6	3/7	3/8	3/8	4/9	4/10	4/11	5/12	5/13	5/14	6/15	6/16	6/16
6	1/2	2/3	2/4	3/5	3/6	4/7	4/8	4/9	5/10	5/11	6/12	6/13	7/14	7/15	8/16	8/17	9/18	10/19
7	2/3	2/4	2/5	3/6	4/8	4/8	5/10	5/11	6/12	6/13	7/14	7/15	8/16	8/17	9/18	10/19	10/20	11/21
8	2/3	2/4	2/6	4/7	4/8	4/9	5/10	6/11	7/12	8/13	8/14	8/15	9/16	10/17	10/18	11/19	12/20	12/21
9	2/4	3/5	3/7	4/8	5/10	5/11	6/13	7/14	8/16	8/17	9/19	10/20	10/22	11/23	12/25	12/26	13/28	14/29
10	2/4	3/6	3/8	4/9	5/11	6/13	6/14	7/15	8/16	8/17	9/18	10/19	10/20	11/21	12/22	12/23	13/24	14/25

Problem 2.2. How must one play to have a winning strategy?

Investigation.

In the table it is figured out the minimal and maximal number of moves, which can be made in a rectangle with the respective measures.

From the results inscribed in the table one can easily see that for a rectangle $P(m,n)$ with measures bigger than 3.3, 3.4, and 4.4 each of the two players has the chance to win. That is why a hypothesis for winning strategy does not exist, but can be given some directions about winning a game.

Hypothesis

1. Playing for the win, the player must aspire to enclose an area in which he is sure that can be made exactly two moves. That has to be reached in minimal number of moves to reduce the chance for losing.
2. If he wastes many moves to do that, he must try to play in a way so that there can be placed an even number of pentominoes after his move.
3. If it is impossible to analyse the situation, he seeks to act in a way to complicate the position, i.e. to put the other player in a condition harder than his own.

On ***Figure 37*** and ***Figure 38*** are shown some possible outcomes of a game on a rectangle $P(4,8)$.

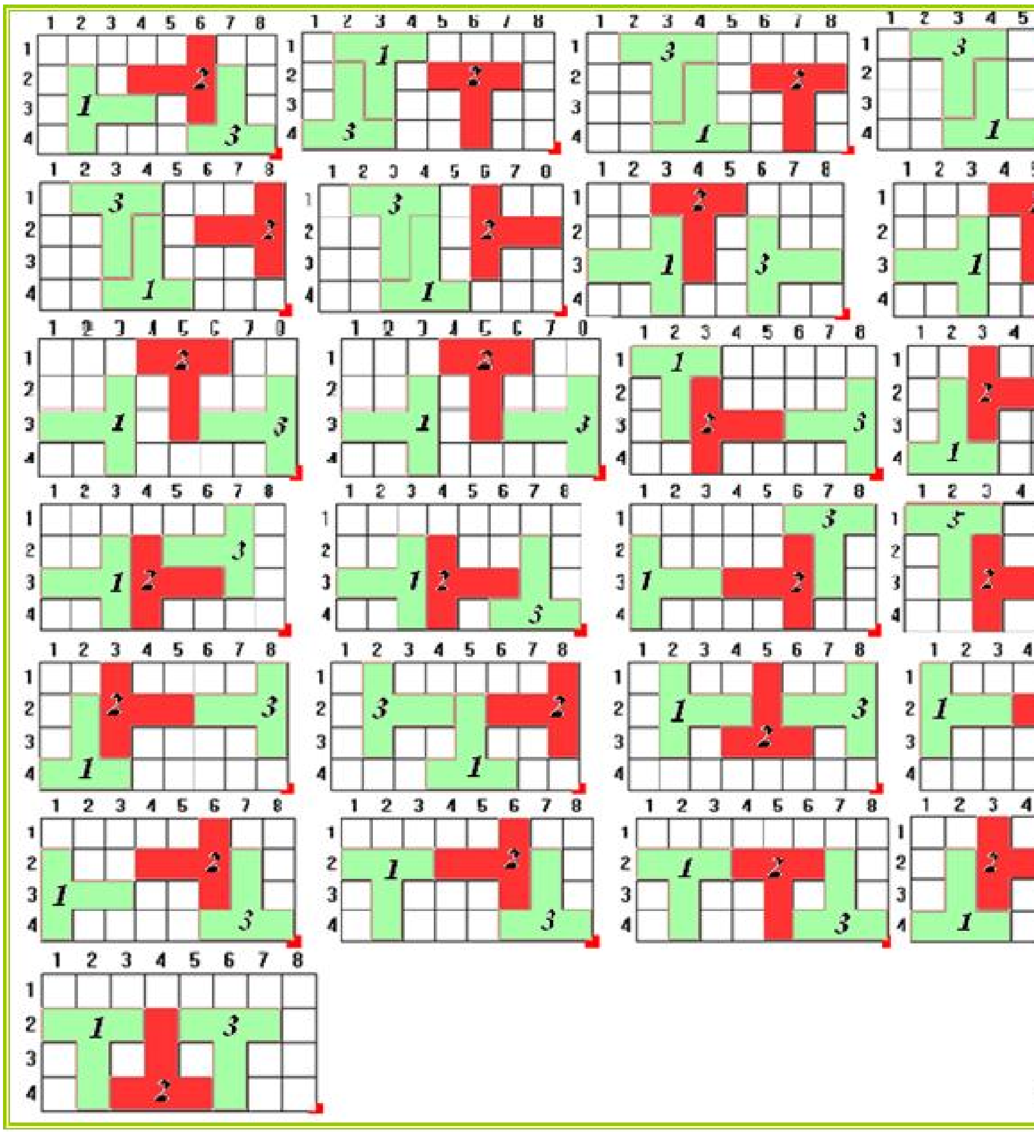


Figure 37.

Figure 37 shows the games, where the first player wins and ***Figure 38*** these, where wins the second player.

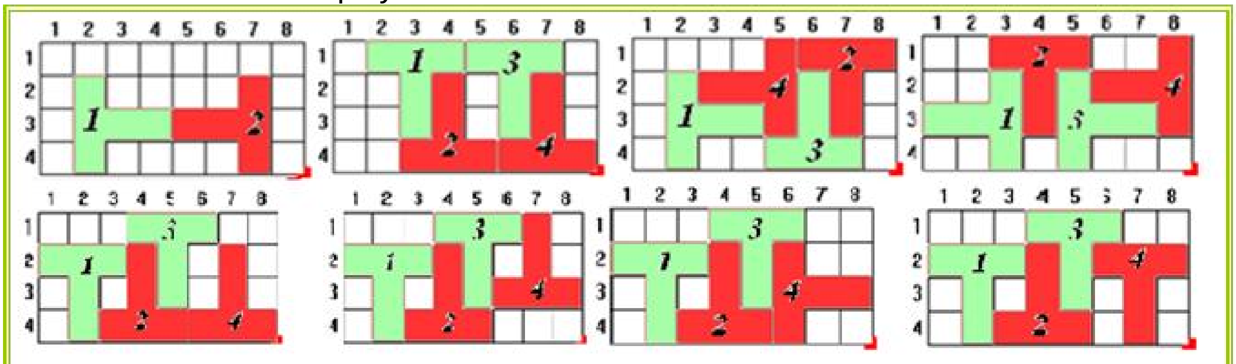
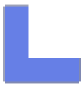


Figure 38.

PART 3.

We will mark pentomino of the type  with V. Without loss of generality let $n \geq m$.

Problem 3.1. Find $V(3m,n)$ in a rectangle if:

A) *m*=1 B) *m*=2

Investigation.

On ***Figure 39*** and ***Figure 40*** is shown the number and how can be placed the V-minoes in a stripe $P(3,n)$. The placement in the other cases for $P(3m,n)$ gives an idea for the estimation of this number.

A) $m=1$

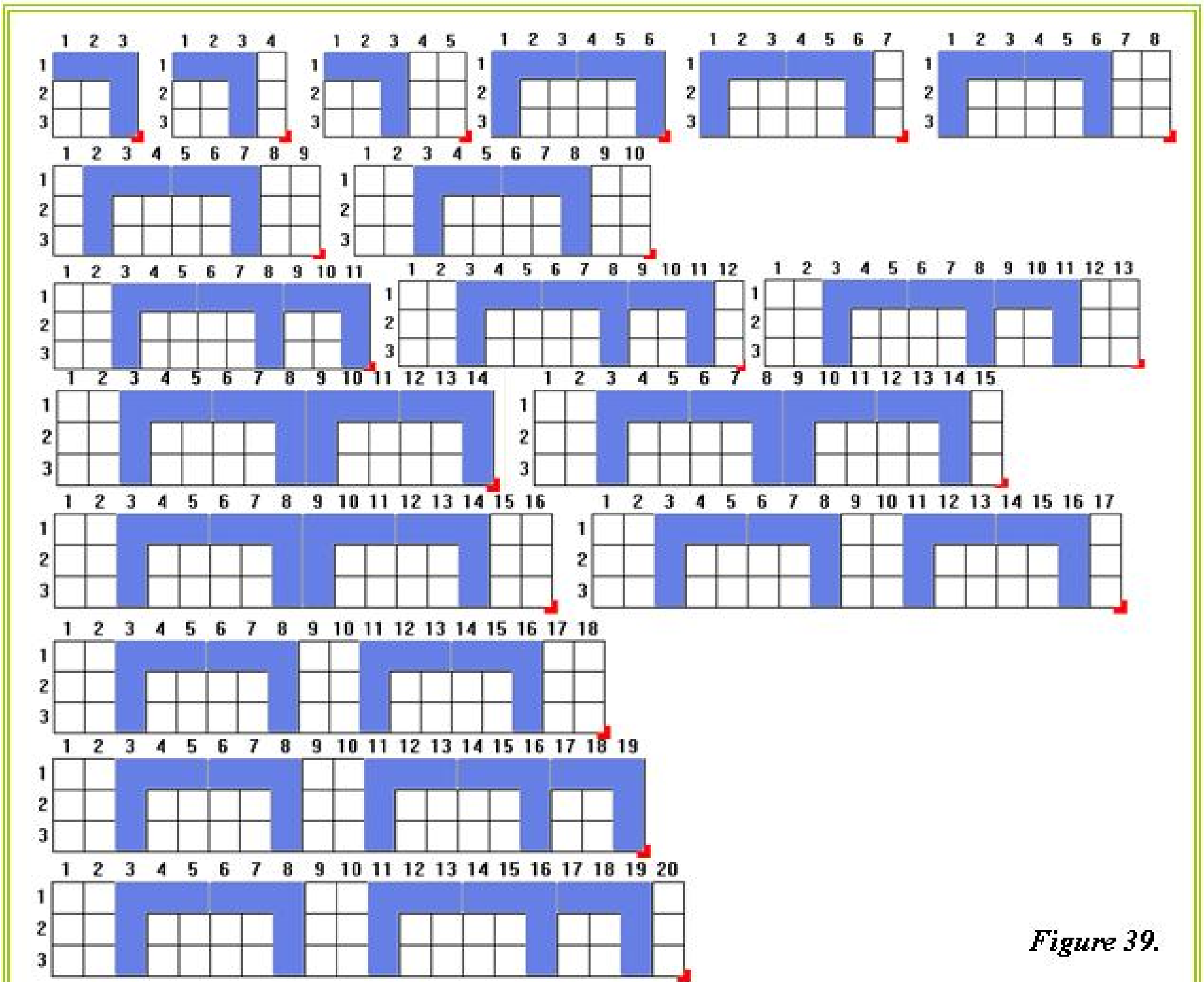


Figure 39.

B) $m = 2$

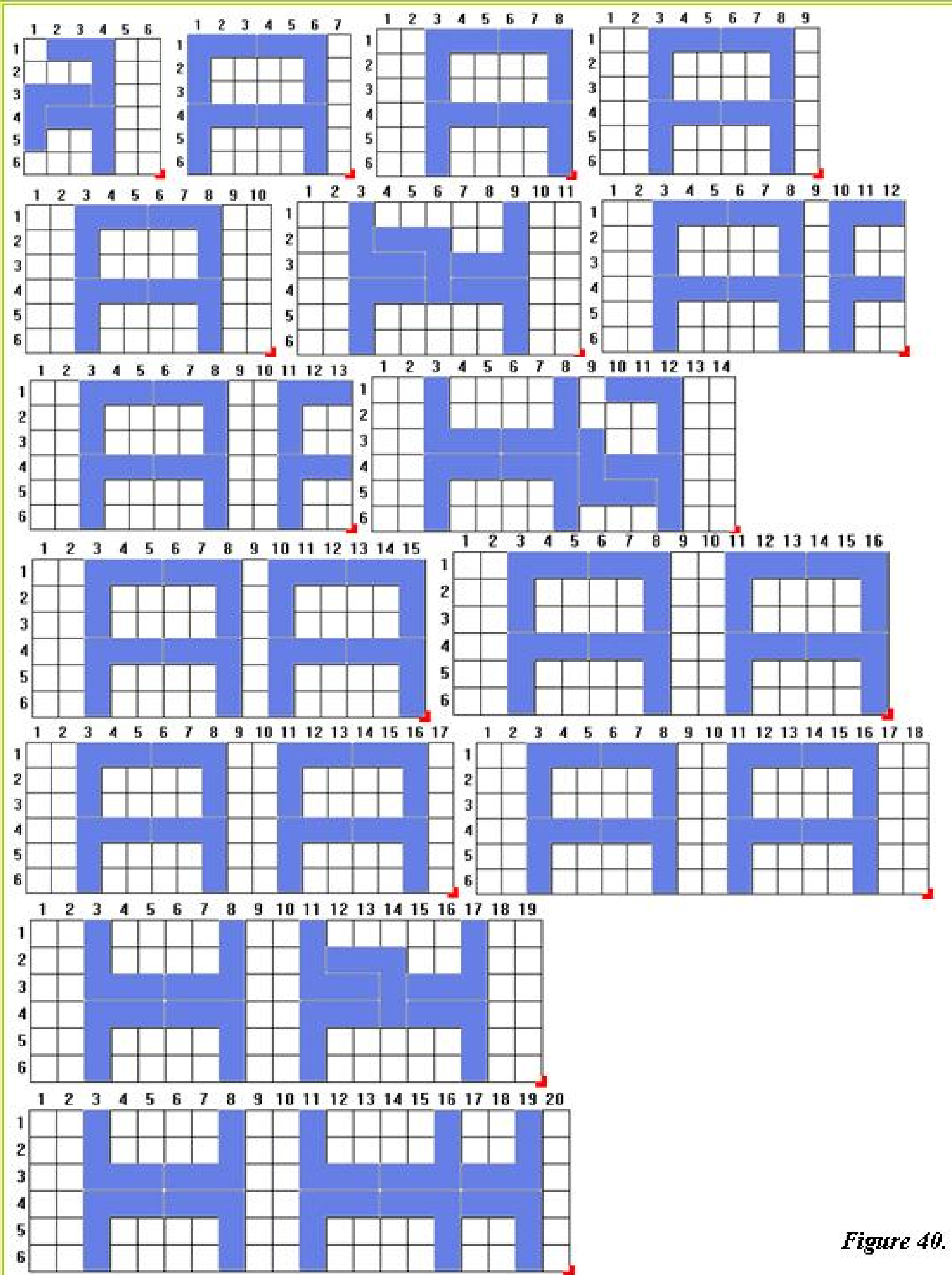


Figure 40.

Conclusion 3.1.

$$V(3,\ n) \le k+1,\ 4k+3 \le n \le 4k+6$$

$$V(3,n) \le V(3m,n) \le m(k+1),\ 4k+3 \le n \le 4k+6,\ k \in Z_0^+,\ n > 3m,\ m \in N$$
$$mV(3,n) - 1 \le V(3m,n) \le mV(3,n) + 1$$

Problem $3.1'$**.** Give an algorithm for constructing suitable placements of $V(3m,n)$, where $m \in N$ for the estimation made.

Algorithm 3.1.

1) Note that as long as the orientation of the V-mino is important, in the algorithm we will explicitly define its orientation  and indicate the type of the couple they make .

2) For $V(3,n)$ we start putting pentominoes after leaving k empty columns ($n \equiv k(\mathrm{mod}\,3),\ k \ne 0$).

3) If $k = 0$, we add a new pentomino symmetrically placed to the previous.

4) After placing such a couple of pentominoes, we detach the obtained rectangle $P(3,8)$ and repeat steps 2) and 3).

5) For $V(3m,n)$ follow steps from 1) to 4) and paste m alike to the obtained strip $V(3,n)$ downwards.

6) If $V(3m,n-8) \equiv 2(\mathrm{mod}\,3)$, place a new pentomino between the two strips $P(3,8)$.

7) Determine the number of needed pentominoes using the estimation: $mV(3,n) - 1 \le V(3m,n) \le mV(3,n) + 1$.

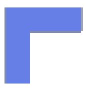
Commentary.

As long as the estimation is non-symmetrical towards *m* and *n*, we will consider rectangles of the type $(3m,n)$, where $3m < n$.

Problem ***3.2.*** Give an algorithm for constructing suitable placements of $V(4m,n), m \in N$.

Algorithm 3.2.

1) In the algorithm we use V with orientation  (or , where is necessary to from couples).

2) For $V(4,n)$ we start placing  compactly to the side of the rectangle as leaving the first row empty.

3) If $n \equiv k(\mathrm{mod}\,3),\ k \ne 0$, add a new pentomino in the free angle of the rectangle.

4) If $k = 0$ put a pentomino symmetrically to the one placed last.

5) Place the next ones by following steps from 2) to 4), but now orientated 

6) Detach the obtained rectangle $P(4,12)$ and follow steps from 2) to 5).

7) For $V(4m,n)$ observe steps from 1) to 6) and paste m alike to the obtained strip $V(4,n)$ downwards.

8) Determine the number of needed pentominoes by the estimation:

$V(4,n) \le k,\ 3k \le n \le 3(k+1),\ k \in \mathrm{N}$

$mV(4,n) - 1 \le V(4m,n) \le mV(4,n) + 1$

In ***Figure 41.*** is shown an example for optimal placing of pentominoes following the algorithm.

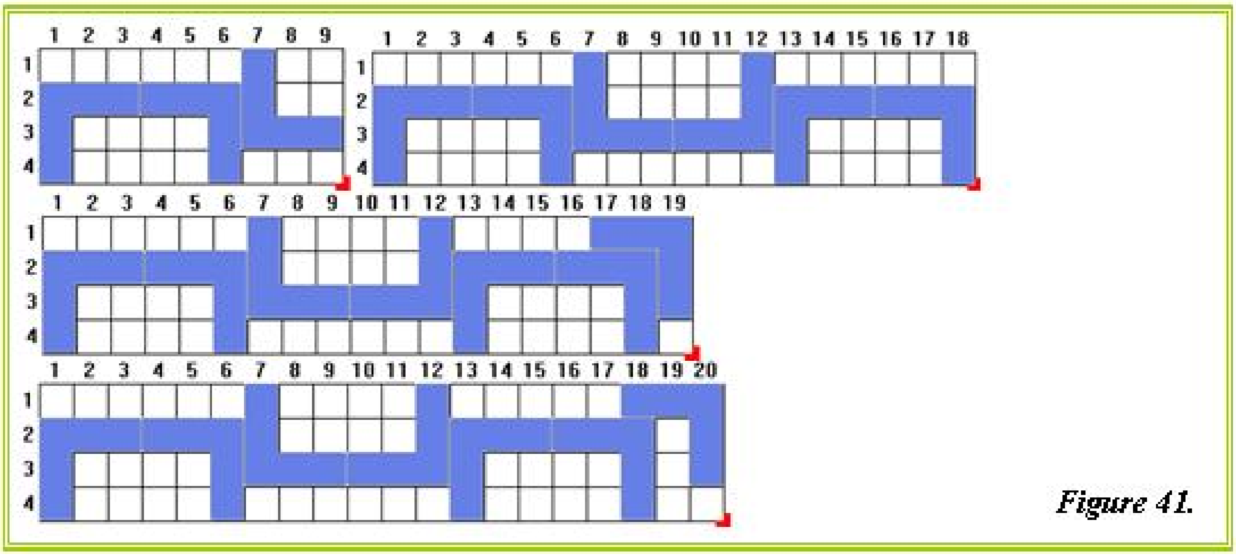


Figure 41.

Problem 3.3. Give an algorithm for constructing suitable placements of $V(5,n)$.

Algorithm 3.3:

1) For $V(5,n)$ we start placing  compactly to the left side of the rectangle as we leave the first two rows empty.

2) If $n \equiv k(\bmod 3),\ k \neq 0$, place a new pentomino in the free angle of the rectangle and move it with one row downwards.

3) If $k = 0$ put a V-mino symmetrically placed to the previous one.

4) Place the next pentominoes by following steps from 2) to 4), but now orientated this way .

5) Detach the got rectangle $P(4,12)$ observe steps from 2) to 5).

6) Determine the number of pentominoes, which are necessary for the covering of the rectangle by the estimation:

$mV(4,n) - 1 \le V(4m,n) \le mV(4,n) + 1$.

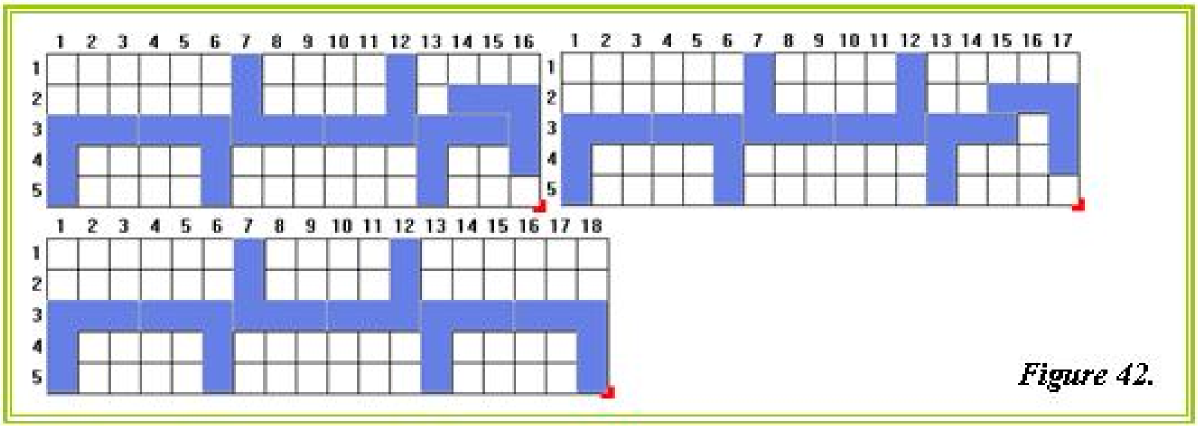


Figure 42.

Figure 42 illustrates the algorithms application.

Problem 3.4. Find $V(m,n)$ if $3 \le m \le 10$ and $3 \le n \le 20$

Solution.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1	1	1	2	2	2	2	2	3	3	3	4	4	4	4	4	5	5
4	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7
5	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7
6	2	2	2	3	4	4	4	4	6	6	6	7	7	8	8	8	9	10
7	2	3	3	4	5	5	6	6	8	8	9	9	10	11	11	12	13	13
8	2	3	3	4	5	5	6	6	8	8	9	9	10	11	11	12	13	13
9	2	3	3	4	6	6	6	6	8	8	10	10	10	12	12	12	14	14
10	2	4	4	4	6	6	6	8	8	8	10	10	10	12	12	12	14	14

Hypothesis.

The optimal placements of V-mino made until now are a basis for getting the following estimations:

$mV(5,n)-1 \le V(5m,n) \le mV(5,n)+1,\ m \in \mathrm{N}$,

$mV(7,n)-1 \le V(7m,n) \le mV(7,n)+1,\ m \in \mathrm{N}$

$V(m = al+bp, n) \le aV(l,n)+bV(p,n),\ a,\ b \in \mathrm{N}$

Commentary.

For the last estimation we have already explained how it works on page 14.

Example:

$$a=2,\ l=3,\ b=3,\ p=4 \Rightarrow V(18,n) \le 2V(3,n)+3V(4,n)$$

$$a=1,\ l=3,\ b=2,\ p=4 \Rightarrow V(11,n) \le V(3,n)+2V(4,n)$$

$$a=1,\ l=5,\ b=2,\ p=7 \Rightarrow V(19,n) \le V(5,n)+2V(7,n)$$

Problem 3.5. What is the minimal and maximal number of possible moves about rectangles with measures:

A) $3 \le m \le 10,\ 3 \le n \le 10$

B) $10 < m,\ 10 < n$

min/ max	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1/1	1/2	1/2	2/2	2/3	2/4	2/4	2/4	3/5	3/6	3/6	4/6	4/7	4/8	4/8	4/8	5/9	5/10
4	½	2/2	2/3	2/4	3/4	3/5	3/6	4/6	4/7	4/8	5/9	5/9	5/10	6/11	6/11	6/12	7/13	7/14
5	½	2/3	2/3	2/4	3/5	3/6	3/8	4/9	4/9	4/11	5/12	5/12	5/13	6/14	6/15	6/16	7/17	7/18
6	2/2	2/4	2/4	3/6	4/7	4/8	4/10	4/10	6/11	6/12	6/13	715	7/15	8/17	8/18	8/19	9/20	10/21
7	2/3	3/4	3/5	4/7	5/8	5/9	6/11	6/12	8/13	8/15	9/17	9/17	10/19	11/20	11//22	12/23	13/24	13/25
8	2/4	3/5	3/6	4/8	5/9	5/12	6/12	6/12	8/13	8/15	9/17	9/19	10/19	11/21	11/23	12/24	13/26	13/27
9	2/4	3/6	3/8	4/10	6/11	6/12	6/13	6/14	8/16	8/18	10/19	10/20	10/23	12/24	12/26	12/28	14/29	14/30
10	2/4	4/6	4/9	4/10	6/12	6/12	6/14	8/17	8/20	8/21	10/22	10/25	10/26	12/29	12/30	12/33	14/34	14/34

Commentary. We have already made a hypothesis about the best way one could play in Part 2.

PART 4.

We will mark a pentomino of the type  with I. Without loss of generality let $n \geq m$.

Problem 4.1. Give an algorithm for constructing suitable placements of $I(m,n)$ for the estimation made.

Algorithm 4.1.

1) For $I(m,n)$ start placing  as leaving k empty columns ($n \equiv k \pmod 5$, $k \neq 0$).

2) In every row *i*, $i \equiv 0 \pmod 5$, put an I-mino without leaving an empty column.

3) If after that there is enough place in this row, add a new pentomino.

4) For each new column move the pentomino with k squares, $n \equiv k \pmod 5$, $k < 4$.

5) Place one vertical orientated pentomino and $m-5a$ horizontal ones, $a \in \mathrm{N}$.

6) If $m,n \geq 6$ we can make the following construction  and put pentominoes around it, as leaving k empty squares, $n \equiv k \pmod 5$.

The application of the algorithm is shown on ***Figure 43***.

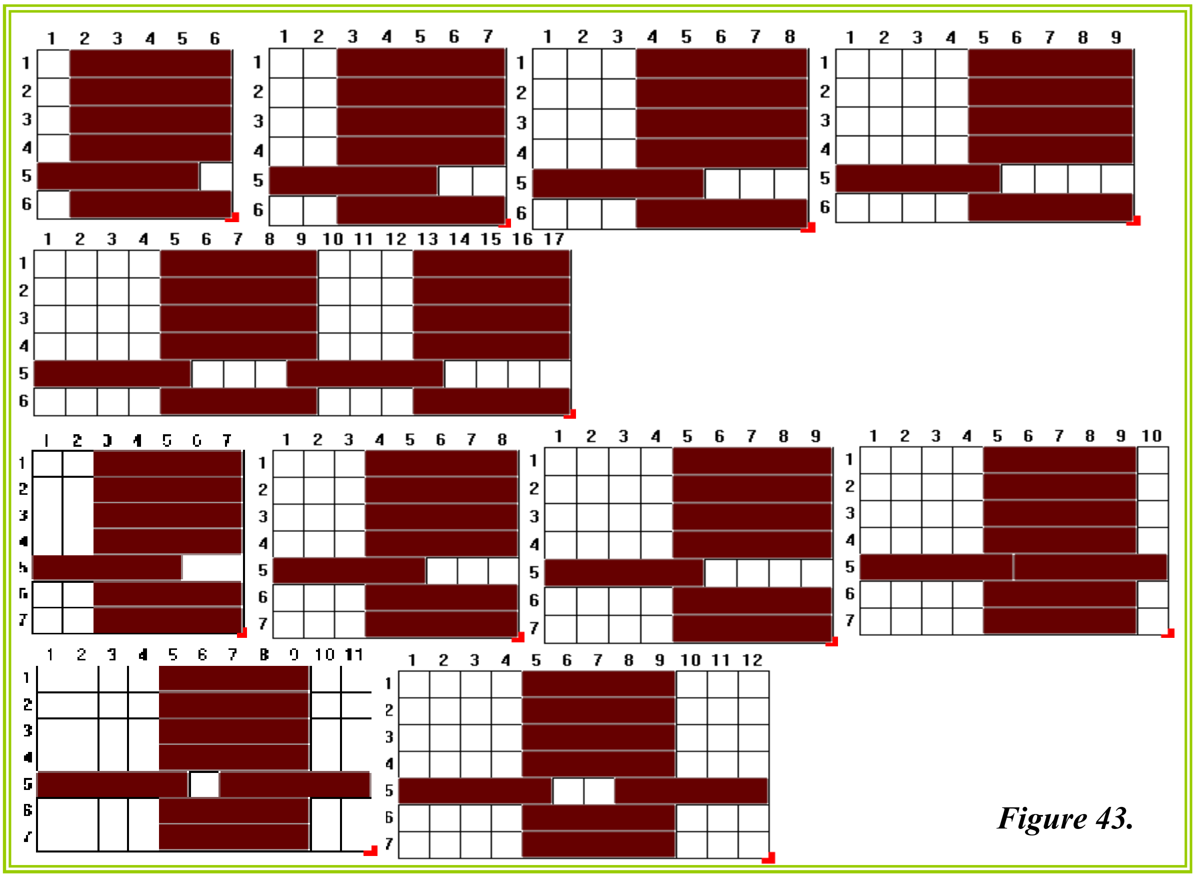


Figure 43.

Problem 4.2. Find $I(m,n)$ for $1 \le m \le 10$ and $1 \le n \le 20$

Solution.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
2	0	0	0	0	2	2	2	2	2	2	2	2	2	4	4	4	4	4	4	4
3	0	0	0	0	3	3	3	3	3	3	3	3	3	6	6	6	6	6	6	6
4	0	0	0	0	4	4	4	4	4	4	4	4	4	8	8	8	8	8	8	8
5	1	2	3	4	5	5	5	5	5	6	6	6	6	10	10	10	10	10	11	11
6	1	2	3	4	5	4	5	6	6	7	7	7	7	7	10	11	12	12	12	13
7	1	2	3	4	5	5	6	7	7	8	8	8	8	11	12	13	14	14	15	16
8	1	2	3	4	5	6	7	8	8	9	9	9	9	12	13	14	15	16	17	18
9	1	2	3	4	5	6	7	8	9	10	10	10	10	13	14	15	16	16	17	18
10	1	2	3	4	6	7	8	9	10	12	12	12	12	14	15	16	17	20	21	22

The following equation gives the minimal number of I-minoes in a rectangle with measures $m.n,\ m \in [1;4],\ n > 4$ and we use that $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \ge x\}$ (a ceiling function)

$$I(m,n) = m\left\lceil \frac{n-4}{9} \right\rceil$$

Example:

$$I(1,5) = \left\lceil \frac{5-4}{9} \right\rceil = \left\lceil \frac{1}{9} \right\rceil = 1,\ I(1,6) = \left\lceil \frac{6-4}{9} \right\rceil = \left\lceil \frac{2}{9} \right\rceil = 1,\ \dots,\ I(1,13) = \left\lceil \frac{13-4}{9} \right\rceil = \left\lceil \frac{9}{9} \right\rceil = 1$$

$$I(1,14) = \left\lceil \frac{14-4}{9} \right\rceil = \left\lceil \frac{10}{9} \right\rceil = 2,\ \dots,\ I(1,22) = \left\lceil \frac{22-4}{9} \right\rceil = \left\lceil \frac{18}{9} \right\rceil = 2,$$

$$I(4,5) = 4\left\lceil \frac{5-4}{9} \right\rceil = 4\left\lceil \frac{1}{9} \right\rceil = 4,\ \dots,\ I(1,22) = 4\left\lceil \frac{20-4}{9} \right\rceil = 4\left\lceil \frac{16}{9} \right\rceil = 8$$

Problem 4.3. What is the minimal and maximal number of possible moves about rectangles with measures:

A) $1 \le m \le 10,\ 1 \le n \le 10$

B) $n > 10$

Solution.

min/ max	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
2	0	0	0	0	2	2	2	2	2	2	2	2	2	4	4	4	4	4	4	4
3	0	0	0	0	3	3	3	3	3	3	3	3	3	6	6	6	6	6	6	6
4	0	0	0	0	4	4	4	4	4	4	4	4	4	8	8	8	8	8	8	8
5	1	2	3	4	5	5/6	5/7	5/8	5/9	6/10	6/11	6/12	6/13	10/14	10/15	10/16	10/17	10/18	11/19	11/20
6	1	2	3	4	5/6	4/7	5/8	6/9	6/10	7/12	7/13	7/14	7/15	7/16	10/18	11/19	12/20	12/21	12/22	13/24
7	1	2	3	4	5/7	5/8	6/9	7/10	7/11	8/14	8/15	8/16	8/17	11/18	12/21	13/22	14/23	14/24	15/25	16/28
8	1	2	3	4	5/8	6/9	7/10	8/11	8/12	9/16	9/17	9/18	9/19	12/20	13/24	14/25	15/26	16/27	17/28	18/32
9	1	2	3	4	5/9	6/10	7/11	8/12	9/13	10/18	10/19	10/20	10/21	13/22	14/27	15/28	16/29	16/30	17/31	18/36
10	1	2	3	4	6/10	7/12	8/14	9/16	10/18	12/20	12/22	12/24	12/26	14/28	15/30	16/32	17/34	20/36	21/38	22/40

Commentary. We have already made a hypothesis about the best way one could play in Part 2.

PART 5.

We will mark a pentomino of the type  with L. Without loss of generality let $n \geq m$.

Problem 5.1. Give an algorithm for constructing suitable placements of $L(3m,n), m \in \mathrm{N}$ for the estimation, which has been made.

Algorithm 5.1.

1) Note that it is important how the pentomino is orientated and that is why in the algorithm we will explicitly define its orientation 

2) For $L(3,n)$ place  without leaving an empty column but let the first row is empty.

3) If $n \equiv k \pmod 4$, $k \neq 0$ put a new pentomino in the free row.

4) For $L(3m,n)$ paste a new strip $P(3,n)$ symmetrically downwards.

5) Determine the number of pentominoes needed by the estimation: $mL(3,n) - 1 \leq L(3m,n) \leq mL(3,n) + 1$.

Commentary.

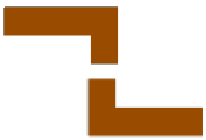
As long as the estimation is non-symmetrical towards *m* and *n*, we will consider rectangles of the type $(3m,n)$, where $3m < n$.

Problem 5.2. Give an algorithm for constructing suitable placements of $L(4m,n), m \in \mathrm{N}$.

Algorithm 5.2.

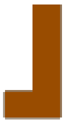
1) In the algorithm we use L with orientation  (or , where is necessary to form couples).

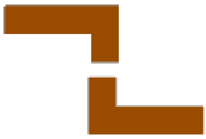
2) For $L(4,n)$ place  without leaving an empty column or row.

3) Put a pentomino to make a couple .

4) For $L(4m,n)$ paste symmetrically a new strip $P(4,n)$ downwards.

5) Place an L-mino with an orientation .

6) Put  , if $n-8a \equiv k \pmod 7, a \in \mathrm{N}, k \neq 0$.

7) If $n-8a \equiv 0 \pmod 7, a \in \mathrm{N}$ place  again.

8) For $L(4m,n)$ paste symmetrically a strip $P(4,n)$ downwards.

9) Determine the number of pentominoes , which are necessary for covering a rectangle $P(m,n)$ using the following estimation:

$$L(4,n) \le k, \ 2k \le n \le 2(k+1), \ k \in \mathrm{N}$$

$$mL(4,n)-1 \le L(4m,n) \le mL(4,n)+1$$

Problem 5.3. Find $L(m,n)$ for $2 \le m \le 10$ and $2 \le n \le 20$

Solution.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5
3	0	0	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5
4	1	1	2	2	2	3	3	4	4	4	5	5	6	6	6	7	7	8	8
5	1	2	2	3	3	3	4	5	5	6	6	7	7	8	8	9	9	9	10
6	1	2	2	3	3	3	4	5	5	6	6	7	7	8	8	9	9	10	10
7	1	2	3	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
8	2	2	3	4	4	4	5	6	7	8	8	9	10	11	12	13	13	13	14
9	2	3	4	5	5	6	6	7	8	9	10	11	12	12	13	14	15	16	17
10	2	3	4	5	5	5	7	8	9	10	11	12	13	14	15	15	16	17	18

The following equation gives the minimal number of L-minoes in a rectangle with measures $m.n, \ m=2, \ n>3$ and we use that $\lceil x \rceil := \min\{n \in \mathrm{Z} \mid n \ge x\}$ (a ceiling function)

$$L(m,n)=\left\lceil \frac{n}{4} \right\rceil$$

Example:

$$L(2,4)=\left\lceil \frac{4}{4} \right\rceil=1 \ , L(2,5)=\left\lceil \frac{5}{4} \right\rceil=1 \ ,.., L(2,4)=\left\lceil \frac{8}{4} \right\rceil=2 \ ,\ldots, L(2,20)=\left\lceil \frac{20}{4} \right\rceil=5$$

Or we can use the estimation:
$L(2,n) \le k,\ 3k+1 \le n \le 3(k+1)+1,\ k \in \mathrm{Z}_0^+$

The following equation gives the minimal number of L-minoes in a rectangle with measures $m.n,\ m=3,\ n>4$ and we use that $[x] := \min\{n \in \mathrm{Z} \mid n \ge x\}$ (a ceiling function)

$$L(3,n) = \left[\frac{n}{4}\right]$$

Example:

$$L(3,4) = \left[\frac{4}{4}\right] = 1,\ L(3,5) = \left[\frac{5}{4}\right] = 2,\ L(3,6) = \left[\frac{6}{4}\right] = 2,\ L(3,5) = \left[\frac{5}{4}\right] = 2, \ldots, L(3,9) = \left[\frac{9}{4}\right] = 3$$

For $L(4,n)$ and $L(6,n)$

$L(4,n) \le k+1,\ 2k \le n \le 2(k+1)$

$L(6,n) \le k+1,\ 2k \le n \le 2(k+1)$

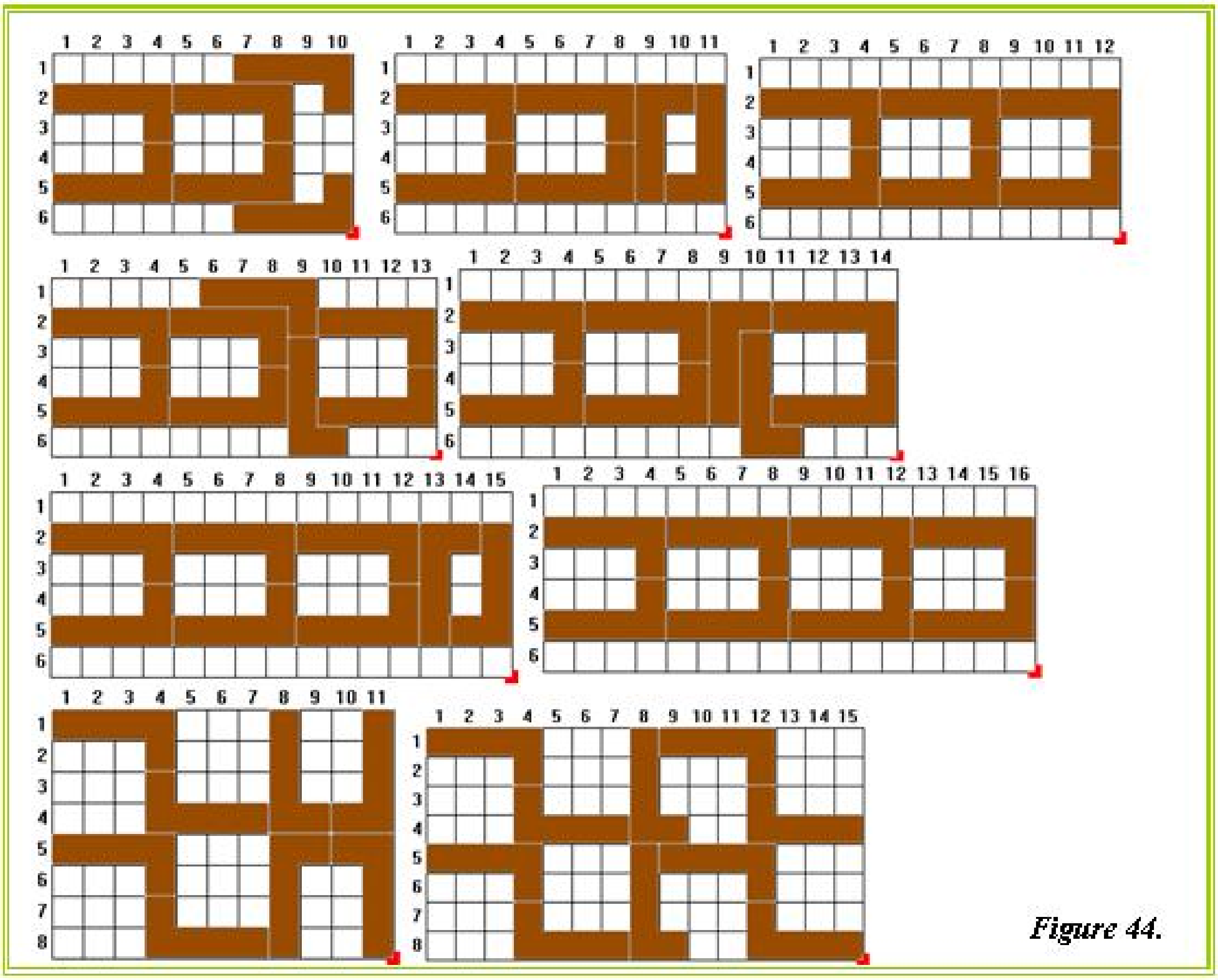


Figure 44.

<u>Hypothesis.</u>

The optimal placements of L-mino made until now are a basis for getting the following estimations:

$$L(m = al + bp, n) \le aL(l, n) + bL(p, n),\ a,\ b \in \mathrm{N}$$

Commentary.

For the last estimation we have already explained how it works on page 14.

Problem 5.4. What is the minimal and maximal number of possible moves in rectangles with measures:

A) $2 \le m \le 10,\ 2 \le n \le 10$

B) $n > 10$

Solution.

We can consider that for small values of *m and n*, the minimal placements found and the shown instructions are a proof – all of them are got by scrutinizing all the possible situations.

min/ max	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0	1	1/2	1/2	1/2	2	2/3	2/4	2/4	3/4	3/4	3/5	3/6	4/6	4/6	4/6	4/7	5/8
3	0	0	1/2	2	2/3	2/3	2/3	3/4	3/5	3/5	3/6	4/6	4/7	4/7	4/8	5/8	5/9	5/9	5/9
4	1	1/2	2	2/4	2/4	3/5	3/6	4/6	4/8	4/8	5/9	5/10	6/10	6/12	6/12	7/13	7/14	8/14	8/16
5	1/2	2/2	2/4	3/4	3/6	3/6	4/8	5/8	5/10	6/10	6/12	7/12	7/14	8/14	8/16	9/16	9/18	9/18	10/20
6	1/2	2/3	2/4	3/6	3/6	3/8	4/9	5/10	5/12	6/12	6/14	7/15	7/16	8/18	8/18	9/20	9/21	10/22	10/23
7	1/2	2/3	3/5	3/6	3/8	4/9	4/10	5/12	5/14	6/14	6/16	7/17	7/19	8/20	8/22	9/23	9/24	10/26	10/28
8	2	2/3	3/6	4/8	4/9	4/10	5/12	6/14	7/16	8/16	8/18	9/20	10/23	11/24	12/24	13/26	13/28	13/30	14/32
9	2/3	3/4	4/6	5/8	5/10	6/14	6/14	7/15	8/16	9/19	10/21	11/24	12/24	12/26	13/27	14/28	15/30	16/32	17/34
10	2/4	3/5	4/8	5/9	5/12	5/13	7/16	8/17	9/20	10/21	11/24	12/25	13//28	14/29	15/32	15/33	16/35	17/37	18/40

Commentary. We have already made a hypothesis about the best way one could play in Part 2.

Summary

Part 1.

1. We have found $T(3m,n)$ and the suitable placements of T-minoes in a rectangle for $m=1$ and $m=2$.
2. We have given an idea for estimating this number.
3. An algorithm for constructing $T(3m,n)$, where $m \in \mathrm{N}$ has been given for the made estimation.
4. $T(4m,n)$ for $m=1$ and $m=2$ have been found, which is the basis of making an algorithm for constructing suitable minimal covering and of giving an estimation for $T(4m,n)$.
5. The values of *m* and *n*, for which we can place k T-minoes, $1 \le k \le 10$ in a rectangle $P(m,n)$ are determined.
6. The cases, where there is significant meaning in cutting up rectangles $P(m,n)$ into suitable strips, are analyzed and are given some examples.
7. We have found $T(m,n)$ for $3 \le m \le 10$ and $3 \le n \le 20$.
8. The optimal placements of T-minoes are a basis for making other estimations.
9. For small values of *m* and *n*, we can consider $T(m,n)$ which have been found yet and the shown instructions a proof – all of them are got by scrutinizing all the possible situations.

Part 2.

1. We have figured up the minimal and maximal number of possible moves on rectangles with measures $3 \le m \le 10,\ 3 \le n \le 10$ and $10 < m,\ 10 < n$.
2. A hypothesis has been given for a possible winning strategy.
3. We have shown examples about some possible outcomes in a game on a rectangle $P(4,8)$.

Part 3.

1. The previous questions are studied for other polyominoes.
2. Similar conclusions have been made for pentominoes L, I, V.
3. We have found $V(3m,n)$ and the suitable placements of V-mino in a rectangle for $m=1$ and $m=2$.
4. The placement in the other cases for $P(3m,n)$ gives an idea for the estimation of this number.
5. An algorithm for constructing $V(3m,n)$, where $m \in \mathrm{N}$ has been given for the made estimation.
6. We have made the commentary that as long as the estimation is non-symmetrical towards *m* and *n*, we will consider rectangles of the type $(3m,n)$, where $3m < n$.
7. An algorithm for constructing suitable placements of $V(4m,n), m \in \mathrm{N}$ has been made.

8. We have given algorithm for constructing suitable placements of $V(5,n)$.
9. We have found $V(m,n)$ by $3 \le m \le 10$ and $3 \le n \le 20$.
10. A hypothesis for the optimal placements of V-mino becomes a basis for getting estimations for this number.
11. We have figured up the minimal and maximal number of possible moves about rectangles with measures $3 \le m \le 10,\ 3 \le n \le 10$ and $10 < m,\ 10 < n$.

Part 4.

1. An algorithm for constructing $I(m,n)$, where $m \in \mathrm{N}$ has been given for the made estimation.
2. We have made the commentary that as long as the estimation is non-symmetrical towards *m* and *n*, we will consider rectangles of the type $(3m,n)$, where $3m < n$.
3. We have found $I(m,n)$ by $1 \le m \le 10$ and $1 \le n \le 20$.
4. A hypothesis for the optimal placements of I-mino becomes a basis for getting an equation for finding this number in cases $m \in [1;4],\ n > 4$.
5. We have figured up the minimal and maximal number of possible moves about rectangles with measures $1 \le m \le 10,\ 1 \le n \le 10$ and $10 < m,\ 10 < n$.

Part 5.

1. An algorithm for constructing $L(3m,n), m \in \mathrm{N}$, has been given for the made estimation.
2. We have made the commentary that as long as the estimation is non-symmetrical towards *m* and *n*, we will consider rectangles of the type $(3m,n)$, where $3m < n$.
3. An algorithm for constructing $L(4m,n), m \in \mathrm{N}$, has been given for the made estimation.
4. We have found $L(m,n)$ by $2 \le m \le 10$ and $2 \le n \le 20$.
5. A hypothesis for the optimal placements of L-mino becomes a basis for getting estimations for this number by $m = 4,6$ and an equation by $m = 2,3,\ n > 4$.
6. We have figured up the minimal and maximal number of possible moves about rectangles with measures $2 \le m \le 10,\ 2 \le n \le 10$ and $10 < m,\ 10 < n$.
7. We can consider that for small values of *m and n*, the minimal placements found and the shown instructions are a proof – all of them are got by scrutinizing all the possible situations