I^{st} International Tournament
of Young Mathematicians
27th June – 3rd July 2009, Paris, France

PROBLEM FOUR

MINIMALITY OF INSCRIBED POLYGONS

We say that a polygon *P* is inscribed in a polygon *Q* if the vertices of *P* lie on edges of *Q*, no two on the same edge.

1. A triangle *T* is inscribed in a triangle *ABC*, so that *ABC* is divided into four triangles: T_1, T_2, T_3 and *T*.

(a) Is it always true that area(*T*) $\geq$ min{area(T_1), area(T_2), area(T_3)}?

(b) Can *T* have a bisector (median, perimeter, angle, inscribed or circumscribed circle, etc.) smaller than all bisectors (medians, perimeters, angles, inscribed or circumscribed circles, etc.) of the triangles T_1, T_2, T_3?

2. A convex polygon $P = P_1P_2 \ldots P_m$ is inscribed in a convex polygon $Q = Q_1Q_2 \ldots Q_n$, where $3 \leq m \leq n$, so that *Q* is divided into $m + 1$ parts. Can the polygon *P* possess a minimality property among them (for instance, can *P* have the smallest area, perimeter, angle, diagonal, etc.)?

3. Formulate and investigate 3-dimensional analogs of the problem.

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SOME PROBLEMS FOR INSCRIBED CONVEX POLYGONS IN OTHER CONVEX POLYGONS

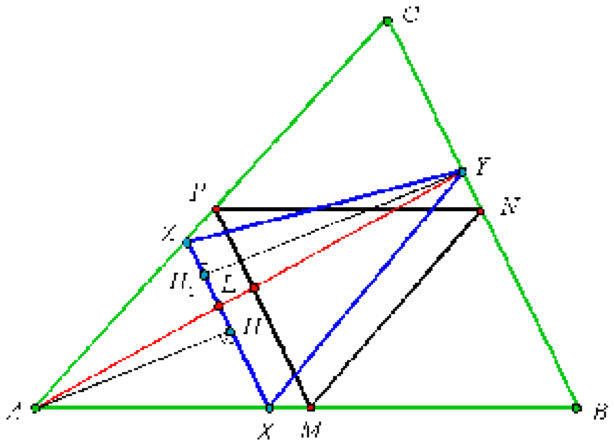
PART ONE

Problem 1. In a triangle ABC is inscribed a triangle T so that ABCs divided to four triangles - T, T_1, T_2, T_3. Prove that $S_T \geq \min(S_{T_1}, S_{T_2}, S_{T_3})$.

Solution. Let T has vertices X, Y, Z and M, N, P be the midpoints of AB, BC and AC. Let us consider the following three groups of segments:

$$\{AP, AM\},\ \{BM, BN\},\ \{CN, CP\}$$

Let two of the points X, Y, Z lye in one of the groups. Without loss of generality let $X \in AM$ and $Y \in AP$. Let H and H_1 be the projections of A and Y on XZ and let $AY \cap XZ = L$ and $AY \cap PM = R$. Since R is the midpoint of AY and $L \in ZX$ and XZ is inside AMP then $AL < LY$. Then $AH < YH_1$. So $S_{AXZ} < S_{XYZ}$ or in other words $S_{XYZ} \geq \min\{S_{AXZ}, S_{BXY}, S_{CYZ}\}$.



Let now in each of the groups $\{AP, AM\}$, $\{BM, BN\}$, $\{CN, CP\}$ lye exactly one of the points X, Y and Z. Without loss of generality let $X \in AM$, $Y \in BN$ and $Z \in CP$.

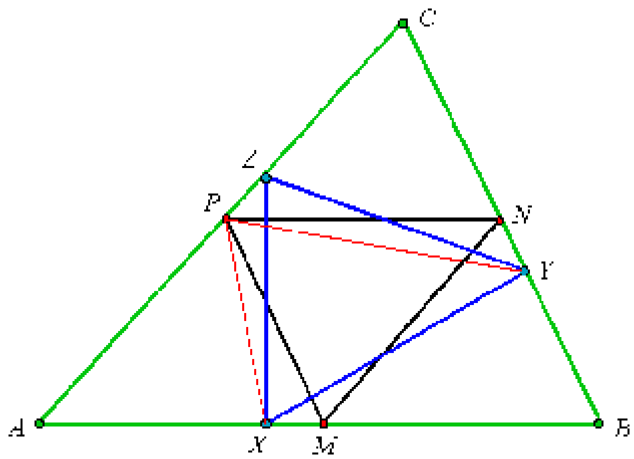
Since $MN \| AC$, $X \in AM$, $Y \in BN$, we have that YX intersect the line AC in point D such that A is between C and D.

Then $S_{XYZ} \geq S_{XPY}$.

Analogously $S_{XPY} \geq S_{XNP} = \frac{1}{4} S_{ABC}$, so

$S_{XYZ} \geq \frac{1}{4} S_{ABC}$.

Then $S_{XYZ} \geq \min\{S_{AXZ}, S_{BXY}, S_{CYZ}\}$.

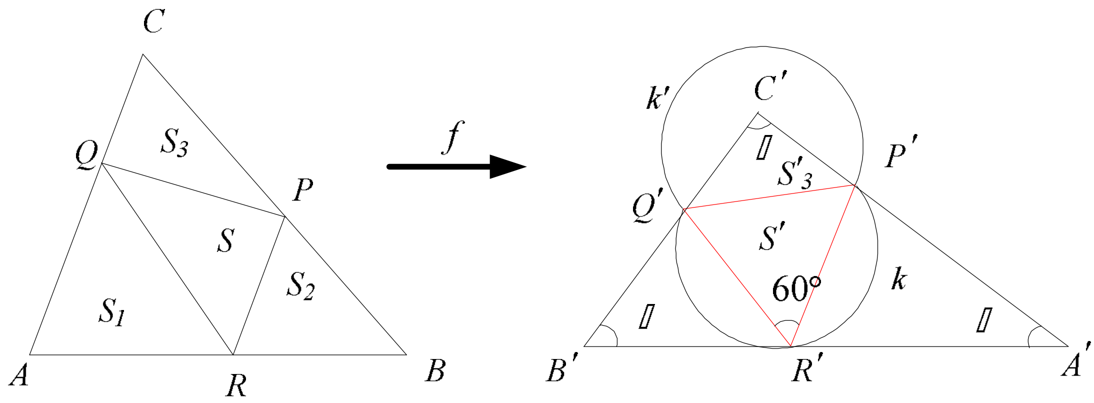


Alternative solution

Let $S_1 = S_{AQR}$, $S_2 = S_{BRP}$, $S_3 = S_{CPQ}$, $S_{PQR} = S$. Prove that $\min\{S_1, S_2, S_3\} \leq S$.

With affine transformation f which preserves *proportions* on lines we can transform any triangle XYZ in equilateral triangle $X'Y'Z'$. Moreover if $M \in XY$ and $N \in XZ$ we have $\frac{XM}{XY} = \frac{X'M'}{X'Y'}$, so $\frac{S_{XMN}}{S_{XYZ}} = \frac{S_{X'M'N'}}{S_{X'Y'Z'}}$.

Let $f(PQR) = P'Q'R'$, where $P'Q'R'$ is equilateral and $f(ABC) = A'B'C'$.



Let $a \le b \le g$ be the angles of $A'B'C'$. Then $a \le 60^0$ and $g \ge 60^0$.

Let arcs k and k' be the locus of points M such that $P'MQ' = 60°$. Then C' is inside or on k' and $S_3' \le S'$. Since $\frac{S_3'}{S'} = \frac{S_3}{S}$ then $S_3 \le S$.

Problem 2. In a triangle ABC is inscribed a triangle T so that ABC is divided to four triangles - T, T_1, T_2, T_3. Let $R(XYZ) = R$, $R(AXZ) = R_A$, $R(BYX) = R_B$ and $R(CYZ) = R_C$ be the radii of the circumscribed circles of the triangles XYZ, AXZ, BYX, CYZ. Prove that if:

(a) ABC is acute triangle, then

$$R(XYZ) \ge \min\{R(AXZ), R(BYX), R(CYZ)\}.$$

(b) ABC is not acute, there is a triangle T for which

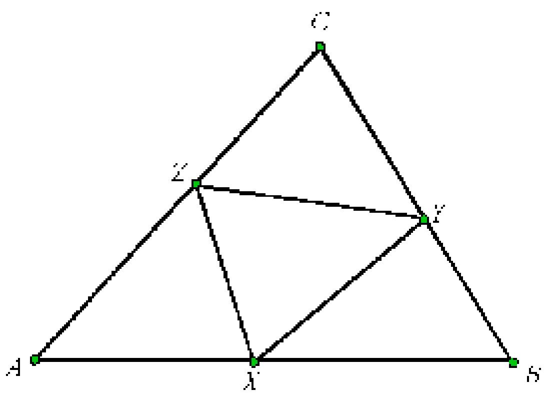
$$R(XYZ) \le \min\{R(AXZ), R(BYX), R(CYZ)\},$$

where X, Y, Z are the vertices of T.

Solution. (a) Let $\angle YXZ = \gamma_1$, $\angle XYZ = \alpha_1$, $\angle YZX = \beta_1$, $\angle ZCY = \gamma$, $\angle XAZ = \alpha$ and $\angle YBX = \beta$. We will prove that $R \ge \min\{R_A, R_B, R_C\}$. Let us assume the contrary. From the low of sines for the triangles XYZ and AXZ we have that

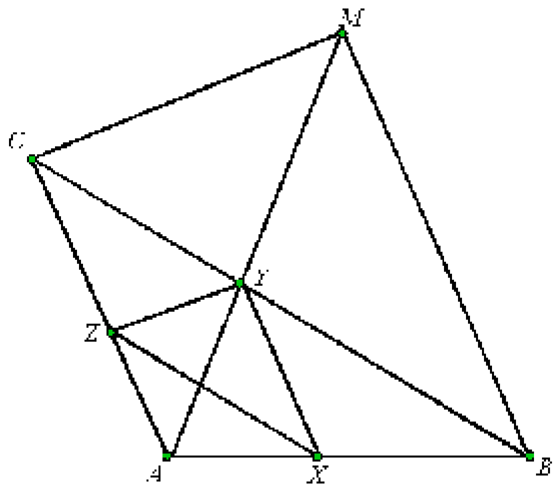
$$\frac{XZ}{\sin\alpha} = 2R_A; \frac{XZ}{\sin\alpha_1} = 2R.$$

Then from our assuming we get that $\sin\alpha_1 > \sin\alpha$. Analogously we have



$\sin\beta_1 > \sin\beta$ and $\sin\gamma_1 > \sin\gamma$.
If XYZ is acute triangle, then $\alpha_1 > \alpha$, $\beta_1 > \beta$ and $\gamma_1 > \gamma$.
Then $180° = \alpha + \beta + \gamma < \alpha_1 + \beta_1 + \gamma_1 = 180°$ which is contradiction.
Let XYZ is not an acute triangle. Without loss of generality we can assume that $\alpha_1 \geq 90°$. Then we have: $\beta_1 + \gamma_1 \leq 90°$, $\sin\alpha_1 = \sin(\beta_1 + \gamma_1) > \sin\alpha$, $\beta_1 + \gamma_1 > \alpha$, $2(\beta_1 + \gamma_1) > \alpha + \beta + \gamma = 180°$, $\beta_1 + \gamma_1 > 90°$, which is a contradiction.

(b) Now let the triangle ABC be nonacute and without loss of generality the nonacute angle is $\angle A$ i.e. $\alpha \geq 90°$. We will construct a triangle XYZ such that $R \leq \min\{R_A, R_B, R_C\}$. Since $\alpha \geq 90°$, we have that on the side BC externally for ABC we can construct a right angled triangle BCM for which $\angle CBM \geq \gamma$ and $\angle BCM \geq \beta$. Let $AM \cap BC = Y$. Let us consider a



homothety H_A with center A and coefficient $\frac{AY}{AM}$. Then $H_A(C) = Z$, $Z \in AC$ and $H_A(B) = X$ $X \in AB$. Then the triangle XYZ is inscribed in ABC, and it is similar to triangle BCM. Then from $90° \geq \angle ZXY = \angle CBM > \gamma$, $90° \geq \angle XZY = \angle BCM > \beta$ and $\alpha \geq 90° = \angle XYZ$. Then $\sin\alpha_1 \geq \sin\alpha$, $\sin\beta_1 \geq \sin\beta$ and $\sin\gamma_1 \geq \sin\gamma$. So from the law of sines we have that $R(XYZ) \leq \min\{R(AXZ), R(BYX), R(CYZ)\}$.

So if ABC is acute we always have $R(XYZ) \geq \min\{R(AXZ), R(BYX), R(CYZ)\}$, and if it is nonacute we constructed an example for which $R(XYZ) \leq \min\{R(AXZ), R(BYX), R(CYZ)\}$.

Problem 3. In a triangle ABC is inscribed a triangle T, so that ABC is divided to four triangles - T, T_1, T_2, T_3. Prove that T doesn't have an altitude, which is with less length than all of the altitudes of T_1, T_2, T_3.

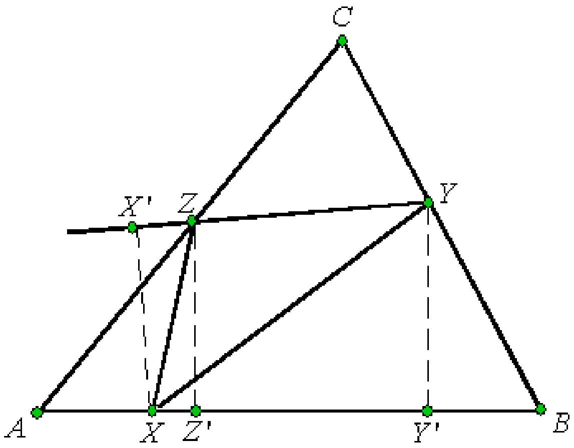
Solution. Let us consider the altitudes of the four triangles XYZ, AXZ, BYX, CYZ. If the triangle XYZ has an altitude, which has less length that all altitudes of AXZ, BYX, CYZ, we can consider the smallest altitude in XYZ. Without loss of generality let it be XX', $X' \in YZ$. Then $ZY > XZ$ and $ZY > XY$. Let the altitudes form the vertices Y and Z in the triangles AXZ and BYX be ZZ' and YY' respectively.

Let us consider the case when X is between Y' and Z'. From our assuming we have that $ZZ' > XX'$ and $YY' > XX'$. The triangles XZZ' and $XX'Z$ are right angled, so from $ZZ' > XX'$ we have $\angle ZXZ' > \angle XZX'$. Analogously we get $\angle YXY' > \angle XYX'$. Then

$$\angle Z'XZ + \angle YXY' > \angle XZY + XYZ = 180^\circ - \angle YXZ = \angle Z'XZ + \angle YXY',$$

which is a contradiction.

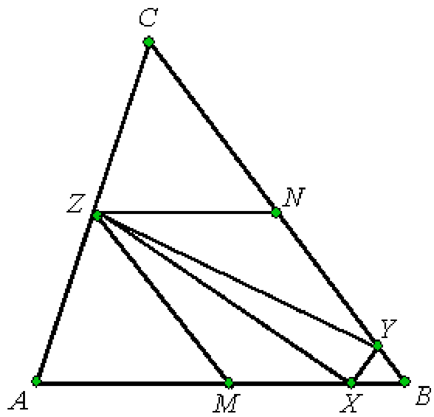
Let us consider the case when Z' is between X and Y'. Since $YY' > XX'$, then $\angle YXY' > \angle XYX'$. So we have $\angle ZYY' < 90^\circ$ or equivalently $\angle YZZ' > 90^\circ$. Since Z' is between X and Y' we have $\angle XZY > Z'ZY > 90^\circ$, so $XY > XZ$, which is a contradiction. The case when Y' is between Z' and X is completely analogous.



So it is not possible XYZ to have an altitude which is with less length than all altitudes of AXZ, BYX, CYZ.

Problem 4. In a triangle ABC is inscribed another triangle T so that it divides ABC in four triangles - T, T_1, T_2, T_3. Prove that there exists T, with an angle which is smaller than all angles of T_1, T_2, T_3.

Solution. Let us consider the angles in AXZ, BYX, CYZ and XYZ. In every triangle there are at least two acute angles. Let in ABC they are $\angle ABC$ and $\angle ACB$. Let the points M, N and P be the midpoints of the sides AB, BC and AC. Let $X \in MB$, $Y \in BN$ and $Z \equiv P$. Then the measure of $\angle XZY$ can be arbitrary small. Then it is obvious that there is a position of X and Y, for which $\angle XZY$ is less than all angles of AXZ, BYX, CYZ.



Problem 5. On the sides BC, AC and AB of a triangle ABC are given the points A_1, B_1 and C_1. Prove that $S_{ABC}S^2_{A_1B_1C_1} \geq 4S_{AB_1C_1}S_{A_1BC_1}S_{A_1B_1C}$.

Solution. Let $\dfrac{AC_1}{AB} = \alpha$, $\dfrac{BA_1}{BC} = \beta$ and $\dfrac{CB_1}{CA} = \gamma$. Then

$S_{A_1BC_1} = S_{ABC}\alpha(1-\gamma)$, $S_{BA_1C_1} = S_{ABC}(1-\alpha)\beta$, $S_{CA_1B_1} = S_{ABC}\gamma(1-\beta)$

and $S_{A_1B_1C_1} = S_{ABC}(1-\alpha(1-\gamma)-\beta(1-\alpha)-\gamma(1-\beta))$.

Then it is enough to prove that

$$(1-\alpha(1-\gamma)-\beta(1-\alpha)-\gamma(1-\beta))^2 \geq 4\alpha\beta\gamma(1-\alpha)(1-\beta)(1-\gamma)$$

or $(1-\alpha-\beta-\gamma+\alpha\gamma+\beta\gamma+\alpha\beta)^2 \geq 4\alpha\beta\gamma(1-\alpha-\beta-\gamma+\alpha\gamma+\beta\gamma+\alpha\beta-\alpha\beta\gamma)$.

The last is equivalent to $(1-\alpha-\beta-\gamma+\alpha\gamma+\beta\gamma+\alpha\beta-2\alpha\beta\gamma)^2 \geq 0$ which is obviously true.

Problem 6. On the sides BC, AC and AB of a triangle ABC are given the points A_1, B_1 and C_1. Prove that $S_{A_1B_1C_1} \geq \min(S_{A_1B_1C}, S_{A_1BC_1}, S_{AB_1C_1})$. (This problem is equivalent to Problem 1. but we will give a second solution.)

Solution. Let us assume the contrary - $S_{A_1B_1C_1} < \min(S_{A_1B_1C}, S_{A_1BC_1}, S_{AB_1C_1})$ and let $S_{AB_1C_1} \leq S_{A_1BC_1} \leq S_{A_1B_1C}$. Then using the result from the previous problem we get that

$$S_{ABC}S^2_{AB_1C_1} > S_{ABC}S^2_{A_1B_1C_1} \geq 4S_{AB_1C_1}S_{A_1BC_1}S_{A_1B_1C} \geq$$

$$\geq S^2_{AB_1C_1}4S_{A_1B_1C} > S^2_{AB_1C_1}(S_{AB_1C_1}+S_{A_1BC_1}+S_{A_1B_1C}+S_{A_1B_1C_1}) = S_{ABC}S^2_{A_1B_1C_1}$$

Then we have that $S_{ABC}S^2_{AB_1C_1} > S_{ABC}S^2_{AB_1C_1}$ which is a contradiction with our assuming.

Problem 7. On the sides BC, AC and AB of a triangle ABC are given the points A_1, B_1 and C_1. Prove that $\min(S_{AB_1C_1}, S_{A_1BC_1}, S_{A_1B_1C}) \leq \frac{1}{4}S_{ABC}$.

Solution. From the previous problem we have that $\min(S_{AB_1C_1}, S_{A_1BC_1}, S_{A_1B_1C}) = \min(S_{AB_1C_1}, S_{A_1BC_1}, S_{A_1B_1C}, S_{A_1B_1C_1}) \leq$

$$\leq \frac{S_{AB_1C_1}+S_{A_1BC_1}+S_{A_1B_1C}+S_{A_1B_1C_1}}{4} = \frac{S_{ABC}}{4}$$

Problem 8. On the sides BC, AC and AB of a triangle ABC are given the points A_1, B_1 and C_1. Prove that $\sqrt{S_{AB_1C_1}} + \sqrt{S_{A_1BC_1}} + \sqrt{S_{A_1B_1C}} \leq \frac{3}{2}\sqrt{S_{ABC}}$.

Solution. Let $\frac{AC_1}{AB} = \alpha$, $\frac{BA_1}{BC} = \beta$ and $\frac{CB_1}{CA} = \gamma$. Then

$S_{A_1BC_1} = S_{ABC}\alpha(1-\gamma)$, $S_{BA_1C_1} = S_{ABC}(1-\alpha)\beta$, $S_{CA_1B_1} = S_{ABC}\gamma(1-\beta)$

and $S_{A_1B_1C_1} = S_{ABC}(1-\alpha(1-\gamma)-\beta(1-\alpha)-\gamma(1-\beta))$.

We have that

$$\sqrt{S_{AB_1C_1}} + \sqrt{S_{A_1BC_1}} + \sqrt{S_{A_1B_1C}} = \sqrt{S_{ABC}}\left(\sqrt{\alpha(1-\gamma)} + \sqrt{\beta(1-\alpha)} + \sqrt{\gamma(1-\beta)}\right) \leq$$

$$\leq \sqrt{S_{ABC}}\left(\frac{\alpha+1-\gamma}{2} + \frac{\beta+1-\alpha}{2} + \frac{\gamma+1-\beta}{2}\right) = \frac{3}{2}\sqrt{S_{ABC}}$$

Problem 9. On the sides BC, AC and AB of a triangle ABC are given the points A_1, B_1 and C_1. Prove that $S_{A_1B_1C_1}\left(\frac{1}{S_{AB_1C_1}}+\frac{1}{S_{A_1BC_1}}+\frac{1}{S_{A_1B_1C}}\right)\geq 3$.

Solution. Let $\frac{AC_1}{AB}=\alpha$, $\frac{BA_1}{BC}=\beta$ and $\frac{CB_1}{CA}=\gamma$. Then

$S_{A_1BC_1}=S_{ABC}\alpha(1-\gamma)$, $S_{BA_1C_1}=S_{ABC}(1-\alpha)\beta$, $S_{CA_1B_1}=S_{ABC}\gamma(1-\beta)$

and $S_{A_1B_1C_1}=S_{ABC}(1-\alpha(1-\gamma)-\beta(1-\alpha)-\gamma(1-\beta))$.

It is enough to show that

$$(1-\alpha(1-\gamma)-\beta(1-\alpha)-\gamma(1-\beta))\left(\frac{1}{\alpha(1-\gamma)}+\frac{1}{\beta(1-\alpha)}+\frac{1}{\gamma(1-\beta)}\right)\geq 3$$

Or in other words

$$A=\left((1-\alpha)(1-\beta)(1-\gamma)+\alpha\beta\gamma\right)\left(\frac{1}{\alpha(1-\gamma)}+\frac{1}{\beta(1-\alpha)}+\frac{1}{\gamma(1-\beta)}\right)\geq 3.$$

We can see that

$$A=\frac{(1-\alpha)(1-\beta)}{\alpha}+\frac{\beta\gamma}{1-\gamma}+\frac{(1-\beta)(1-\gamma)}{\beta}+\frac{\alpha\gamma}{1-\alpha}+\frac{(1-\alpha)(1-\gamma)}{\gamma}+\frac{\alpha\beta}{1-\beta}=$$

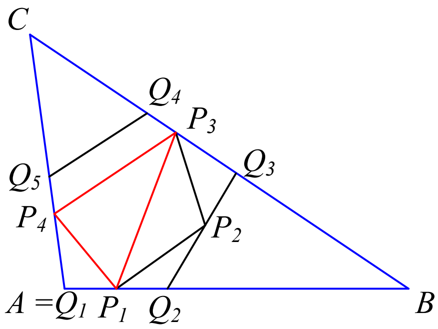
$$=\frac{1-\beta}{\alpha}+\beta-1+\ldots+\frac{\alpha}{1-\beta}+\alpha=$$

$$=\left(\frac{1-\beta}{\alpha}+\frac{\alpha}{1-\beta}\right)+\left(\frac{1-\gamma}{\beta}+\frac{\beta}{1-\gamma}\right)+\left(\frac{1-\alpha}{\gamma}+\frac{\gamma}{1-\alpha}\right)-3\geq 3$$

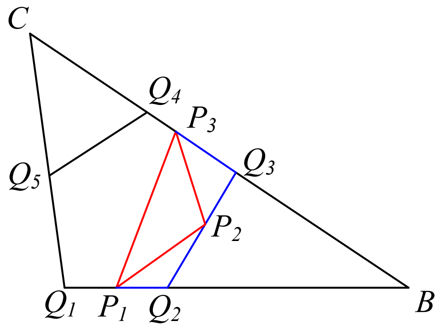
which completes the proof.

PART TWO

Problem 10. A convex polygon $P = P_1P_2...P_m$ is inscribed in another convex polygon $Q = Q_1Q_2...Q_n$ where $3 \le m \le n$ so that Q is divided in $m+1$ parts - $P_1P_2...P_m$, L_1, L_2,... L_m. Prove that $S_{P_1P_2...P_m} \ge \min(S_{L_1}, S_{L_2},...S_{L_m})$



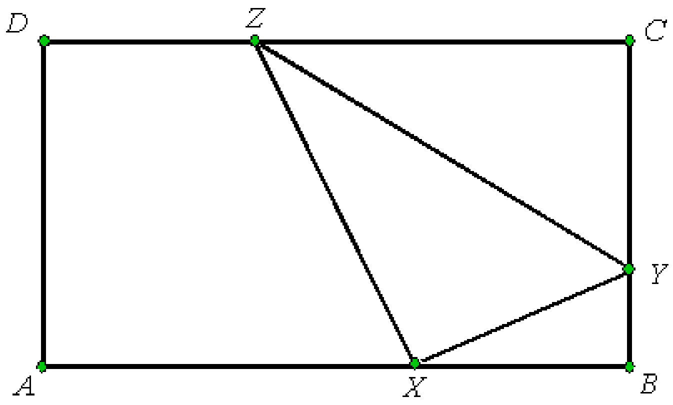
Solution. Since $Q_1Q_2...Q_n$ is divided in $m+1$ parts, no two points P_i and P_j, lye on one side of the Q. If P_a, P_b, P_c are arbitrary vertices of P then $S_{P_aP_bP_c} < S_P$. Let P_a, P_b, P_c lie on three sides of Q such that P_a, P_b, P_c are inner points of the triangle ABC formed by these lines. Then from problem 1 we have that the area of $P_aP_bP_c$ is not less than one of the areas of the other three parts. If there is no such triangle, i.e. P_a, P_b, P_c are not inner points of the triangle ABC, we consider the quadrilateral $P_aQ_mQ_nP_c$. Since $P_aQ_m \cap Q_nP_c = B$ then



$$S_{P_aP_bP_c} \ge \min(S_{P_aQ_mP_b}, S_{P_bQ_nP_c}, S_{P_aBP_b}).$$

So our assumption is true.

Problem 11. Given a rectangle $ABCD$ and a triangle XYZ inscribed in it so that $ABCD$ is divided in 4 parts. Prove that the perimeter of $XYZ - P_{XYZ}$ is not less than the minimal perimeter of the other three parts.



Solution. Let us consider the right-angled trapezium $XBCZ$ and without loss of generality we can assume that $XB < CZ$. Let us consider the symmetric point K of B toward the midpoint of XY i.e. we supplement XYB to a rectangle. Let $P \in XZ$ so that $YP \| CD$. Then $YP > XB$ i.e. K is internal of XYZ. So from that follows that $P_{XYZ} > P_{XBY}$.

Problem 12. In a rectangle P_1 is inscribed a rectangle P_2 with sides c and d, $c \le d$. Prove that the angle α between two lines which contain two of the sides of P_1 and P_2 has the following property - $\sin 2\alpha \ge \frac{c}{d}$.

Solution. Let a and b be the sides of P_1 and α can be the angle between the lines containing the bigger sides of the two rectangles. We have that $d = b\cos\alpha + a\sin\alpha \ge c = b\sin\alpha + a\cos\alpha$ and $b > b\cos\alpha + a\sin\alpha$ so $\alpha < 45°$ and $b > a\frac{\sin\alpha}{1-\cos\alpha}$.

Then we have that

$$\sin 2\alpha - \frac{c}{d} = \sin 2\alpha - \frac{b\sin\alpha + a\cos\alpha}{b\cos\alpha + a\sin\alpha} =$$

$$= \frac{\cos 2\alpha(b\sin\alpha - a\cos\alpha)}{b\cos\alpha + a\sin\alpha} \ge \frac{a\cos 2\alpha}{b\cos\alpha + a\sin\alpha} \ge 0$$

Then we have that $\sin 2\alpha \ge \frac{c}{d}$.

Problem 13. In a convex polygon $A = A_1A_2...A_n$ is inscribed a convex polygon $B = B_1B_2...B_n$ so that $\angle B_nB_1A_1 = \angle B_2B_1A_2, \angle B_1B_2A_2 = \angle B_3B_2A_3, ...$. Prove that for arbitrary points $C_1, C_2, ... C_n$ on A_1A_2, A_2A_3, …, A_nA_1 we have that

$$B_1B_2 + B_2B_3 + ... + B_{n-1}B_n + B_nB_1 \le C_1C_2 + C_2C_3 + ... + C_nC_1.$$

Solution. Let $\angle B_nB_1A_1 = \angle B_2B_1A_2 = \beta_1, \angle B_1B_2A_2 = \angle B_3B_2A_3 = \beta_2, ...$ Trough the vertices of the polygon we draw lines parallel to $B_1B_n, ..., B_{n-1}B_n$. Let C_1' and C_1'' be projections of the point C_1 on the lines l_1 and l_2. We make the points C_2' and C_2'' analogously and so on. Using the idea from Problem 5 we have

$$C_1C_2 + C_2C_3 + ... + C_nC_1 \ge C_1''C_2' + C_2''C_3 + ... =$$

$$= (A_2C_1\cos\beta_1 + A_2C_2\cos\beta_2) + ... + (C_1A_1\cos\beta_1 + C_nA_1\cos\beta_n) =$$

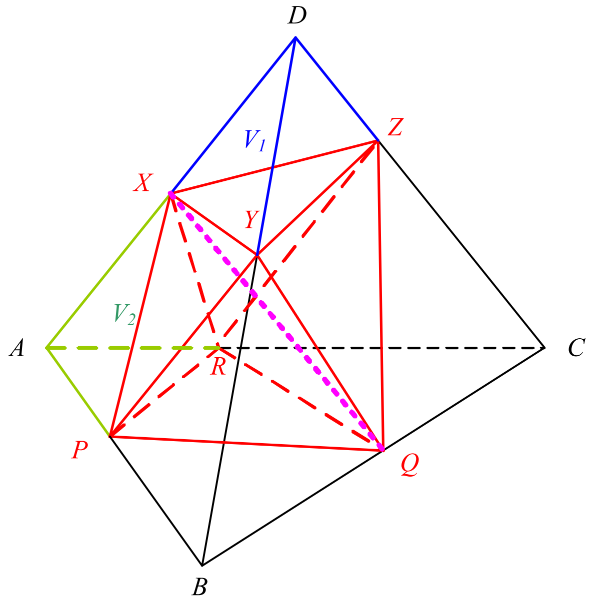
$$= A_1A_2\cos\beta_1 + A_2A_3\cos\beta_2 + ... + A_nA_1\cos\beta_n =$$

$$= (A_2B_1\cos\beta_1 + B_1A_1\cos\beta_1) + ... + (A_1B_n\cos\beta_n + A_nB_n\cos\beta_n).$$

So the problem is solved.

PART THREE

Problem 14. Given a tetrahedron $ABCD$, in which is inscribed a octahedron $XYZPQR$(dividing $ABCD$ in 5 parts). Denote the volumes of the parts with $V_{XYZPQR} = V$, $V_{XYZD} = V_1$, $V_{XAPR} = V_2$, $V_{YBPQ} = V_3$, $V_{ZCQR} = V_4$ and $V^* = \min\{V_i + V_j\}, i \neq j = 1,2,3,4$. Prove that $V > V^*$.



Solution.

From Problem 1 we have

$S_{PQR} \geq \min\{S_{APR}; S_{BPQ}; S_{CQR}\}$.

Let $S_{PQR} \geq S_{APR}$. Then $V_{XPQR} \geq V_{XAPR}$, since tetrahedrons $APRX$ and $PQRX$ have common altitude .

$S_{YQZ} \geq \min\{S_{BQY}; S_{CQZ}; S_{DYZ}\}$

Let $S_{YQZ} \geq S_{DYZ}$. Then $V_{XYZQ} \geq V_{XDZY}$.

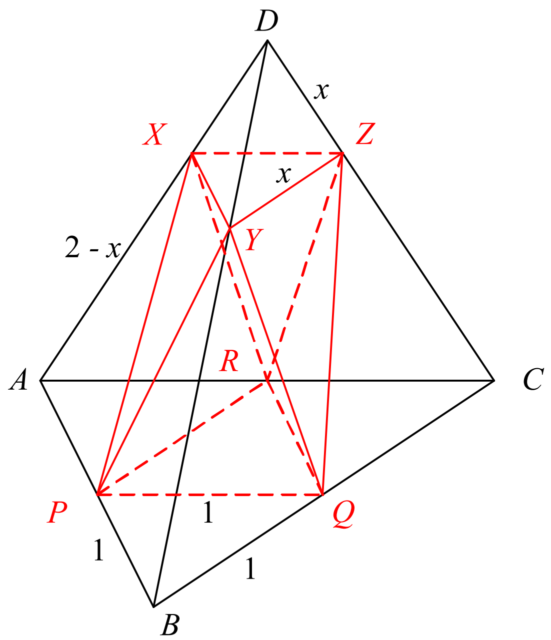
Obviously

$V_{XYZPQR} > V_{XPQR} + V_{XYZQ}$.

So $V > V^* = V_1 + V_2$.

Note: It easy to construct an example where $V < \max\{V_i + V_j\}, i \neq j = 1,2,3,4$.

Problem 15. Given a regular tetrahedron $ABCD$, in which is inscribed a octahedron $XYZPQR$ such that $DX = DY = DZ = x$ and P, Q, R are the midpoints of AB, BC, CA respectively . Denote the surface areas of the 5 parts with $S_{XYZPQR} = S$, $S_{XAPR} = S_1$, $S_{YBPQ} = S_2$, $S_{ZCQR} = S_3, S_{XYZD} = S_4$. Prove that for some x $S < S_1, S_2, S_3$.



Solution. Since $ABCD$ is regular, obviously $S_1 = S_2 = S_3$, and $DXYZ$ is also regular tetrahedron with edge x. Assume that $AB = 2$. It is easy to calculate:

$$S = S_{PQR} + S_{XYZ} + 3S_{PXY} + 3S_{PQY}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}x^2 + \frac{3\sqrt{3}}{4}x(2-x) + \frac{3}{4}\sqrt{4x^2 - 12x + 11}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}(3x - x^2) + \frac{3}{4}\sqrt{4x^2 - 12x + 11}$$

$$S_1 = S_{PBQ} + 2S_{PBY} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}(2-x)$$

Now we are looking for x, such

that the inequality

$$\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}(3x-x^2)+\frac{3}{4}\sqrt{4x^2-12x+11}<\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}(2-x)$$

or

$$\sqrt{4x^2-12x+11}<\sqrt{3}(x^2-4x+2)$$

holds.

Since $\sqrt{11}<2\sqrt{3}$, for positive values of x, close to 0, the inequality holds.

Conjecture. Denote the surface areas of the 5 parts with $S_{XYZPQR}=S$, $S_{XYZD}=S_1$, $S_{XAPR}=S_2$, $S_{YBPQ}=S_3$, $S_{ZCQR}=S_4$. Then $S>\min\{S_1,S_2,S_3,S_4\}$.

SUMMARY

The following results are achieved:

PART ONE
1. It is proved that if a triangle is inscribed in other triangle its area is not less than the minimal area of the other three triangles.
2. The problem with the radiuses of the circumscribed circles of the four triangles is considered.
3. The problem with the angles and the altitudes is considered.
4. Some other problems about the area of an inscribed triangle in other triangle in another triangle are considered.

PART TWO
The analogue of Problem 1 for inscribed polygon in another polygon (Problem 11) is solved.
Some other particular cases for elements of minimality are considered.

PART THREE
The analogue of Problem 1 for the volume of inscribed octahedron in tetrahedron is solved.
A conjecture for the surface area of inscribed octahedron in tetrahedron is deduced.

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