I[st] International Tournament of Young Mathematicians
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PROBLEM THREE

MONOTONIC SQUARES

We say that a positive integer $a = a_k a_{k-1} ... a_1 a_0$, where $0 \le a_i \le 9$ are the digits of a in base 10, is an *increasing square* if $a = b^2$ for some integer b and $a_k \le a_{k-1} ... \le a_1 \le a_0$. For instance: $13456 = 116^2$.

If we have the reverse inequalities $a_k \ge a_{k-1} ... \ge a_1 \ge a_0$, then the square a is called *decreasing*. For instance: $8874441 = 2979^2$.

Let $a = b^2$ and $c = d^2$ be two squares with the following representations in base 10: $a = a_k a_{k-1} ... a_1 a_0$ $b = b_l b_{l-1} .. b_1 b_0$ $c = c_m c_{m-1} ... c_1 c_0$ $d = d_n d_{n-1} ... d_1 d_0$. We say that a pair of squares a and c is ordered, and write a **p** c, if the sequence $a_k, a_{k-1}, ..., a_1, a_0$ is a subsequence of $c_m, c_{m-1}, ..., c_1, c_0$, and the sequence $b_l, b_{l-1}, ..., b_1, b_0$ is a subsequence of $d_n, d_{n-1}, ..., d_1, d_0$. For instance: $1154 = 34^2$ **p** $111556 = 334^2$.

A set of squares F is called a family if any pair of squares from F is ordered.

1. Find infinite families of increasing squares.

For instance: $1156 = 34^2$ **p** $111556 = 334^2$ **p** $11115556 = 3334^2$ **p**..

2. Is there any infinite family of decreasing squares?
3. How many elements are in a maximal increasing family? For example, can it have exactly?

(a) One increasing square? (b) Two increasing squares? (A maximal increasing family F is a family of increasing squares, such that any increasing square a with the property "either a **p** c or c **p** a for all $c \in F$`` is already in F)

4. How many elements can a maximal family of decreasing squares have?
5. Investigate the problem in other bases.

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THEORETICAL PART

Let call the smallest element in a family *originative square and* number, which square is monotonic *base number*.

In the work with monotonic squares we are going to use the results of the two following problems:

Problem 1. Prove that there is no square with one and the same digits that is larger than 10.

Solution. Since squares end on $0,1,4,5,6$ or 9, we should examine the cases for the number $\overline{aa...a}$ when $a=1,4,5,6$ and 9.

If $a=1$, $\overline{11..11}=A.100+11$

$A.100\equiv 0(\mathrm{mod}\,4)$, but $11\equiv 3(\mathrm{mod}\,4)\Rightarrow \overline{11...1}\equiv 3(\mathrm{mod}\,4)$. If we have natural number *p*, $p\equiv 0\ or\ 1(\mathrm{mod}\,2)\ \Rightarrow\ p^2\equiv 0\ or\ 1(\mathrm{mod}\,4)\Rightarrow\ \overline{11...1}$ can't be a square.

If $a=4$, $\overline{44...4}=4.\overline{11...1}$, 4 is a square, but $\overline{11...1}$ is not $\Rightarrow$ $\overline{44...4}$ is not a square.

If $a=5$, $\overline{55...5}$ is divided by 5 $\Rightarrow$should be divided by 25.

$\overline{55...5}=A.100+55$, 25/100, but 55 is not divided by 25 $\Rightarrow$ $\overline{55...5}$ is no divided by 25 $\Rightarrow$is not a square.

If $a=6$, $\overline{66...6}=A.100+66$, $66\equiv 2(\mathrm{mod}\,4)\Rightarrow \overline{66...6}\equiv 2(\mathrm{mod}\,4)$ i.e. $\overline{66...6}$ can't be a square.

If $a=9$, $\overline{99...9}=A.100+99$, $99\equiv 3(\mathrm{mod}\,4)\Rightarrow \overline{99...9}\equiv 3(\mathrm{mod}\,4)$, i.e. $\overline{99...9}$ can't be a square.

$\Rightarrow$ we proved that there is no square with one and the same digits that is larger than 10.

Problem 2. Find the last two digits of every monotonic square.

Solution. We know that 0^2=0, 1^2=1, 2^2=4, 3^2=9, 4^2=16, 5^2=25, 6^2=36, 7^2=49, 8^2=64 and 9^2=81, so the last digit of any monotonic square should be 0, 1, 4, 5, 6 or 9.

Obviously increasing squares can not contain 0. If their last digit is 1, means that all digits are units, but there is no numbers, whose digits are only units and is square except $1^2 = 1$.

Consequently the increasing squares can end on 4, 5, 6 or 9. If there is a decreasing square with 9 for last digit, means that all digits are just 9. That is possible only if the square is 9 (3^2=9). So the decreasing squares can end on 0, 1, 4, 5 or 6.

Let the base number of some monotonic square be N=$10a+b$ and $N\equiv k(\mathrm{mod}\,100)$.

If the monotonic square ends on 0, that means $b=0$.

$(10a+0)^2=100a^2\equiv 0(\mathrm{mod}\ 100)\ \Rightarrow$ The monotonic square ends on 00.

If the monotonic square ends on 1, that means $b=1\ or\ 9\ (1^2=1,\ 9^2=81)$

$(10a+1)^2=100a+20a+1\equiv 20a+1\equiv k(\mathrm{mod}\ 100)$. If $a\equiv 0,1,2,3,4(\mathrm{mod}\ 5)\Rightarrow k=21,41,61,81,1$

$(10a+9)^2=100a^2+180a+81\equiv 80a+81\equiv k(\mathrm{mod}\ 100)$. $a\equiv 0,1,2,3,4(\mathrm{mod}\ 5)\Rightarrow k=21,41,61,81,1$

If the monotonic square ends on 4, that means $b=2$ or 8 $(2^2=4,\ 8^2=64)$

$(10a+2)^2=100a^2+40a+4\equiv 40a+4\equiv k(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=4,44,84,24,64$

$(10a+8)^2=100a^2+160a+64\equiv 60a+64\equiv k(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=4,44,84,24,64$

If the monotonic square ends on 5, that means $b=5$ $(5^2=5)$

$(10a+5)^2=100a^2+100a+25\equiv 25(\text{mod }100)\Rightarrow$ The monotonic square ends on 25.

If the monotonic square ends on 6, that means $b=4 or 6$ $(4^2=16,\ 6^2=36)$

$(10a+4)^2=100a^2+160a+16\equiv 60a+16(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=16,76,36,96,56$

$(10a+6)^2=100a^2+360a+36\equiv 60a+36(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=36,56,96,16,76$

If the monotonic square ends on 9, that means $b=3$ or 7 $(3^2=9, 7^2=49)$

$(10a+3)^2=100a^2+60a+9\equiv 60a+9(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=9,69,29,89,49$

$(10a+7)^2=100a^2+140a+49\equiv 40a+49(\text{mod }100)$. $a\equiv 0,1,2,3,4(\text{mod }5)\Rightarrow k=49,89,29,69,9$

So the result is that the monotonic squares can end on: 00, 21, 41, 61, 81, 44, 84, 24, 64, 25, 16, 76, 36, 96, 56, 69, 29, 89, and 49.

The increasing monotonic squares can end on 24, 44, 25, 16, 36, 56, 69, 29, 89 or 49.

The decreasing monotonic squares can end on 00, 21, 41, 61, 81, 44, 84, 64, 76, and 96.

Some cases can be rejected:

1. In the increasing monotonic square ends on 24, the variants for the square are: 22…224, 11..12…24
2. In the increasing monotonic square ends on 29, the variants for the square are: 11…12…29, 22…29
3. In the decreasing monotonic square ends on 84, the variants for the square are: 88…84, 99…98…84
4. In the decreasing monotonic square ends on 76, the variants for the square are: 99…87…76, 99…97…76, 88…87…76
5. In the decreasing monotonic square ends on 96, the variants for the square are: 99…96

Let $2...24=x^2$ $x^2=2...2+2=\dfrac{2\left(10^k-1\right)}{9}+2\Rightarrow 2^{k+1}5^k+16=(3x)^2\Rightarrow$

$2^{k+1}5^k=(3x-4)(3x+4)$. We know that 4/x, so let 4y=3x$\Rightarrow 2^{k-3}5^k=(y-1)(y+1)$, 5^k divides exactly one of the expressions in the brackets. But in this case the expression in the other brackets is too big to be covered by $2^{k-3}\Rightarrow$ the assumption is wrong and 2...24 can not be a square.

Let 1…124 = x^2. $11...12...224 = 1000.a + 124 \equiv 4 (\bmod 8) \Rightarrow$ x is even, can not be divided by 4, i.e. $x = 4k + 2$. I.e. $x^2 = (4k+2)^2 = 4(2k+1)^2 = 16k(k+1) + 4 \,.\, k(k+1)$ is even $\Rightarrow 32 / 16k(k+1)$, i.e. $a^2 \equiv 4 (\bmod 32)$.

$a^2 \equiv 4 (\bmod 32)$ and $a^2 \equiv 4 (\bmod 8)$. By direct calculation we find out that 124, 1124 and 11124 are not squares. If the square includes at least three units, it can be represented that way: $11...100000 + 11124$

$11...1100000 = 11...11.10^5 \Rightarrow$ $2^5 / 10^5$, i.e. $11...100000 \equiv 0 (\bmod 32)$. $11124 \equiv 20 (\bmod 32)$, i.e. we get a contradiction with the statement that $a^2 \equiv 4 (\bmod 32)$.

Analogically we can prove that the monotonic squares can not end on 29, 84, 76 and 96.

PART ONE

Problem 3. Find all monotonic squares in the interval $(10^0;10^7)$.
Solution. First 100 squares are well-known and with close examination we can find the squares to 10^7.

Increasing squares:

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$12^2 = 144$
$13^2 = 169$
$15^2 = 225$
$16^2 = 256$
$17^2 = 289$
$34^2 = 1156$
$35^2 = 1225$
$37^2 = 1369$
$38^2 = 1444$
$67^2 = 4489$
$83^2 = 6889$
$106^2 = 11236$
$107^2 = 11449$

$116^2 = 13456$
$117^2 = 13689$
$167^2 = 27889$
$183^2 = 33489$
$334^2 = 111556$
$335^2 = 112225$
$337^2 = 113569$
$367^2 = 134689$
$383^2 = 146689$
$587^2 = 344569$
$667^2 = 444889$
$1633^2 = 2666689$
$1667^2 = 2778889$
$3334^2 = 11115556$
$3335^2 = 11122225$
$3337^2 = 11135569$
$3367^2 = 11336689$
$3383^2 = 11444689$
$3667^2 = 13446889$
$4833^2 = 23357889$
$6667^2 = 44448889$

$16667^2 = 277788889$
$33334^2 = 1111155556$
$33335^2 = 1111222225$
$33337^2 = 1111355569$
$33367^2 = 1113356689$
$33667^2 = 1133466889$
$36667^2 = 1344468889$
$66667^2 = 4444488889$
$105462^2 = 11122233444$
$125167^2 = 15666777889$
$166667^2 = 27777888\ 889$
$333334^2 = 111111555556$
$333335^2 = 111112222225$
$333337^2 = 111113555569$
$333367^2 = 111133556689$
$333667^2 = 111333666889$
$336667^2 = 113344668889$
$366667^2 = 134444688889$
$666667^2 = 444444888889$

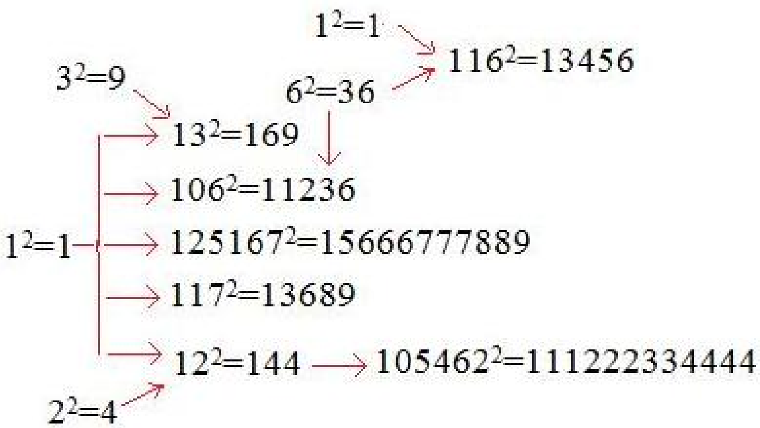
$1666667^2 = 2777778888889$
$3333334^2 = 11111115555556$
$3333335^2 = 11111122222225$
$3333337^2 = 11111135555569$
$3333367^2 = 11111335556689$
$3333667^2 = 11113335666889$
$3336667^2 = 11133346668889$
$3366667^2 = 11334446688889$
$3666667^2 = 13444446888889$
$6666667^2 = 44444448888889$

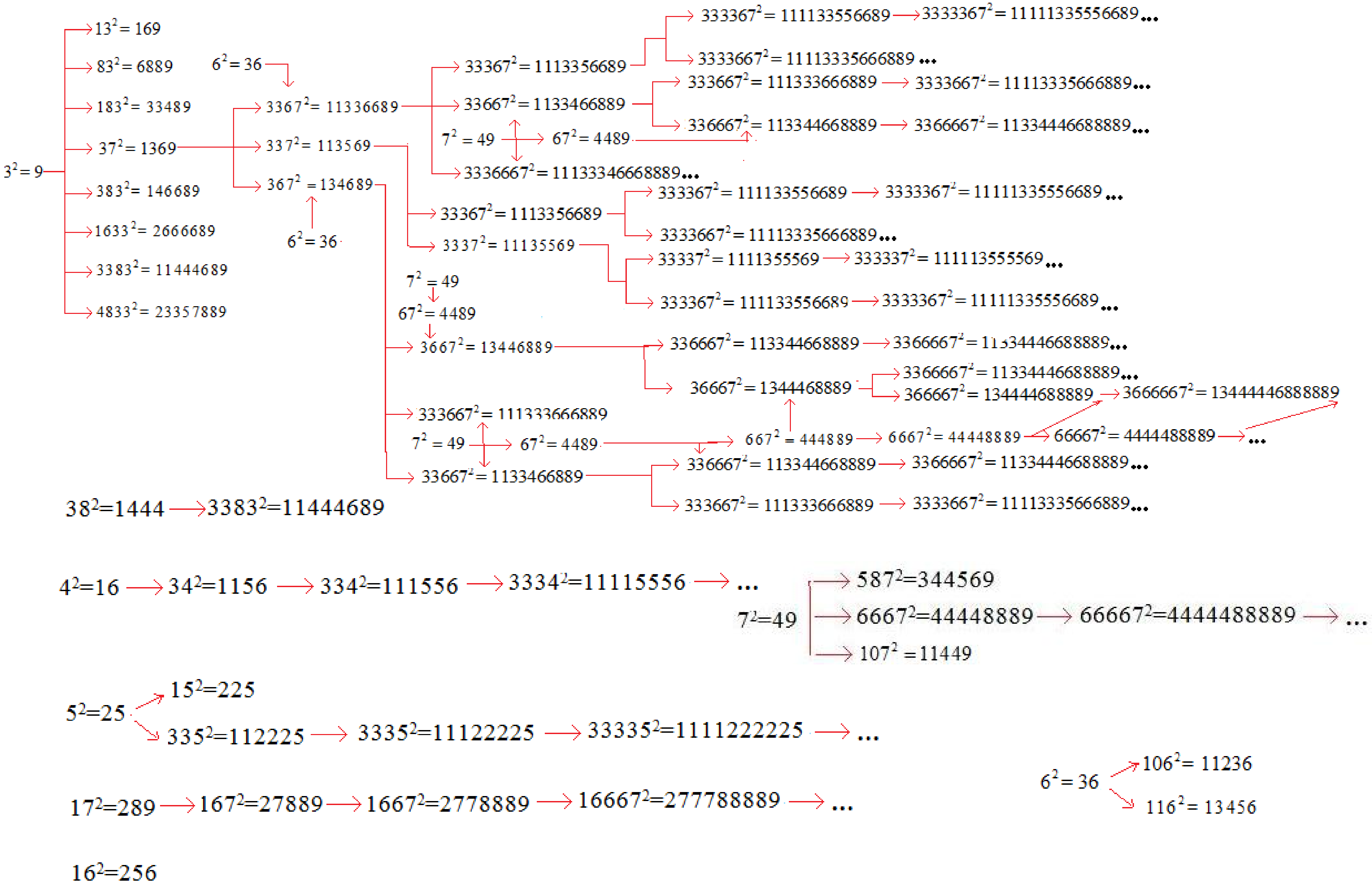
Decreasing squares:

$1^2=1$	$800^2=64.10^4$	$30000^2=9.10^8$	$1000000^2=1.10^{12}$
$2^2=4$	$880^2=7744.10^2$	$31000^2=961.10^6$	$2000000^2=4.10^{12}$
$3^2=9$	$900^2=81.10^4$	$31390^2=9853321.10^2$	$2100000^2=441.10^{10}$
$8^2=64$	$1000^2=1.10^6$	$31621^2=999887641$	$2900000^2=841.10^{10}$
$9^2=81$	$2000^2=4.10^6$	$80000^2=64.10^8$	$2979000^2=8874441.10^6$
$10^2=1.10^2$	$2100^2=441.10^4$	$88000^2=7744.10^6$	$3000000^2=9.10^{12}$
$20^2=4.10^2$	$2900^2=841.10^4$	$90000^2=81.10^8$	$3100000^2=961.10^{10}$
$21^2=441$	$2979^2=8874441$	$100000^2=1.10^{10}$	$3139000^2=9853321.10^6$
$29^2=841$	$3000^2=9.10^6$	$200000^2=4.10^{10}$	$3162100^2=999887641.10^4$
$30^2=9.10^2$	$3100^2=961.10^4$	$210000^2=441.10^8$	$8000000^2=64.10^{12}$
$31^2=961$	$3139^2=9853321$	$290000^2=841.10^8$	$8800000^2=7744.10^{10}$
$80^2=64.10^2$	$8000^2=64.10^6$	$297900^2=8874441.10^4$	$9000000^2=81.10^{12}$
$88^2=7744$	$8800^2=7744.10^4$	$300000^2=9.10^{10}$	$10000000^2=1.10^{14}$
$90^2=81.10^2$	$9000^2=81.10^6$	$310000^2=961.10^8$	
$100^2=1.10^4$	$10000^2=1.10^8$	$313900^2=9853321.10^4$	
$200^2=4.10^4$	$20000^2=4.10^8$	$316210^2=999887641.10^2$	
$210^2=441.10^2$	$21000^2=441.10^6$	$800000^2=64.10^{10}$	
$290^2=841.10^2$	$29000^2=841.10^6$	$880000^2=7744.10^8$	
$300^2=9.10^4$	$29790^2=8874441.10^2$	$900000^2=81.10^{10}$	
$310^2=961.10^2$			

Problem 4. Find infinite families of increasing squares.

Solution. Using the results by Problem 1, we can find all infinite families whose originative squares are with based number less than 10^7. We are going to present them by schemes. They will help us to see exactly how many families with one, two, three or countless elements are there.





3²= 9
13²= 169
83²= 6889
183²= 33489
37²= 1369
383²= 146689
1633²= 2666689
3383²= 11444689
4833²= 23357889
6²= 36
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33667²= 1133466889
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67²= 4489
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333667²= 111333666889
7² = 49
67²= 4489
667² = 444889
6667²= 44448889
66667²= 4444488889
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33667²= 1133466889
336667²= 113344668889
3366667²= 11334446688889...
333667²= 111333666889
3333667²= 11113335666889...
38²=1444
3383²=11444689
4²=16
34²=1156
334²=111556
3334²=11115556
...
7²=49
587²=344569
6667²=44448889
66667²=4444488889
...
107² =11449
5²=25
15²=225
335²=112225
3335²=11122225
33335²=1111222225
...
6²= 36
106²= 11236
116²= 13456
17²=289
167²=27889
1667²=2778889
16667²=277788889
...
16²=256

Comment. Among the monotonic squares with base numbers in the interval from 10^0 to 10^{18}, does not exist originative squares in bigger values. We can make the hypothesis that with the results from Problem 1 we can find all families of monotonic squares.

This is confirmed by a computer program that finds all monotonic squares with base number less than 10^{18}.

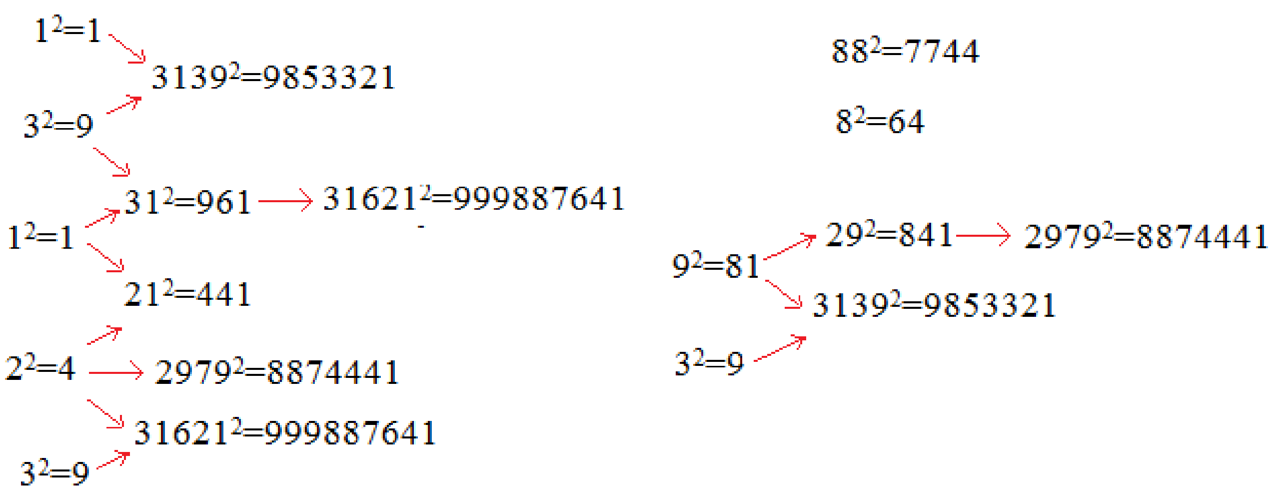
(Application 1, Application 2, Application 3).

Problem 5. Find infinite decreasing families in the interval $(10^0; 10^{18})$.

Solution. The trivial decreasing families, which are formed just by adding a 0 for last digit to the base number of some square that is already in the family, are easy to found.

$1^2 = 1$ **p** $10^2 = 100$ **p** $100^2 = 10\ 000$ **p** $1\ 000^2 = 1\ 000\ 000$ ...
$2^2 = 4$ **p** $20^2 = 400$ **p** $200^2 = 40\ 000$ **p** $2\ 000^2 = 4\ 000\ 000$ ...
$3^2 = 9$ **p** $30^2 = 900$ **p** $300^2 = 90\ 000$ **p** $3\ 000^2 = 9\ 000\ 000$ ...
$8^2 = 64$ **p** $80^2 = 6\ 400$ **p** $800^2 = 640\ 000$ **p** $8\ 000^2 = 64\ 000\ 000$ ...
$9^2 = 81$ **p** $90^2 = 81\ 00$ **p** $900^2 = 810\ 000$ **p** $9\ 000^2 = 81\ 000\ 000$ ...

All decreasing families which originative square is less than 10^{18}, and do not have a 0 for a last digit are:



Problem 6. How many elements can have a maximal decreasing family?

Solution. If p is decreasing square and $p = q^2$, means that p can be always included in the following infinite family: q^2 **p** $\overline{q0}^2$ **p** $\overline{q00}^2$ … So the maximal decreasing family for a square is always with countless elements. This kind of families we will call *trivial families.*

Comment. Every of the families found in Problem 5 can be transformed in infinite family by adding zeros to the base numbers of the elements. We see that the decreasing monotonic squares are formed by these kinds of base numbers –

$a.10^k$, a = 1, 2, 3, 8, 9, 21, 31, 88, 29, 2979 or 31621, $k \in (0, \infty)$.

Comment. We can notice that in the increasing monotonic squares there is symmetry in forming the squares and no new families arise out of bigger values, so in Problem 2 we have found all families

There are two interesting families:

$1^2=1 \searrow$, $2^2=4 \nearrow$ $\quad 21^2=441 \longleftrightarrow 144=12^2 \quad \nearrow 1^2=1$, $\searrow 2^2=4$

$1^2=1 \searrow$, $3^2=9 \nearrow$ $\quad 169=13^2 \longleftrightarrow 31^2=961 \quad \nearrow 1^2=1$, $\searrow 3^2=9$

In the analyzed interval (10^0; 10^{18}) this are the only “mirror” families.

PART TWO

Problem 7. Is there any maximal increasing family with one element?
Solution. In Problem 2 we have found all increasing families. We found a family with one element and we can assume that there is no other element in the squares that are out of the analyzed interval. In the interval (10^0; 10^{18}) the only family with one element is $\mathbf{16^2=256}$. According to our hypothesis there is no other family with one element out of the interval.

Problem 8. Is there any maximal increasing family with two elements?
Solution. By Problem 2 we have the following results for families with two elements:

$1^2 = 1$ **p** $13^2 = 169$
$1^2 = 1$ **p** $106^2 = 11236$
$1^2 = 1$ **p** $116^2 = 13456$
$6^2 = 36$ **p** $116^2 = 13456$
$6^2 = 36$ **p** $106^2 = 11236$
$1^2 = 1$ **p** $117^2 = 13689$
$1^2 = 1$ **p** $125167^2 = 15666777889$
$1^2 = 1$ **p** $107^2 = 11449$
$3^2=9$ **p** $1633^2=2666689$
$3^2=9$ **p** $183^2=33489$

$5^2 = 25$ **p** $15^2 = 225$
$3^2 = 9$ **p** $13^2 = 169$
$3^2 = 9$ **p** $4833^2 = 23357889$
$3^2 = 9$ **p** $83^2 = 6889$
$3^2 = 9$ **p** $383^2 = 146689$
$3^2=9$ **p** $3383^2 = 11444689$
$38^2 = 1444$ **p** $3383^2 = 11444689$
$7^2 = 49$ **p** $587^2 = 344569$
$7^2 = 49$ **p** $107^2 = 11449$

Problem 9. Is there any maximal increasing family with three elements?
Solution. In the analyzed interval there are two families with three elements:

$1^2 = 1$ **p** $12^2 = 144$ **p** $105462^2 = 111222334444$
$2^2 = 4$ **p** $12^2 = 144$ **p** $105462^2 = 111222334444$

Comment. If we think that the one digit numbers ($1^2 = 1, 2^2 = 4$ *and* $3^2 = 9$) are not monotonic squares, the results from Problem 6, 7 and 8 are going to be:
Maximal increasing families with one element:

$16^2 = 256$, $13^2 = 169$, $117^2 = 13689$, $125167^2 = 15666777889$,
$4833^2 = 23357889$, $83^2 =6889$, $83^2 = 6889$, $383^2 = 146689$, $3383^2 = 11444689$.

Maximal increasing families with two elements:

$6^2 = 36$ **p** $116^2 = 13456$
$6^2 = 36$ **p** $106^2 = 11236$
$5^2 = 25$ **p** $15^2 = 225$
$38^2=1444$ **p** $3383^2=11444689$

$7^2 = 49$ **p** $587^2 = 344569$
$7^2 = 49$ **p** $107^2 = 11449$
$12^2 = 144$ **p** $105462^2 = 111222334444$.

Maximal increasing families with three elements are not founded.
Comment. From Problem 4 we make the following observation:

Increasing monotonic families can be divided on three groups: with one, two, three (if we count the one digit squares for monotonic) and countless elements.

PART THREE

Problem 10. Find all monotonic squares in the interval $(10^0; 10^{18})$ in base p, $p < 10$.
Solution. We made a computer program to calculate all monotonic squares in base p for $p < 10$.

10.1. Increasing squares:

10.1.1. Base 2

$1^2 = 1$

10.1.2. Base 3

$1^2 = 1$

$2^2 = 11$

$102^2 = 11111$

10.1.3. Base 4

$1^2 = 1$

10.1.4. Base 5

$1^2 = 1$
$2^2 = 4$
$3^2 = 14$
$12^2 = 144$
$13^2 = 224$
$23^2 = 1134$
$33^2 = 2244$
$103^2 = 11114$
$223^2 = 111334$
$2223^2 = 11113334$
$3233^2 = 22234444$
$22223^2 = 1111133334$
$222223^2 = 111111333334$
$2222223^2 = 11111113333334$
$22222223^2 = 1111111133333334$
$222222223^2 = 111111111333333334$
$2222222223^2 = 11111111113333333334$
$22222222223^2 = 1111111111133333333334$

10.1.5. Base 6

$1^2 = 1$
$2^2 = 4$
$3^2 = 13$
$4^2 = 24$
$12^2 = 144$
$14^2 = 244$
$44^2 = 3344$
$104^2 = 11224$
$114^2 = 13444$
$242^2 = 112244$

10.1.6. Base 7

$1^2 = 1$
$2^2 = 4$
$3^2 = 12$
$4^2 = 22$
$5^2 = 34$
$12^2 = 144$
$26^2 = 1111$
$55^2 = 4444$
$104^2 = 11122$
$105^2 = 11334$
$412^2 = 233344$
$260055^2 = 111144444444$

10.1.7. Base 8

$1^2 = 1$
$2^2 = 4$
$3^2 = 11$
$6^2 = 44$
$12^2 = 144$
$32^2 = 1244$
$52^2 = 3344$
$106^2 = 11444$
$306^2 = 114444$

10.1.8. Base 9

$1^2 = 1$
$2^2 = 4$
$4^2 = 17$
$5^2 = 27$
$12^2 = 144$
$15^2 = 237$
$32^2 = 1134$
$34^2 = 1277$
$35^2 = 1357$
$45^2 = 2267$
$55^2 = 3377$
$105^2 = 11127$
$115^2 = 13337$
$145^2 = 22367$
$445^2 = 222667$

$1054^2 = 1122257$
$1445^2 = 2223667$
$2534^2 = 6666677$
$4445^2 = 22226667$
$5455^2 = 33347777$
$14445^2 = 222236667$
$44445^2 = 2222266667$
$144445^2 = 22222366667$
$444445^2 = 222222666667$
$1444445^2 = 2222223666667$
$4444445^2 = 22222226666667$
$14444445^2 = 222222236666667$
$44444445^2 = 2222222266666667$
$144444445^2 = 22222222366666667$

10.2. Decreasing squares:

10.2.1.Base 2

$$\overline{1\underbrace{0\ldots0}_{n}}^2 = \overline{1\underbrace{0\ldots0}_{2n}} = 1.2^{2n}$$

10.2.2. Base 3

$$\overline{1\underbrace{0\ldots0}_{n}}^2 = \overline{1\underbrace{0\ldots0}_{2n}} = 1.3^{2n}$$

$$\overline{2\underbrace{0\ldots0}_{n}}^2 = \overline{11\underbrace{0\ldots0}_{2n}} = 11.3^{2n}$$

$$\overline{12\underbrace{0\ldots0}_{n}}^2 = \overline{221\underbrace{0\ldots0}_{2n}} = 221.3^{2n}$$

$$\overline{102\underbrace{0\ldots0}_{n}}^2 = \overline{11111\underbrace{0\ldots0}_{2n}} = 11111.3^{2n}$$

10.2.3. Base 4

$$\overline{1\underbrace{0\ldots0}_{n}}^2 = \overline{1\underbrace{0\ldots0}_{2n}} = 1.4^{2n}$$

$$\overline{2\underbrace{0\ldots0}_{n}}^2 = \overline{10\underbrace{0\ldots0}_{2n}} = 10.4^{2n}$$

$$\overline{3\underbrace{0\ldots0}_{n}}^2 = \overline{21\underbrace{0\ldots0}_{2n}} = 21.4^{2n}$$

$$\overline{12\underbrace{0\ldots0}_{n}}^2 = \overline{210\underbrace{0\ldots0}_{2n}} = 210.4^{2n}$$

$$\overline{31\underbrace{0\ldots0}_{n}}^2 = \overline{2221\underbrace{0\ldots0}_{2n}} = 2221.4^{2n}$$

$$\overline{122\underbrace{0\ldots0}_{n}}^2 = \overline{22210\underbrace{0\ldots0}_{2n}} = 22210.4^{2n}$$

10.2.4. Base 5

$\overline{10\underbrace{\ldots}_{n}00}^{2} = \overline{10\underbrace{\ldots}_{2n}00} = 1.5^{2n}$

$\overline{20\underbrace{\ldots}_{n}00}^{2} = \overline{40\underbrace{\ldots}_{2n}00} = 4.5^{2n}$

$\overline{40\underbrace{\ldots}_{n}00}^{2} = \overline{310\underbrace{\ldots}_{2n}00} = 31.5^{2n}$

$\overline{140\underbrace{\ldots}_{n}00}^{2} = \overline{3110\underbrace{\ldots}_{2n}00} = 311.4^{2n}$

$\overline{210\underbrace{\ldots}_{n}00}^{2} = \overline{4410\underbrace{\ldots}_{2n}00} = 441.5^{2n}$

$\overline{41140\underbrace{\ldots}_{n}00}^{2} = \overline{331111110\underbrace{\ldots}_{2n}00} = 33111111.4^{2n}$

10.2.5. Base 6

$\overline{10\underbrace{\ldots}_{n}00}^{2} = \overline{10\underbrace{\ldots}_{2n}00} = 1.6^{2n}$

$\overline{20\underbrace{\ldots}_{n}00}^{2} = \overline{40\underbrace{\ldots}_{2n}00} = 4.6^{2n}$

$\overline{50\underbrace{\ldots}_{n}00}^{2} = \overline{410\underbrace{\ldots}_{2n}00} = 41.6^{2n}$

$\overline{150\underbrace{\ldots}_{n}00}^{2} = \overline{3210\underbrace{\ldots}_{2n}00} = 321.6^{2n}$

$\overline{210\underbrace{\ldots}_{n}00}^{2} = \overline{4410\underbrace{\ldots}_{2n}00} = 441.6^{2n}$

$\overline{22410\underbrace{\ldots}_{n}00}^{2} = \overline{55553210\underbrace{\ldots}_{2n}00} = 5555321.6^{2n}$

10.2.6. Base 7

$\overline{10\underbrace{\ldots}_{n}00}^{2} = \overline{10\underbrace{\ldots}_{2n}00} = 1.7^{2n}$

$\overline{20\underbrace{\ldots}_{n}00}^{2} = \overline{40\underbrace{\ldots}_{2n}00} = 4.7^{2n}$

$\overline{40\underbrace{\ldots}_{n}00}^{2} = \overline{220\underbrace{\ldots}_{2n}00} = 22.7^{2n}$

$\overline{60\underbrace{\ldots}_{n}00}^{2} = \overline{510\underbrace{\ldots}_{2n}00} = 51.7^{2n}$

$\overline{160\underbrace{\ldots}_{n}00}^{2} = \overline{3310\underbrace{\ldots}_{2n}00} = 331.7^{2n}$

$\overline{210\underbrace{\ldots}_{n}00}^{2} = \overline{4410\underbrace{\ldots}_{2n}00} = 441.7^{2n}$

$\overline{240\underbrace{\ldots}_{n}00}^{2} = \overline{6420\underbrace{\ldots}_{2n}00} = 642.7^{2n}$

$\overline{260\underbrace{\ldots}_{n}00}^{2} = \overline{11110\underbrace{\ldots}_{2n}00} = 1111.7^{2n}$

$\overline{550\underbrace{\ldots}_{n}00}^{2} = \overline{44440\underbrace{\ldots}_{2n}00} = 4444.7^{2n}$

$\overline{2040\underbrace{\ldots}_{n}00}^{2} = \overline{422220\underbrace{\ldots}_{2n}00} = 42222.7^{2n}$

$\overline{2260\underbrace{\ldots}_{n}00}^{2} = \overline{554110\underbrace{\ldots}_{2n}00} = 55411.7^{2n}$

$\overline{550026\underbrace{0\ldots0}_{n}}^{2} = \overline{444444441111\underbrace{0\ldots0}_{2n}} = 444444441111.7^{2n}$

10.2.7. Base 8

$\overline{1\underbrace{0\ldots0}_{n}}^{2} = \overline{1\underbrace{0\ldots0}_{2n}} = 1.8^{2n}$

$\overline{2\underbrace{0\ldots0}_{n}}^{2} = \overline{4\underbrace{0\ldots0}_{2n}} = 4.8^{2n}$

$\overline{3\underbrace{0\ldots0}_{n}}^{2} = \overline{11\underbrace{0\ldots0}_{2n}} = 11.8^{2n}$

$\overline{4\underbrace{0\ldots0}_{n}}^{2} = \overline{20\underbrace{0\ldots0}_{2n}} = 20.8^{2n}$

$\overline{5\underbrace{0\ldots0}_{n}}^{2} = \overline{31\underbrace{0\ldots0}_{2n}} = 31.8^{2n}$

$\overline{6\underbrace{0\ldots0}_{n}}^{2} = \overline{44\underbrace{0\ldots0}_{2n}} = 44.8^{2n}$

$\overline{7\underbrace{0\ldots0}_{n}}^{2} = \overline{61\underbrace{0\ldots0}_{2n}} = 61.8^{2n}$

$\overline{14\underbrace{0\ldots0}_{n}}^{2} = \overline{220\underbrace{0\ldots0}_{2n}} = 220.8^{2n}$

$\overline{21\underbrace{0\ldots0}_{n}}^{2} = \overline{441\underbrace{0\ldots0}_{2n}} = 441.8^{2n}$

$\overline{210\underbrace{0\ldots0}_{n}}^{2} = \overline{44100\underbrace{0\ldots0}_{2n}} = 44100.8^{2n}$

$\overline{23\underbrace{0\ldots0}_{n}}^{2} = \overline{551\underbrace{0\ldots0}_{2n}} = 551.8^{2n}$

$\overline{24\underbrace{0\ldots0}_{n}}^{2} = \overline{620\underbrace{0\ldots0}_{2n}} = 620.8^{2n}$

$\overline{26\underbrace{0\ldots0}_{n}}^{2} = \overline{744\underbrace{0\ldots0}_{2n}} = 744.8^{2n}$

$\overline{51\underbrace{0\ldots0}_{n}}^{2} = \overline{3221\underbrace{0\ldots0}_{2n}} = 3221.8^{2n}$

$\overline{64\underbrace{0\ldots0}_{n}}^{2} = \overline{5220\underbrace{0\ldots0}_{2n}} = 5220.8^{2n}$

$\overline{66\underbrace{0\ldots0}_{n}}^{2} = \overline{5544\underbrace{0\ldots0}_{2n}} = 5544.8^{2n}$

$\overline{72\underbrace{0\ldots0}_{n}}^{2} = \overline{6444\underbrace{0\ldots0}_{2n}} = 6444.8^{2n}$

$\overline{73\underbrace{0\ldots0}_{n}}^{2} = \overline{6631\underbrace{0\ldots0}_{2n}} = 6631.8^{2n}$

$\overline{75\underbrace{0\ldots0}_{n}}^{2} = \overline{7211\underbrace{0\ldots0}_{2n}} = 7211.8^{2n}$

$\overline{164\underbrace{0\ldots0}_{n}}^{2} = \overline{32220\underbrace{0\ldots0}_{2n}} = 32220.8^{2n}$

$\overline{227\underbrace{0\ldots0}_{n}}^{2} = \overline{54421\underbrace{0\ldots0}_{2n}} = 54421.8^{2n}$

$\overline{244\underbrace{0\ldots0}_{n}}^{2} = \overline{64420\underbrace{0\ldots0}_{2n}} = 64420.8^{2n}$

$\overline{264\underbrace{0\ldots0}_{n}}^{2} = \overline{77220\underbrace{0\ldots0}_{2n}} = 77220.8^{2n}$

$\overline{265\underbrace{0\ldots0}_{n}}^{2} = \overline{77771\underbrace{0\ldots0}_{2n}} = 77771.8^{2n}$

$\overline{603\underbrace{0\ldots0}_{n}}^{2} = \overline{444411\underbrace{0\ldots0}_{2n}} = 444411.8^{2n}$

$\overline{732\underbrace{0\ldots0}_{n}}^{2} = \overline{666644\underbrace{0\ldots0}_{2n}} = 666644.8^{2n}$

$\overline{1664\underbrace{0\ldots0}_{n}}^{2} = \overline{3333220\underbrace{0\ldots0}_{2n}} = 3333220.8^{2n}$

$$\overline{23060\underbrace{\ldots00}_{n}}^{2} = \overline{55444440\underbrace{\ldots00}_{2n}} = 5544444.8^{2n}$$

$$\overline{26140\underbrace{\ldots00}_{n}}^{2} = \overline{75422200\underbrace{\ldots00}_{2n}} = 7542220.8^{2n}$$

$$\overline{76140\underbrace{\ldots00}_{n}}^{2} = \overline{743322200\underbrace{\ldots00}_{2n}} = 74332220.8^{2n}$$

10.2.8. Base 9

$$\overline{10\underbrace{\ldots00}_{n}}^{2} = \overline{10\underbrace{\ldots00}_{2n}} = 1.9^{2n}$$

$$\overline{30\underbrace{\ldots00}_{n}}^{2} = \overline{100\underbrace{\ldots00}_{2n}} = 10.9^{2n}$$

$$\overline{70\underbrace{\ldots00}_{n}}^{2} = \overline{540\underbrace{\ldots00}_{2n}} = 54.9^{2n}$$

$$\overline{210\underbrace{\ldots00}_{n}}^{2} = \overline{4410\underbrace{\ldots00}_{2n}} = 441.9^{2n}$$

$$\overline{20\underbrace{\ldots00}_{n}}^{2} = \overline{40\underbrace{\ldots00}_{2n}} = 4.9^{2n}$$

$$\overline{60\underbrace{\ldots00}_{n}}^{2} = \overline{400\underbrace{\ldots00}_{2n}} = 40.9^{2n}$$

$$\overline{80\underbrace{\ldots00}_{n}}^{2} = \overline{710\underbrace{\ldots00}_{2n}} = 71.9^{2n}$$

$$\overline{230\underbrace{\ldots00}_{n}}^{2} = \overline{5400\underbrace{\ldots00}_{2n}} = 540.9^{2n}$$

$$\overline{260\underbrace{\ldots00}_{n}}^{2} = \overline{7100\underbrace{\ldots00}_{2n}} = 710.9^{2n}$$

$$\overline{280\underbrace{\ldots00}_{n}}^{2} = \overline{8310\underbrace{\ldots00}_{2n}} = 831.9^{2n}$$

$$\overline{710\underbrace{\ldots00}_{n}}^{2} = \overline{55510\underbrace{\ldots00}_{2n}} = 5551.9^{2n}$$

$$\overline{830\underbrace{\ldots00}_{n}}^{2} = \overline{76400\underbrace{\ldots00}_{2n}} = 7640.9^{2n}$$

$$\overline{2330\underbrace{\ldots00}_{n}}^{2} = \overline{555100\underbrace{\ldots00}_{2n}} = 55510.9^{2n}$$

$$\overline{2530\underbrace{\ldots00}_{n}}^{2} = \overline{664400\underbrace{\ldots00}_{2n}} = 66440.9^{2n}$$

$$\overline{7630\underbrace{\ldots00}_{n}}^{2} = \overline{6531100\underbrace{\ldots00}_{2n}} = 653110.9^{2n}$$

$$\overline{283780\underbrace{\ldots00}_{n}}^{2} = \overline{8555222210\underbrace{\ldots00}_{2n}} = 855522221.9^{2n}$$

$$\overline{872560\underbrace{\ldots00}_{n}}^{2} = \overline{85552222100\underbrace{\ldots00}_{2n}} = 8555222210.9^{2n}$$

$$\overline{270\underbrace{\ldots00}_{n}}^{2} = \overline{7640\underbrace{\ldots00}_{2n}} = 764.9^{2n}$$

$$\overline{630\underbrace{\ldots00}_{n}}^{2} = \overline{44100\underbrace{\ldots00}_{2n}} = 4410.9^{2n}$$

$$\overline{770\underbrace{\ldots00}_{n}}^{2} = \overline{66440\underbrace{\ldots00}_{2n}} = 6644.9^{2n}$$

$$\overline{860\underbrace{\ldots00}_{n}}^{2} = \overline{83100\underbrace{\ldots00}_{2n}} = 8310.9^{2n}$$

$$\overline{2510\underbrace{\ldots00}_{n}}^{2} = \overline{653110\underbrace{\ldots00}_{2n}} = 65311.9^{2n}$$

Problem 11. Are there any monotonic squares in binary number system?
Solution. In the table are presented the squares form 2^0 to 2^5 in binary number system.

$X_{(10)}$	$X_{(2)}$	$X_{(2)}^2$
1	1	1
2	10	100
3	11	1001
4	100	10000
5	101	11001
6	110	100100
7	111	110001
8	1000	1000000
9	1001	1010001
10	1010	1100100
11	1011	1111001
12	1100	10010000
13	1101	10101001
14	1110	11000100
15	1111	11100001
16	10000	100000000

$X_{(10)}$	$X_{(2)}$	$X_{(2)}^2$
17	10001	100100001
18	10010	101000100
19	10011	101101001
20	10100	110010000
21	10101	110111001
22	10110	111100100
23	10111	1000010001
24	11000	1001000000
25	11001	100111001
26	11010	1010100100
27	11011	1011011001
28	11100	1100010000
29	11101	1101001001
30	11110	1110000100
31	11111	1111000001
32	100000	10000000000

We see that the monotonic squares are with base number that is a power of 2 in decimal number system.

Problem 12. Prove that there are no squares of natural numbers, which are with one and the same digits in binary number system.
Solution. We are going to proof that every square have a 0 in the last two digits.

If the base number ends on 0, the last two digits of the square are going to be 00. If the base number ends on 11 or 01, the square is going to end on 01. So in binary number system there is no square with same digits.

Comment. Number of decimals is always 0. The squares in binary number system can not be increasing.

Problem 13. Prove that the monotonic squares in binary number system can be only of the kind 10^k.
Solution. We are looking for monotonic squares of that kind: $A=\overline{\underbrace{11\ldots1}_{m}\underbrace{0\ldots00}_{n}}$, for $m,n\geq 0$.

If A is a square, n must be even. If the base number ends on 1 the last digit of the square will be 1 again. So the monotonic square needs to be of the kind 11…1, but it is impossible. So the base number needs to end on 0, and the monotonic square on 00.

We can present the monotonic square that way: $A=\underbrace{\overline{11\ldots1}}_{m}.10^{n}$. n is even, so 10^{n} is square. If A is a square $\underbrace{\overline{11\ldots1}}_{m}$ needs to be square, too. From the solution of Problem 11 we know that this is possible only when $m=1$. I.e. the monotonic squares in binary number system have that form: 10^{k}.

Conclusion. In binary number system there is only one family and it is infinite decreasing family:

$$10^2 = 100 \;\mathbf{p}\; 100^2 = 10000 \;\mathbf{p}\; 1000^2 = 1000000\ldots$$

Number of that kind $1\underbrace{0\ldots0}_{n}$ in decimal number system is:

$1.2^{n} + 0.2^{n-1} + 0.2^{n-2} + \ldots + 0.2^{0} = 2^{n}$.

I.e. the monotonic squares in binary number system are with base numbers 2^{n} in decimal number system.

Problem 14. Are there any strongly increasing families in ternary number system?

Solution. In the table are shown the squares of the numbers from 3^0 to 3^3 in ternary number system.

$X_{(10)}$	$X_{(3)}$	$X_{(3)}^2$
1	1	1
2	2	11
3	10	100
4	11	121
5	12	221
6	20	1100
7	21	1211
8	22	2101
9	100	10000
10	101	10201
11	102	11111
12	110	12100
13	111	20021
14	112	21021

$X_{(10)}$	$X_{(3)}$	$X_{(3)}^2$
15	120	22100
16	121	100011
17	122	101201
18	200	110000
19	201	111101
20	202	112211
21	210	121100
22	211	122221
23	212	201120
24	220	210100
25	221	212011
26	222	221001
27	1000	1000000

Obviously
$$\left.\begin{array}{l} A_{(3)} = \overline{a_1 \ldots a_{n-1} 0} \equiv 0 \pmod 3 \\ A_{(3)} = \overline{a_1 \ldots a_{n-1} 1} \equiv 1 \pmod 3 \\ A_{(3)} = \overline{a_1 \ldots a_{n-1} 2} \equiv -1 \pmod 3 \end{array}\right\rangle A_{(3)}^{\;2} \equiv 0,1 \pmod 3$$

So strongly increasing squares (families) do not exist. Because numbers $(3^n+2)^2 = 3^{2n} + 3^{n+1} + 3^n + 3^0$ are equal to a numbers of the following type: $11\ldots11_{(3)}$ only when n=0, 2, there exist two increasing families with two elements:

$1^2=1$**p**$102^2=11111$ and $2^2=11$**p**$102^2=11111$.

Problem 15. Find the last digits of the monotonic squares in ternary number system.

Solution. Let look over the possibilities for the last digits of the base number in ternary number system.

In the table are shown all of the possibilities for the last digits in ternary number system. Form them only the following combinations can be last digits of the monotonic square: 2100, 111, 10000, 1100, 221, 000000, 211.

Last digits of the based number	Last digits of the square	Last digits of the based number	Last digits of the square
111	021	200	10000
120	2100	211	221
102	111	000	000000
100	10000	012	221
122	201	021	211
222	001	022	101
201	101	011	121
210	1100		

Comment. Using the results of Problem 10 and Problem 15 we can say that all infinite trivial families are already found in 10.2.2. Except them there are some other families:

$1^2=1$**p**$102^2=11111$ $\quad$ $2^2=11$**p**$102^2=11111$ $\quad$ $1^2=1$**p**$12^2=221$

Problem 16. Find all increasing families in base p, where $3<p<10$.

Solution. We are going to use the results in Problem 10.1.

Let p=4

The last digits of these squares can be only 0 or 1.Using the algorithm from Problem 1 it follows that there are no other increasing squares, except $1^2=1$.

Let p=5

The last digits of these squares can be only 0, 1 or 4. We are looking for increasing squares, so their last digits can be just 1 or 4.

We found the following families:

$33^2=2244$

$3233^2=22234444$

$2^2=4$ $23^2=1134$ $223^2=111334...$

$13^2=224$

$3^2=14$

$12^2=144$

$1^2=1$

$103^2=11114$

$3^2=14$

Let p=6

$4^2=24$

$104^2=11224$

$44^2=3344$

$1^2=1$ $114^2=13444$

$3^2=13$

$12^2=144$

$2^2=4$

$242^2=112244$

$4^2=24$

$14^2=244$

Let p=7

$4^2=22$

$4^2=22$

$12^2=144$

$104^2=11122$

$2^2=4$

$412^2=233344$

$1^2=1$ $12^2=144$

$260055^2=111144444444$

$105^2=11334$

$26^2=1111$

$5^2=34$

$3^2=12$ $55^2=4444$ $105^2=11334$

Let p=8

$52^2=3344$

$2^2=4$ $32^2=1244$

$3^2=11$

$6^2=44$

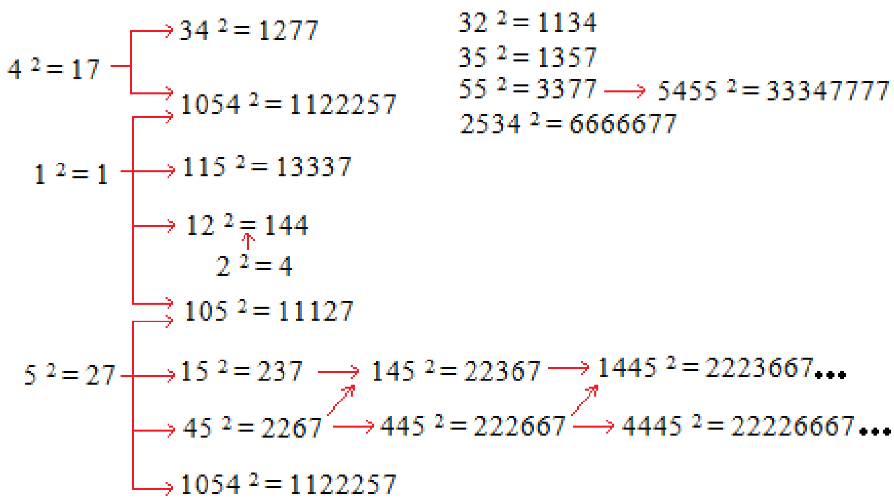
$12^2=144$

$306^2=114444$

$1^2=1$

$106^2=11444$

Let p=9

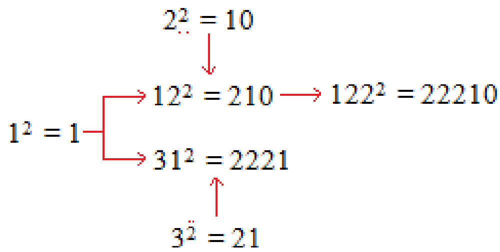


Problem 17. Find all non trivial decreasing families in p-base, where $3<p<10$.

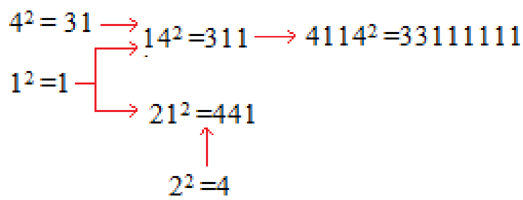
Solution. We are going to use the results from Problem 10.2.

In Problem 10.2 every line gives the elements of trivial decreasing infinite family. Because of the parameters of the program, we are not sure for the existing of other infinite decreasing families. We are going to present just families with elements, which are in the analyzed interval. Everyone in the families in these schemes can be infinite.

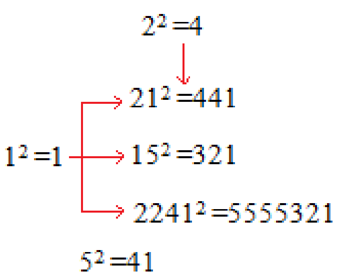
Let p=4



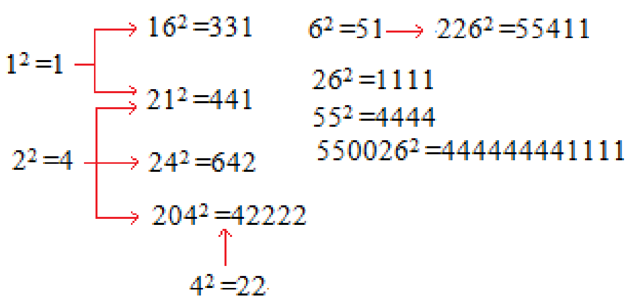
Let p=5



Let p=6



Let p=7



Let p=8

$4^2=20 \to 14^2=220 \to 164^2=32220 \to 1664^2=3333220$

$14^2=220 \to 7614^2=74332220$

$4^2=20 \to 64^2=5220 \to 2614^2=7542220$

$4^2=20 \to 7614^2=74332220$

$4^2=20 \to 24^2=620 \to 244^2=64420$

$4^2=20 \to 264^2=77220$

$3^2=11 \to 603^2=444411$

$6^2=44 \to 603^2=444411$

$6^2=44 \to 66^2=5544$

$6^2=44 \to 2306^2=5544444$

$6^2=44 \to 26^2=744$

$2^2=4 \to 26^2=744$

$14^2=220 \to 2614^2=7542220$

$2^2=4 \to 2614^2=7542220$

$2^2=4 \to 2306^2=5544444$

$2^2=4 \to 732^2=666644$

$2^2=4 \to 244^2=64420$

$2^2=4 \to 72^2=64444$

$2^2=4 \to 227^2=54421$

$2^2=4 \to 21^2=441$

$1^2=1 \to 21^2=441$

$1^2=1 \to 51^2=3221$

$23^2=551$
$73^2=6631$
$75^2=7211$
$265^2=77771$

$7^2=61 \to 73^2=6631$

$5^2=31 \to 51^2=3221$

Let p=9

$1^2=1 \to 251^2=65311$

$1^2=1 \to 71^2=5551$

$1^2=1 \to 21^2=441$

$2^2=4 \to 21^2=441$

$2^2=4 \to 23^2=540$

$2^2=4 \to 27^2=764$

$2^2=4 \to 253^2=66440$

$6^2=40 \to 63^2=4410$

$3^2=10 \to 63^2=4410$

$3^2=10 \to 233^2=55510$

$3^2=10 \to 763^2=653110$

$7^2=54$
$8^2=71$
$26^2=710$
$28^2=831$
$77^2=6644$
$83^2=7640$
$86^2=8310$
$28378^2=855522221$
$87256^2=8555222210$

SUMMARY

THEORETICAL PART

1. Definition of originative square.
2. Definition of base number.
3. Proof that there is no square with one and the same digits that is larger than 10.
4. The last two digits of every monotonic square are found.

PART ONE

1. All monotonic squares in the interval $(10^0;10^7)$ are found.
2. All infinite families of increasing squares are found.
3. A computer program that finds all monotonic squares with base number less than 10^{18} was made.
4. All infinite decreasing families are found.
5. The number of elements in a maximal decreasing family was found.
6. Two "mirror" families in the interval $(10^0; 10^{18})$ are found.

PART TWO

1. The increasing families with one element are found.
2. The increasing families with two elements are found.
3. The increasing families with three elements are found.
4. Grouping the families, looking on the number of the elements, if the one digit numbers ($1^2 = 1,\ 2^2 = 4$ *and* $3^2 = 9$) are not a monotonic squares

PART THREE

1. All monotonic squares in the interval $(10^0; 10^{18})$ in base p system, $p < 10$ are found.
2. The monotonic squares in binary number system in the interval from 2^0 to 2^5 are found.
3. Proof that there are no squares of natural numbers, which are with the same digits in binary number system.
4. Proof that the number of decimals of the squares in binary number system is always 0.
5. Proof that the squares in binary number system can not be increasing.
6. Proof that the monotonic squares in binary number system can be only of the kind 10^k and all decreasing infinite families are found.
7. The strongly increasing families in ternary number system are found.
8. The last digits of the squares in ternary number system are found.
9. Proof that there are no increasing squares in ternary number system
10. All increasing families in base p, where $3<p<10$ are found.
11. All non trivial decreasing families in base p, where $3<p<10$ are found.

APPLICATIONS

1. All decreasing monotonic squares with base number from 10^0 to 10^{18}
2. All increasing monotonic squares with base number from 10^0 to 10^{18}
3. The sources of the computer programs.

APPLICATIONS

Application 1.

All decreasing monotonic squares with based number from 10^0 to 10^{18}. The squares are a result of a computer program (Application 3)

$1^2=1$
$2^2=4$
$3^2=9$
$8^2=64$
$9^2=81$
$10^2=10^2$
$20^2=4.10^2$
$30^2=9.10^2$
$80^2=64.10^2$
$90^2=81.10^2$
$21^2=441$
$31^2=961$
$88^2=7744$
$29^2=841$
$100^2=961.10^4$
$200^2=4.10^4$
$300^2=9.10^4$
$800^2=64.10^4$
$900^2=81.10^4$
$210^2=441.10^2$
$310^2=961.10^2$
$880^2=7744.10^2$
$290^2=841.10^2$
$1000^2=1.10^6$
$2000^2=4.10^6$
$3000^2=9.10^6$
$8000^2=64.10^6$
$9000^2=81.10^6$
$2100^2=441.10^4$
$3100^2=961.10^4$
$8800^2=7744.10^4$
$2900^2=841.10^4$
$3139^2=9853321$
$2979^2=8874441$
$10000^2=1.10^8$
$20000^2=4.10^8$
$30000^2=9.10^8$
$80000^2=64.10^8$
$90000^2=81.10^8$
$21000^2=441.10^6$
$31000^2=961.10^8$
$88000^2=7744.10^8$
$29000^2=841.10^8$
$31390^2=9853321.10^2$
$29790^2=8874441.10^2$
$31621^2=999887641$
$100000^2=1.10^{10}$
$200000^2=4.10^{10}$
$300000^2=9.10^{10}$
$800000^2=64.10^{10}$
$900000^2=81.10^{10}$
$210000^2=441.10^8$
$310000^2=961.10^8$
$880000^2=7744.10^8$
$290000^2=841.10^8$
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$297900^2=8874441.10^4$
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$3000000^2=9.10^{12}$
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$2100000^2=441.10^{10}$
$3100000^2=961.10^{10}$
$8800000^2=7744.10^{10}$
$2900000^2=841.10^{10}$
$3139000^2=9853321.10^6$
$2979000^2=8874441.10^6$
$3162100^2=999887641.10^4 10000$
$10000000^2=1.10^{14}$
$20000000^2=4.10^{14}$
$30000000^2=9.10^{14}$
$80000000^2=64.10^{14}$
$90000000^2=81.10^{14}$
$21000000^2=441.10^{12}$
$31000000^2=961.10^{12}$
$88000000^2=744.10^{12}$
$29000000^2=841.10^{12}$
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$8000000000^2=64.10^{18}$
$9000000000^2=81.10^{18}$
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$9000000000000000^2=81.10^{30}$
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$800000000000000000^2=64.10^{34}$
$900000000000000000^2=81.10^{34}$
$210000000000000000^2=441.10^{32}$
$310000000000000000^2=961.10^{32}$
$880000000000000000^2=7744.10^{32}$
$290000000000000000^2=841.10^{32}$
$313900000000000000^2=9853321.10^{28}$
$297900000000000000^2=8874441.10^{28}$
$316210000000000000^2=999887641.10^{28}$

Application 2.

All increasing monotonic squares with base number from 10^0 to 10^{18}. The squares are a result of a computer program (Application 3)

$1^2=1$
$2^2=4$
$3^2=9$
$4^2=16$
$5^2=25$
$6^2=36$
$7^2=49$
$12^2=144$
$13^2=169$
$83^2=6889$
$34^2=1156$
$15^2=225$
$35^2=1225$
$16^2=256$
$17^2=289$
$37^2=1369$
$67^2=4489$
$38^2=1444$
$183^2=33489$
$383^2=146689$
$334^2=111556$
$335^2=112225$
$106^2=11236$
$116^2=13456$
$107^2=11449$
$117^2=13689$
$337^2=113569$
$167^2=27889$
$367^2=134689$
$667^2=444889$
$587^2=344569$
$1633^2=2666689$
$4833^2=23357889$
$3383^2=11444689$
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$3335^2=11122225$
$3337^2=11135569$
$3367^2=11336689$
$1667^2=2778889$
$3667^2=13446889$
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$333666666666666667^2=111333444444444444666888888888888889$

$336666666666666667^2=113344444444444444668888888888888889$

$166666666666666667^2=27777777777777777888888888888888889$

$366666666666666667^2=134444444444444444688888888888888889$

$666666666666666667^2=44444444444444444488888888888888888889$

Application 3.

Source of the computer program, giving all monotonic squares form 10^0 to 10^{18}.

```
#include <cstdio>
#include <cstring>
using namespace std;

unsigned long long a[200];
unsigned long long b[200], br;
int n;
unsigned long long p = 0;

bool check ()
{
 br = 0;
p = 0;
 int i;
        //printf("num ");
        for (i = n-1; i >= 0; i--)
        {
                p =  p * 10 + a[i];
        //      printf("%lld",a[i]);
        }
      // printf("\n");
 long long ost = 0;
        for (i = 0; i < n; i++)
        {
                ost += p*a[i];
                b[br++] = ost%10;
                ost/=10;
                if (br > 1 && b[br-1] > b[br-2]) return false;

        }
while (ost){ b[br++] = ost%10; ost/=10; if (br > 1 && b[br-1] > b[br-2]) return false; }
        return true;

}
void print ()
{
printf ("%lld ^2 -> ", p);
int i;
   for (i = br-1; i>=0; i--)
   {
   printf ("%lld", b[i]);
   }
  printf ("\n"); fflush (stdout);

}
void rec (int pos, long long ost, int last)
{
        if (pos == n )
                {
                        if (a[pos-1]!= 0 && check ())
                                {
                                        print ();
                                }
                   return;
                }
        int i, j;
        for (i = 0; i <10; i++)
        {
                long long ost2 = ost;
                a[ pos ] = i;
                int k = pos;
                for (j = 0; j <= pos; j++, k--)
                                ost2 += a[j] * a[k];

                int temp = ost2%10;
                if (temp <= last) rec (pos+1, ost2/10, temp);
        }


}

int main ()
{


        for(n=1;n<=18;n++)
        {
         //scanf ("%d", &n);
        printf("%d\n",n);
        printf("\n");
        memset(a,0,sizeof(a));
        memset(b,0,sizeof(b));
        p=0;br=0;
        rec (0, 0, 10);
```

```
printf("\n");
}

//abcd*abcd

return 0;
}
```

Application 4.

Source of the computer program, giving all monotonic squares in the base p, $1<p<10$ in the interval $(10^0;10^{18})$.

```
#include<cstdio>
#include<cstring>
using namespace std;
#define uul unsigned long long
int base;
int s[200];
int s1[200];

void check(uul d)
{
 uul x=d*d;
 memset(s,0,sizeof(s));
 memset(s1,0,sizeof(s1));
 int l=-10;
 int br=0;
// printf("num %d\n",d);
 while(x)
 {
  //printf("%d\n",br);
  if(br!=0)
   if(x%base>l)
    return;
  l=x%base;
  s[br++]=l;
  x/=base;
  //printf("%d\n",x);
 }
// printf("YES %lld\n",d);
 int i;
 int br1=0;
 do
 {
//  printf("%d\n",br1);
  s1[br1++]=d%base;
  d/=base;
 }while(d);
 for(i=br1-1;i>=0;i--)
  printf("%d",s1[i]);
 printf(" ^2 ==> ");
 for(i=br-1;i>=0;i--)
  printf("%d",s[i]);
 printf("\n");
}
int main()
{
 uul i;
 for(base=2;base<=9;base++)
 {
 printf("base %d\n",base);
 for(i=1;i<=100000000;i++)
 {
  //printf("%lld\n",i);
  check(i);
 }
 }
}
```