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PROBLEM TWO

FUNCTIONAL EQUATION

Let k be a constant real number.

1. Find some (all) functions $f : R \to R$ with the property $f(f(x) + kx) = xf(x)$ for all real numbers x.

2. Find all solutions $f : R \to R$ of the functional equation

(1) $$f(f(x) + f(y) + kxy) = xf(y) + yf(x),\ x, y \in R.$$

Consider the case when f is (a) a polynomial, (b) a continuous function, (c) an arbitrary function.

3. Let $n > 2$ be a positive integer. Find all functions $f : R \to R$ such that

$$f(f(x_1) + f(x_2) + \ldots + f(x_n) + kx_1x_2 \ldots x_n) = x_1f(x_2) + x_2f(x_3) + \ldots + x_nf(x_1)$$

for all $x_1, x_2, \ldots, x_n \in R$.

4. Suggest and investigate other generalisations of the functional equation (1).

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Problem 1. Find some functions f : R→ R with the property $f(f(x) + kx) = xf(x)$ for all real numbers x.

Analysis. Let $f(x)$ be continuous and let $k=0$, i.e. the equation gets the type $f(f(x)) = xf(x)$. Obviously, if $f(x)$ is constant, then $f \equiv 0$. Let $f(x) = y$. From the equation $f(f(x)) = xf(x)$ we have that $f(y) = xy$. Hence we obtain $f(f(y)) = f(xy)$. Since $f(f(y)) = yf(y) = f(x)f(y)$, then $f(xy) = f(x)f(y)$. So it is well-known that $f(x) = x^a$. Then from the condition we have that $x^{a^2} = x.x^a$. Hence $a = \dfrac{1 \pm \sqrt{5}}{2}$.

Solution. For every k the equation $f(f(x)+kx) = xf(x)$ has a solution $f = \begin{cases} f(x)=0, x \neq 0 \\ f(0)=a, a \in R \end{cases}$. If $k = 0$ we have also solutions: $\begin{matrix} f_1(x) = x^{\frac{1-\sqrt{5}}{2}} \\ f_2(x) = x^{\frac{1+\sqrt{5}}{2}} \end{matrix}$.

Problem 2. Find all solutions $f : \mathrm{R} \to \mathrm{R}$ of the functional equation

(I) $$f(f(x) + f(y)) = xf(y) + yf(x),\ x, y \in \mathrm{R}$$

Solution. Put $x = y = 0$ in (I) and we get $f(2f(0)) = 0$.

Let $f(0) = 0$. Then, by $x = 0$ we have $f(f(y)) = 0$.

Now in (I) we put $x = f(x)$ and get

$f(x)f(y) = f(x)f(y) + yf(f(x)) =$

$= f(f(f(x) + f(y)) = f(f(y)) = 0$

From here and from $f(x)f(x) = 0$ it follows that $f \equiv 0$.

Now let $f(0) \neq 0$. From $f(2f(0)) = 0$ and the substitution $x = y = 2f(0)$ we get $f(0) = 0$, which contradicts to the assumption.

Therefore $f \equiv 0$.

Problem 3. Find all numbers k, for which the functional equation

(II) $$f(f(x) + f(y) + kxy) = xf(y) + yf(x),\ x, y \in \mathrm{R}$$

has a solution $f : \mathrm{R} \to \mathrm{R}$.

Solution. From problem 2 follows that by $k = 0$ the equation (II) has a solution.

Let $k \neq 0$ and $kf(x) = h(kx)$ for each $x \in R$ and $h : R \to R$. Then substitute in (II) with $x = \dfrac{x}{k}$ $\left(y = \dfrac{y}{k} \right)$ and we get

$$f\left(f\left(\frac{x}{k}\right)+f\left(\frac{y}{k}\right)+\frac{xy}{k^2}\right)=\frac{x}{k}f\left(\frac{y}{k}\right)+\frac{y}{k}f\left(\frac{x}{k}\right).$$

(III) or
$$h(h(x)+h(y)+xy)=\frac{xh(y)+yh(x)}{k}.$$

Here we put $x=0$ and get that $h(h(0)+h(y))=\frac{yh(0)}{k}$.

Again we consider two cases:

1) If $h(0)=0$, then $h(h(y))=0$ for each $y\in$ R.

Put $x=-\frac{h(y)}{y}=p$, $y\neq 0$ in $h(h(x)+h(y)+xy)=\frac{xh(y)+yh(x)}{k}$ and we get $\frac{-h^2(y)}{ky}+\frac{yh\left(\frac{-h(y)}{y}\right)}{k}=h(h(-\frac{h(y)}{y}))=0$,

i.e. $p^2=\left(\frac{h(y)}{y}\right)^2=h\left(-\frac{h(y)}{y}\right)=h(p)$.

Substituting in (III) $x=-y=p$ we obtain

$$h\left(p^2+h(-p)-p^2\right)=\frac{ph(-p)-p^3}{k},$$

and from $h(h(y))=0$ it follows that $kh(h(-p))=k.0=ph(-p)+p^3$ or $h(-p)=p^2$.

Then from (III) by $x=y=p$ and $x=y=p$ we have $-p^3=kh\left(3p^2\right)=p^3$, wherefrom we determine $p=0$. Therefore for each $y\neq 0$, $y\in$ R is fulfilled $x=-\frac{h(y)}{y}=0$, i.e. $h\equiv 0$.

We have $f\equiv 0$.

2) Let $h(0)\neq 0$. From $h(h(0)+h(y))=\frac{yh(0)}{k}$ follows that the function h is both injective and surjective, i.e. it is bijective. From (III) by $x=y=0$ we have that $h(2h(0))=0$. Again we put $x=y=2h(0)$ there and get $h(4h^2(0))=0$. Then we have that $2h(0)=4h^2(0)$ and since $h(0)\neq 0$ it follow that $h(0)=\frac{1}{2}$.

Now in (III) we put $x=0$. Then $h\left(h(y)+\frac{1}{2}\right)=\frac{y}{2k}$, wherefrom when $y=0$ we get $h(1)=0$, i.e. $h\left(\frac{1}{2}\right)=\frac{1}{2k}$, and when $y=\frac{1}{2}$ – $h\left(h\left(\frac{1}{2}\right)+\frac{1}{2}\right)=\frac{1}{4k}$.

As we take these values when $x=1$ and $y=\frac{1}{2}$ from (III) we get $\frac{1}{4k}=h\left(h\left(\frac{1}{2}\right)+\frac{1}{2}\right)=\frac{1}{k}h\left(\frac{1}{2}\right)=\frac{1}{2k^2}$, wherefrom we find $k=2$.

Problem 4. Find all solutions $f:\mathrm{R}\to\mathrm{R}$ of the functional equation

(IV) $$f(f(x)+f(y)+2xy)=xf(y)+yf(x),\ x,y\in \mathrm{R}.$$

Solution. We will use the symbols and the results from problem 3. The equation (III) is now in the form

(IIIa) $$h(h(x)+h(y)+xy)=\frac{xh(x)+yh(x)}{2}.$$

Let $x=0$. Since $h\left(h(y)+\frac{1}{2}\right)=\frac{y}{4}$, we have $h\left(h\left(h(y)+\frac{1}{2}\right)+\frac{1}{2}\right)=\frac{h(y)+\frac{1}{2}}{4}$, wherefrom when $y=2h(y)$ we have $h\left(h\left(\frac{h(y)}{2}+\frac{1}{2}\right)\right)=\frac{h\left(2h(y)+\frac{1}{2}\right)}{4}$.

When $x=1$ and $y=\frac{1}{2}$ we have $h(y)=2h(h(y)+y)$, wherefrom follows that $h\left(h(h(y)+y)+\frac{1}{2}\right)=h\left(\frac{h(y)+1}{2}\right)=\frac{h(y)+y}{4}$. From the last two results follows $h(y)=h(2h(y))+y+\frac{1}{2}$. In this equation we put $y=h(y)+\frac{1}{2}$ and since $h\left(h(y)+\frac{1}{2}\right)=\frac{y}{4}$ we get $h\left(h(y)+\frac{1}{2}\right)=h\left(2h\left(h(y)+\frac{1}{2}\right)\right)+h(y)+\frac{1}{2}-\frac{1}{2}$.

So we reduce the problem to finding functions $h:\mathrm{R}\to\mathrm{R}$, which are solutions of

(IIIb) $$h(y)=h\left(\frac{y}{2}\right)-\frac{y}{4}.$$

We will use that $h\left(h\left(h(y)+\frac{1}{2}\right)+\frac{1}{2}\right)=\frac{h(y)+\frac{1}{2}}{4}$. First, we put $y=2y-2$ in (IIIb):

$$h(2y-2)=h(y-1)+\frac{1-y}{2}\xrightarrow{+\frac{1}{2}}=4h(y)=4h\left(\frac{y}{2}\right)+y.$$

So $h(y)=\frac{h(y-1)}{4}-\frac{3}{2}(y-1)+\frac{1}{4}$ i.e. $h(y+1)=\frac{h(y)}{4}-\frac{3}{2}(y+1)-\frac{1}{4}$

In (IIIb) we put $y=4y+2$:

$$h\left(\frac{2y+2}{2}+\frac{1}{2}\right)=h(y+1)=\frac{h(4y+2)+\frac{1}{2}}{4}=\frac{h(2y+1)-\frac{2y+1}{4}+\frac{1}{2}}{4}=$$

$$\frac{\frac{h(2y)}{4}-\frac{3}{2}(2y+1)-1-\frac{2y+1}{4}+\frac{1}{2}}{4}=\frac{\frac{h(2y)}{4}-\frac{5}{2}y-\frac{1}{4}}{4}=\frac{\frac{h(2y)}{4}-\frac{21}{8}y-\frac{1}{4}}{4}$$

Equalizing the two equations on $h(y+1)$ we get $h(y)=-\frac{y-1}{2}$.

Substitute and finally get $f(x)=-\frac{1}{2}x+\frac{1}{4}$.

Problem 5. Let $n\geq 3$ be a positive integer. Find all functions $f:\mathrm{R}\to\mathrm{R}$ such that

(V) $\quad f\left(f(x_1)+f(x_2)+\ldots+f(x_n)+kx_1x_2\ldots x_n\right)=x_1f(x_2)+x_2f(x_3)+\ldots+x_nf(x_1)$

for all $x_1, x_2, \ldots, x_n \in \mathrm{R}$.

Solution. We put $x_1=x_n$ and $x_n=x_1$. The left side does not change, wherefrom we get

$x_1f(x_2)+x_2f(x_3)+\ldots+x_nf(x_1)=x_nf(x_2)+x_2f(x_3)+\ldots+x_{n-1}f(x_1)+x_1f(x_n)$

$x_1f(x_2)+x_{n-1}f(x_n)+x_nf(x_1)=x_nf(x_2)+x_{n-1}f(x_1)+x_1f(x_n)$

Put $x_1=x$ and let $x_2=c_1\neq 0$, $x_{n-1}=c_2\neq 0, c_2\neq c_3$, $x_n=c_3\neq 0$.

Then $xf(c_1)+c_2f(c_3)+c_3f(x)=c_3f(c_1)+c_2f(x)+xf(c_3)$, wherefrom we determine $f(x)=\frac{f(c_3)-f(c_1)}{c_3-c_2}x+\frac{c_3f(c_1)-c_2f(c_3)}{c_3-c_2}=mx+p$.

Substitute $f(x_i)=mx_i+n$ in (V) and compare the coefficients of the corresponding terms of the polynomials with respect to $x_1,x_2,\ldots,x_n$ of the two sides of the equation:

$f\left(m(x_1+x_2+\ldots+x_n)+np+kx_1x_2\ldots x_n\right)=\left(m^2(x_1+x_2+\ldots+x_n)+npm+p+mkx_1x_2\ldots x_n\right)$

$f\left(m(x_1+x_2+\ldots+x_n)+np+kx_1x_2\ldots x_n\right)=m(x_1x_2+\ldots+x_nx_1)+p(x_1+x_2+\ldots+x_n)$

$\Rightarrow mk=0, m=0, m^2-p=0, nmp+p=0$

Hence $m=n=0$.

Finally $f\equiv 0$.

SUMMARY

The following results are achieved:

1. The solutions of the equation $f(f(x)+kx)=xf(x)$ are investigated with respect to *k*. The solutions of the equation are found.
2. The values of *k*, for which the functional equation $f(f(x)+f(y)+kxy)=xf(y)+yf(x)$ has solution, are found.
3. The functional equation $f(f(x)+f(y))=xf(y)+yf(x)$ is solved.
4. The functional equation $f(f(x)+f(y)+2xy)=xf(y)+yf(x)$ is solved.
5. The functional equation $f(f(x_1)+f(x_2)+...+f(x_n)+kx_1x_2...x_n)=x_1f(x_2)+x_2f(x_3)+...+x_nf(x_1)$, when $n\geq 3$ is solved.