Ist International Tournament
of Young Mathematicians
27th June – 3rd July 2009, Paris, France

PROBLEM ONE

SPECULAR COLOURINGS

1. What is the maximum number of cells of an $m \times n$ grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

2. Some cells of an $m \times n$ grid are coloured blue. We call such a colouring specular if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by $S(m,n)$ the minimal number of blue cells in a specular colouring of an $m \times n$ grid.

Find $S(m,n)$ or estimate it (give lower and upper bounds).

3. Formulate and investigate 3-dimensional analogs of the problem.

TEAM: **MATH HIGH SCHOOL *NANCHO POPOVICH,*** **BULGARIA**

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PART ONE

Some cells of an $m \times n$ grid are coloured blue. The grid is situated in coordinate system (integer lattice) (Fig. 1). Obviously x-coordinate of each knot is identical to the number of line, with respect to which we coloured the grid.

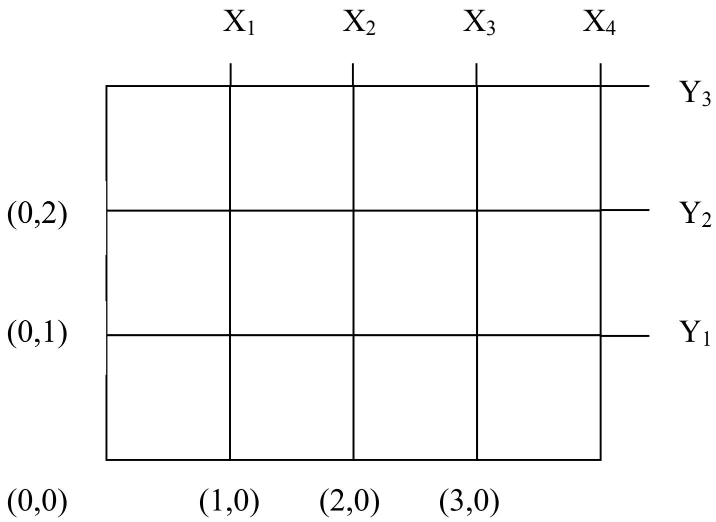


Figure 1

Problem 1. What is the maximum number of cells of an $m \times n$ grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

Solution.

Case 1. At least one of *m* and *n* is even.

We can divide the grid into 1×2 rectangles. The number of these rectangles is $\frac{(m \times n)}{2}$. Consider one of them. If the two cells in the rectangle are coloured, they will be symmetric with respect to the line which divide them (Fig. 2). Consequently in the 1×2 rectangle could be coloured maximum one cell. So the maximum number of coloured cells is equal to the number of rectangles – $\frac{(m \times n)}{2}$.

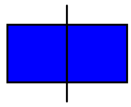


Figure 2

Case 2. *n* and *m* are odd.

We divide the grid into 1×2 rectangles and one cell is empty. By analogy with the previous case, in any rectangle could be coloured at most one cell. Therefore, the maximum number of coloured cell is $\frac{(m \times n + 1)}{2}$.

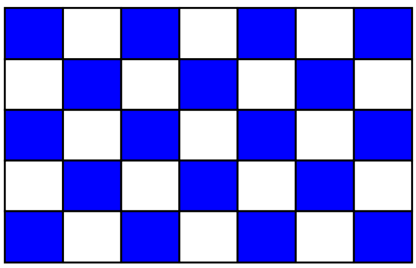


Figure 3

In both cases an appropriate example is chess colouring, because there is no symmetrical cells that are coloured (Figure 3).

PART TWO

Some cells of an $m \times n$ grid are coloured blue. We call such a colouring specular if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by $S(m,n)$ the minimal number of blue cells in a specular colouring of an *m* x *n* grid.

Problem 2. Prove that a specular colouring is minimal when all the coloured cells are located on one row and one column of the grid.

Solution. By intuitive reasoning, it's easy to reach this conclusion.

If the number of coloured cells is minimal, then it is necessary each coloured cell to have symmetrical cells with respect to as much as possible lines

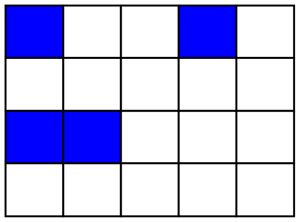


Figure 4a

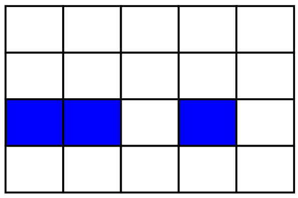


Figure 4b

With respect to the X_1-line the first two cells of a row must be painted. Denote this row with *s*. Consider the X_2-line. Let us assume that two symmetrical cells are situated in the row, which is not s, at least one of the first two cells must be coloured, consequently now we will have at least four coloured cells. (Fig.4a). However, if the two cells which are symmetrical with respect to X_2-line are on the s-row then one of them will be among the first and the second cells on the row, i. e. it has already been coloured. Consequently we will need only three cells because of the first two lines (Fig. 4b). By analogical reasoning inductively we could conclude that the number of coloured cells is minimal when they lie on the same row.

Problem 3. Prove that $S(1,n) \geq 4$, when $n \geq 4$.

Solution. (Figure 5) With respect to the X_1 and X_{n-1} line it's necessary and sufficient to colour the cells $(s,1)$, $(s,2)$ и $(s,n-1)$, (s,n).

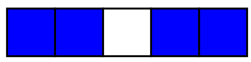


Figure 5

Problem 4. Prove that $S(1,n) \geq 6$, when $n \geq 8$.

Solution. It follows from Problem 3 that there are at least 4 blue cells. They are the first, the second, the *(n-1)*-st и *n*-th cells. We suppose that the remaining n-4 cells are uncoloured. Then there isn't a specular colouring with respect to the X_2 line. Consequently at least one of the third and the fourth cells must be coloured. It's analogical about the *(n-*

2)-th и *(n-3)*-th. Since $n \geq 8$, a specular colouring is possible when there are at least 6 blue cells.

It follows from Problem 4 that when $n \geq 8$ the beginning/the end of the coloured row has one of the following two appearance (Figure 6).

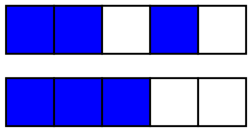


Figure 6

Problem 4a. Prove that $S(1,n) \geq 8$, when $n \geq 12$.

Solution. We examine the first 6 cells. It follows from Problem 4 that at least three of them are coloured. Consider X_3 line. In the both cases when we want to have symmetry with respect to it, it's necessary to have at least one more coloured cell among the remaining 3 cells (Fig. 7). Analogically it's necessary to have at least 4 cells among the 6 cells in the end. Consequently $S(1,n) \geq 8$, when $n \geq 12$.

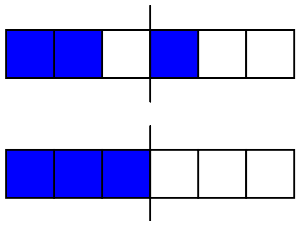


Figure 7

It follows from the results of the programme (shown below) that when $n \geq 20$, $S(1,n) \geq 10$, and when $n \geq 28$, $S(1,n) \geq 12$.

Note that the specular colourings which have symmetry with respect to X_i и Y_j are independent each other. So we could colour by analogy the columns. Since we search the minimal number of blue cells it's expedient to colour a row and a column with a common blue cell (fig.8). (It's obvious that such a colouring exists because every row and every column have at least 4 coloured cells.).

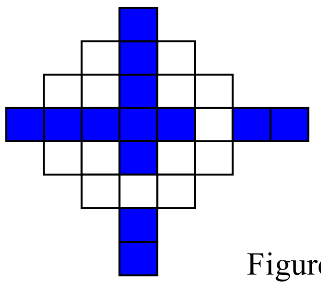


Figure 8

Consequently

$$S(m,n) = S(m,1) + S(1,n) - 1$$

Problem 5. Find $S(1,n)$ or estimate it (give lower and upper bounds).

Solution.

Case 1. When $n = 2,3,4$ must be coloured 2, 3, 4 cells.

Case 2. When $4 < n \leq 11$ It follows from Problem 4 that the minimal value of $n = 6,7,8,9,10$ is

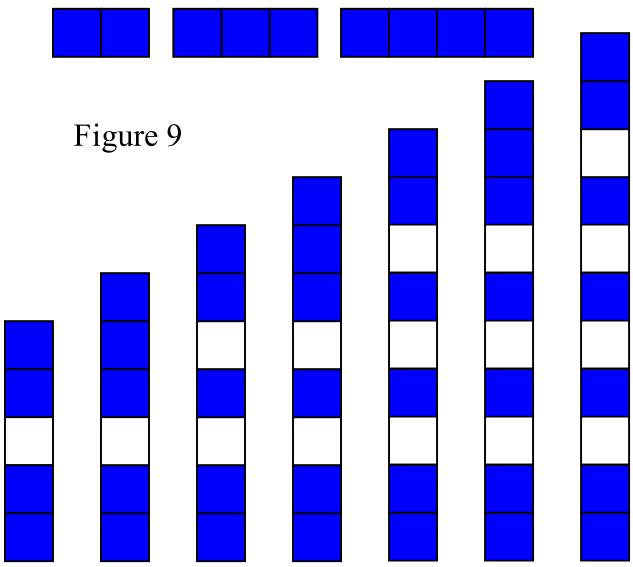


Figure 9

respectively 5, 5, 6, 6, 7. I.e. $S(1,5)=4$, $S(1,6)=5$, $S(1,7)=5$, $S(1,8)=6$, $S(1,9)=6$, $S(1,10)=7$, $S(1,11)=7$ (Fig. 9). We get the colourings when we colour the end 4 cells (the first two and the last two) and all other are coloured through one. Consequently $S(1,n)=4+\left[\frac{n-4}{2}\right]$, $(4<n\leq 11)$.

Consequently $S(m,n)=S(m,1)+S(1.n)-1=4+\left[\frac{n-4}{2}\right]+4+\left[\frac{m-4}{2}\right]-1$ when $(4<n\leq 11$, $4<m\leq 11)$. Some examples are shown on Figure 10.

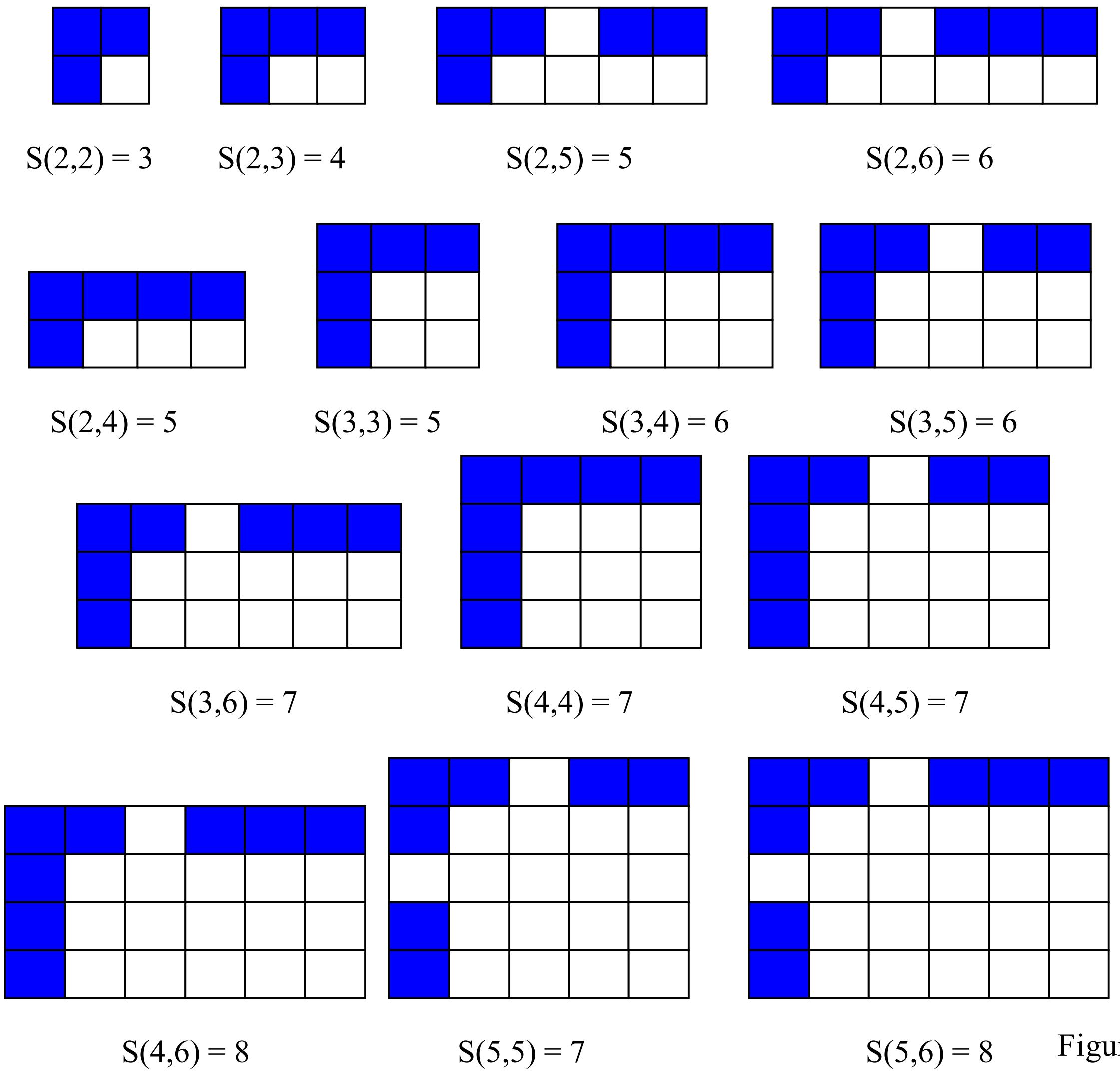
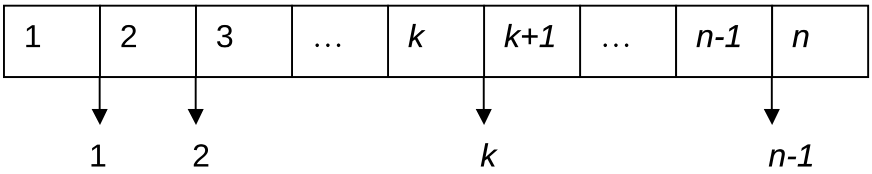


Figure 10

Consider the *k* line and the two cells which are symmetric with respect to it. The numbers of these cells are $k-p$ и $k+1+p$, where *p* is natural number. The sum of these two cells is

$(k-p)+(k+1+p)=2k+1$. Consequently when *k* runs over all values between 1 and *n*, the sum of the two symmetrical cells runs over all odd number between 3 and 2*n*-1.



Example. Consider rectangle 1x6. Then the sums over lines are respectively:

3	5	7	9	11
1 + 2	1 + 4	1 + 6	3 + 6	5 + 6
	2 + 3	2 + 5	4 + 5	
		3 + 4		

From the scheme directly follows the strategy for choice of cells for coloring.

On the grounds of this statement a computer programme was produced which is able to find $S(1,n)$ when $1<n<31$ and describe all possible colourings for a fixed *n*. Here are shown the results from the programme:

*Note: The colourings are presented with 0 and 1 – with 1 are marked the coloured cells whereas with 0 – uncoloured. For each *n* as different „solution" are shown all possible colourings for *n*.

S(1,2) = 2
solution 1 11

S(1,6) = 5
solution 1 111011
solution 2 110111

S(1,3) = 3
solution 1 111

S(1,7) = 5
solution 1 1101011

S(1,4) = 4
solution 1 1111

S(1,8) = 6
solution 1 11101011
solution 2 11011011
solution 3 11100111
solution 4 11010111

S(1,5) = 4
solution 1 11011

S(1,9) = 6
solution 1 110101011

S(1,10) = 7
solution 1 1110101011
solution 2 1101101011
solution 3 1110011011
solution 4 1101011011
solution 5 1110100111
solution 6 1101100111
solution 7 1110010111
solution 8 1101010111

S(1,11) = 7
solution 1 11010101011
solution 2 11100100111

S(1,12) = 8
solution 1 111010101011
solution 2 110110101011
solution 3 111001101011
solution 4 110101101011
solution 5 111100011011
solution 6 111010011011
solution 7 110110011011
solution 8 111001011011
solution 9 110101011011
solution 10 111100100111
solution 11 111010100111
solution 12 110110100111
solution 13 111001100111

S(1,13) = 8
solution 1 1101010101011
solution 2 1101100011011
solution 3 1111000100111
solution 4 1110100100111
solution 5 1110010100111
solution 6 1110010010111
solution 7 1110010001111

S(1,14) = 8
solution 1 11011000011011
solution 2 11100100011011
solution 3 11011000100111
solution 4 11100100100111

S(1,15) = 8

solution 14 110101100111
solution 15 111010010111
solution 16 110110010111
solution 17 111001010111
solution 18 110101010111
solution 19 110110001111
solution 20 111001001111

solution 1 111001000100111

S(1,16) = 9
solution 1 1101100001101011
solution 2 1110010001101011
solution 3 1101101000011011
solution 4 1101011000011011
solution 5 1111000100011011
solution 6 1110100100011011
solution 7 1110010100011011
solution 8 1101100001011011
solution 9 1110010001011011
solution 10 1111001000100111
solution 11 1101101000100111
solution 12 1101011000100111
solution 13 1110100100100111
solution 14 1110010100100111
solution 15 1101100010100111
solution 16 1110010010100111
solution 17 1101100010010111
solution 18 1110010010010111
solution 19 1101100010001111
solution 20 1110010001001111

S(1,17) = 9
solution 1 11011000100011011
solution 2 11110001000100111
solution 3 11101001000100111
solution 4 11100101000100111
solution 5 11100100100100111
solution 6 11100100010100111
solution 7 11100100010010111
solution 8 11100100010001111

S(1,18) = 9
solution 1 111001000100011011
solution 2 110110001000100111

S(1,19) = 9
solution 1 1110010001000100111

S(1,20) = 10
solution 1 11010110000001101011
solution 2 11101001000001101011
solution 3 11100100010001101011
solution 4 11011000101000011011
solution 5 11100100101000011011
solution 6 11011000011000011011
solution 7 11110001000100011011
solution 8 11101001000100011011
solution 9 11100101000100011011
solution 10 11011000010100011011
solution 11 11100100010100011011
solution 12 11100100010001011011
solution 13 11110010001000100111
solution 14 11011010001000100111
solution 15 11010110001000100111
solution 16 11011000101000100111
solution 17 11100100101000100111
solution 18 11100100100100100111
solution 19 11011000010100100111
solution 20 11100100010100100111
solution 21 11011000100010100111
solution 22 11010110000010010111
solution 23 11101001000010010111
solution 24 11011000100010010111
solution 25 11011000100010001111
solution 26 11100100010001001111

S(1,21) = 10
solution 1 110110000101000011011
solution 2 110110001000100011011
solution 3 111100010001000100111
solution 4 111010010001000100111
solution 5 111001010001000100111
solution 6 111001001001000100111
solution 7 111001000101000100111
solution 8 111001000100100100111
solution 9 111001000100010100111
solution 10 111010010000010010111
solution 11 111001000100010010111
solution 12 111001000100010001111

S(1,22) = 10
solution 1 1110010001000100011011
solution 2 1101100010001000100111

S(1,23) = 10
solution 1 11100100010001000100111

S(1,24) = 11
solution 1 111011000001000001101011
solution 2 111010010001000001101011
solution 3 110110000101000001101011
solution 4 111010010000010001101011

S(1,25) = 11
solution 1 1101011000001000001101011
solution 2 1101100010000101000011011
solution 3 1101100001010000100011011
solution 4 1101100010001000100011011

solution 5 11100100010001000110101l
solution 6 1101011000001010000011011
solution 7 1110010001001010000011011
solution 8 1110010010100001000011011
solution 9 1111000100010001000011011
solution 10 1110100100010001000011011
solution 11 1110010100010001000011011
solution 12 1110010001010001000011011
solution 13 1110010001000101000011011
solution 14 1110010001000100010111011
solution 15 1110100100000100001111011
solution 16 1111001000100010001001111
solution 17 1101101000100010001001111
solution 18 1101011000100010001001111
solution 19 1101100010100010001001111
solution 20 1101100001010010001001111
solution 21 1101100010001010001001111
solution 22 1110100100000101001001111
solution 23 1101100010000101001001111
solution 24 1101100010001000101001111
solution 25 1101110000100000100101111
solution 26 1101011000100000100101111
solution 27 1110010010100000100101111
solution 28 1101011000001000100101111
solution 29 1101100010001000100101111
solution 30 1101011000001000001101111
solution 31 1101100010001000100011111
solution 32 1110010001000100010011111
solution 5 1111000100010001000100111
solution 6 1110100100010001000100111
solution 7 1110010100010001000100111
solution 8 1110010010010001000100111
solution 9 1110010001010001000100111
solution 10 1110010001001001000100111
solution 11 1110010001000101000100111
solution 12 1110010001000100100100111
solution 13 1110010001000100010100111
solution 14 1110110000010000010010111
solution 15 1110100100010000010010111
solution 16 1110100100000100010010111
solution 17 1110010001000100010010111
solution 18 1110100100000100000110111
solution 19 1110010001000100010001111

S(1,26) = 11

solution 1 11101001000001000001101011
solution 2 11100100010001000100011011
solution 3 11011000100010001000100111
solution 4 11010110000010000010010111

S(1,27) = 11

solution 1 111001000100010001000100111
solution 2 111010010000010000010010111

S(1,28) = 12

solution 1 1110100100000100000110101011
solution 2 1101011000000110000001101011
solution 3 1111100001000001000001101011
solution 4 1110110001000001000001101011
solution 5 1110101001000001000001101011
solution 6 1110100101000001000001101011
solution 7 1101100011000001000001101011
solution 8 1110110000010001000001101011
solution 9 1110100100010001000001101011
solution 10 1110010010010001000001101011
solution 11 1110100100000101000001101011
solution 12 1110110000010000010001101011
solution 13 1110100100010000010001101011
solution 14 1110100100000100010001101011
solution 15 1110010001000100010001101011
solution 16 1110100100000100000101101011
solution 17 1110010010100000101000011011
solution 18 1101100001010000101000011011
solution 19 1110010001000100101000011011
solution 20 1110100100000100011000011011
solution 21 1110010001001010000100011011
solution 22 1101100010000110000100011011
solution 23 1110010010100001000100011011
solution 24 1111000100010001000100011011
solution 25 1110100100010001000100011011
solution 26 1110010100010001000100011011
solution 27 1110010001010001000100011011
solution 28 1110010001000101000100011011
solution 29 1110010001000100010100011011
solution 30 1101011000001000001100011011
solution 31 1110010001000100010001011011
solution 38 1101011000100010001000100111
solution 39 1101100010100010001000100111
solution 40 1101100001010010001000100111
solution 41 1101100010001010001000100111
solution 42 1110010001001001001000100111
solution 43 1101100010000101001000100111
solution 44 1101100010001000101000100111
solution 45 1101011000001000100100100111
solution 46 1110010010100000010100100111
solution 47 1101100001010000010100100111
solution 48 1101100010001000010100100111
solution 49 1110100100000100001100100111
solution 50 1101100010001000100010100111
solution 51 1111010010000010000010010111
solution 52 1101110010000010000010010111
solution 53 1101011010000010000010010111
solution 54 1101010110000010000010010111
solution 55 1110010011000010000010010111
solution 56 1101110000100010000010010111
solution 57 1101011000100010000010010111
solution 58 1101100001100010000010010111
solution 59 1101011000001010000010010111
solution 60 1110100100001001000010010111
solution 61 1101110000100000100010010111
solution 62 1101011000100000100010010111
solution 63 1101011000001000100010010111
solution 64 1101100010001000100010010111
solution 65 1101011000001000001010010111
solution 66 1101011000001000001001010111
solution 67 1101110000100000100000110111
solution 68 1101011000100000100000110111

solution 32	**1110110000010000010000111011**	**solution 69**	**1101011000001000100000110111**
solution 33	**1110100100010000010000111011**	**solution 70**	**1101011000001000001000110111**
solution 34	**1110100100000100010000111011**	**solution 71**	**1101100010001000100010001111**
solution 35	**1110100100000100000100111011**	**solution 72**	**1110010001000100010001001111**
solution 36	**1111001000100010001000100111**	**solution 73**	**1110100100000100000100101111**
solution 37	**1101101000100010001000100111**	**solution 74**	**1101011000001000001000011111**

S(1,29) = 12

solution 1	**11011100001000001000001101011**	**solution 23**	**11100100010001000100010100111**
solution 2	**11010110001000001000001101011**	**solution 24**	**11111000010000010000010010111**
solution 3	**11010110000010001000001101011**	**solution 25**	**11101100010000010000010010111**
solution 4	**11010110000010000010001101011**	**solution 26**	**11101010010000010000010010111**
solution 5	**11011000010100000101000011011**	**solution 27**	**11101001010000010000010010111**
solution 6	**11011000100010000101000011011**	**solution 28**	**11011000110000010000010010111**
solution 7	**11011000100001010000100011011**	**solution 29**	**11101100000100010000010010111**
solution 8	**11011000010100001000100011011**	**solution 30**	**11101001000100010000010010111**
solution 9	**11011000100010001000100011011**	**solution 31**	**11100100100100010000010010111**
solution 10	**11101001000001000001100011011**	**solution 32**	**11101001000001010000010010111**
solution 11	**11010110000010000010000111011**	**solution 33**	**11101100000100000100010010111**
solution 12	**11110001000100010001000100111**	**solution 34**	**11101001000100000100010010111**
solution 13	**11101001000100010001000100111**	**solution 35**	**11101001000001000100010010111**
solution 14	**11100101000100010001000100111**	**solution 36**	**11100100010001000100010010111**
solution 15	**11100100100100010001000100111**	**solution 37**	**11101001000001000001010010111**
solution 16	**11100100010100010001000100111**	**solution 38**	**11101001000001000001001010111**
solution 17	**11100100010010010001000100111**	**solution 39**	**11101100000100000100000110111**
solution 18	**11100100010001010001000100111**	**solution 40**	**11101001000100000100000110111**
solution 19	**11100100010001001001000100111**	**solution 41**	**11101001000001000100000110111**
solution 20	**11100100010001000101000100111**	**solution 42**	**11101001000001000001000110111**
solution 21	**11101001000001000100100100111**	**solution 43**	**11100100010001000100010001111**
solution 22	**11100100010001000100100100111**	**solution 44**	**11101001000001000001000011111**

S(1,30) = 12

solution 1	**111011000001000001000001101011**	**solution 7**	**110110001000100010001000100111**
solution 2	**111010010001000001000001101011**	**solution 8**	**110111000010000010000010010111**
solution 3	**111010010000010001000001101011**	**solution 9**	**110101100010000010000010010111**
solution 4	**111010010000010000010001101011**	**solution 10**	**110101100000100010000010010111**
solution 5	**111001000100010001000100011011**	**solution 11**	**110101100000100000100010010111**
solution 6	**111010010000010000010000111011**	**solution 12**	**110101100000100000100000110111**

Case 3. $n > 11$

On the basis of the results of the programme we make the following conjecture:

Conjecture. When $n \geq 12$, $S(1,n) \leq 5 + \left[\frac{n}{4}\right]$.

Application

```
#include<cstdio>
#include<algorithm>
using namespace std;
int n;
void solve()
{
 int k=1<<(n+1)-1;
 int i,j,l;
 int res=200000000;
 int c;
 for(i=0;i<=k;i++)
  {
   int bad=0;
   for(j=1;j<=n-1;j++)
     {
      int f=j-1;
      int l=j;
      int is=0;
      while(f>=0&&l<n)
       if((i&1<<f)&&(i&1<<l))
        { is=1;break;}
       else
        { f--;l++;}
      if(is==0)
       { bad=1;break;}
     }
    int num=0;
    if(bad==0)
     {
       for(j=0;j<n;j++)
        if(i&1<<j)
         num++;
       //res=min(res,num);
      if(res>num)
        {
         res=num;
         c=i;
        }
    }
  }
  printf("%d -> %d %d\n",n,res,c);
  int br=0;
  /*for(i=0;i<n;i++)
   if((c&1<<i))
    printf("1");
   else printf("0");
  printf("\n");
 */
 for(i=0;i<=k;i++)
  {
   int bad=0;
   for(j=1;j<=n-1;j++)
     {
      int f=j-1;
      int l=j;
      int is=0;
      while(f>=0&&l<n)
       if((i&1<<f)&&(i&1<<l))
        { is=1;break;}
       else
        { f--;l++;}
      if(is==0)
       { bad=1;break;}
     }
    int num=0;
    if(bad==0)
     {
       for(j=0;j<n;j++)
        if(i&1<<j)
         num++;
       //res=min(res,num);
      if(res==num)
        {
         int l;
         br++;
         printf("solution %d ",br);
         for(l=0;l<n;l++)
          if(i&1<<l)
           printf("1");
          else
           printf("0");
         printf("\n");
        }
     }
  }
}
int main()
{
 //scanf("%d",&n);
 for(n=1;n<=30;n++)
  if(n!=1)
   solve();
  else
   {
     printf("1 -> 1 1\n");
     printf("solution 1 1\n");

   }
 return 0;
}
```

PART THREE

Denote by $B(m,n)$ the number of different minimal specular colourings of an $m \times n$ grid. Since the cells are numbered (they are located in the coordinate system), we will call two colourings different if exist a cell (x, y), which is coloured in the first case and none-coloured in the second case.

Problem 6. Find $B(1,n)$ or estimate it.

Solution. The value of $B(1,n)$ when $n \leq 30$ can be viewed directly from the computer programme.

B(1,2) = 1	B(1,12) = 20	B(1,22) = 2
B(1,3) = 1	B(1,13) = 7	B(1,23) = 1
B(1,4) = 1	B(1,14) = 4	B(1,24) = 32
B(1,5) = 1	B(1,15) = 1	B(1,25) = 19
B(1,6) = 2	B(1,16) = 20	B(1,26) = 4
B(1,7) = 1	B(1,17) = 8	B(1,27) = 2
B(1,8) = 4	B(1,18) = 2	B(1,28) = 74
B(1,9) = 1	B(1,19) = 1	B(1,29) = 44
B(1,10) = 8	B(1,20) = 26	B(1,30) = 12
B(1,11) = 2	B(1,21) = 12	

Problem 6a. When *1xn* grid has a specular colouring, what is the maximum number of the pairs of white cells that are symmetric with respect to one of the lines.

Solution. Denote by $P(n)$ the number of the pairs of white cells that are symmetric.

$P(2) = 0$, $P(3) = 0$, $P(4) = 0$, $P(5) = 0$, $P(6) = 0$, $P(7) = 0$, $P(8) = 1$, $P(9) = 0$, $P(10) = 2$, $P(11) = 4$, $P(12) = 4$ and so on. The values could be obtained directly from the computer programme which is shown above.

Problem 7. Find $B(m,n)$ or estimate it.

Solution. Since the minimum specular colouring of the whole grid contains coloured cells only on one row and one column and they intersect at a single cell, then different colourings could be obtained on the basis of this cell. The number of coloured cells on the row is $S(1,n)$, and on the column - $S(m,1)$. Therefore, the common cell could be selected in $S(m,1) \times S(1,n)$ ways. I.e. depending on the choice of row and column, the grid could be coloured specular in $S(m,1) \times S(1,n)$ ways. But the row and the column may also be coloured in several different ways - $B(m,1)$ the row and $B(1,n)$ - the column. Therefore the total number of colourings is $B(m,n) = S(m,1) \times S(1,n) \times B(m,1) \times B(1,n)$.

PART FOUR

Some cells of an $m \times n$ grid are coloured blue and some are in red. The grid is situated in integer lattice (fig. 1).

Problem 8. What is the maximum number of cells of an $m \times n$ grid that can be coloured blue and red, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid and no two red cells are symmetric with respect to any horizontal or vertical line of the grid?

Solution. All cells could be coloured. An example is shown on figure 11.

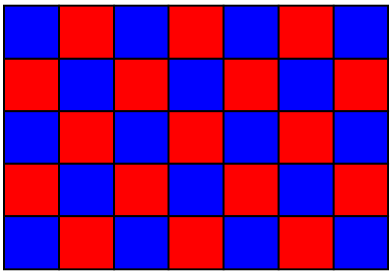


Figure 11

Problem 9. What is the minimal number of cells in a blue and a red specular colouring of an $m \times n$ grid.

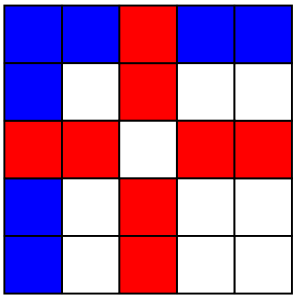
Solution. The colourings are both independent of each other. Therefore, if the minimum number of cells which are necessary for a specular colouring with only one colour are $S(m,n)$, the minimal number of cells for two colours is $S_2(m,n) \geq S(m,n) + S(m,n) = 2S(m,n)$ (The index 2 shows the number of used colours). However, the colouring is not possible for all values of *m* и *n*.

When *m* < 5 or *n* < 5 there is no colouring in two colours, because the entire row and a column of the table are filled with only one colour.

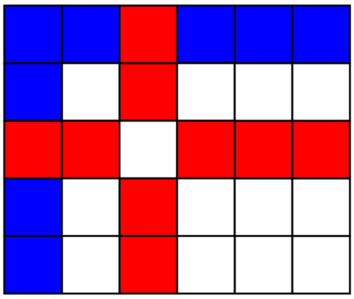
When *m, n*>4 the wanted colouring always exists. To obtain such a colouring, first we should construct a specular colouring with only one colour – all the coloured blue cells are located on one row and one column of the grid. As we want a minimal colouring, coloured cells in the second colour – red – should be also located on one row and one column. Depending on whether the red row and column is intersected in a coloured cell or not, the minimal number of cells is

$$S_2(m,n) = 2S(m,n) \text{ or } S_2(m,n) = 2S(m,n) + 1.$$

The second one would be realized if there is only one specular colouring for a blue row or column i.e. when $B(1,n) = 1$ or $B(m,1) = 1$. Some examples are shown on figure 12.

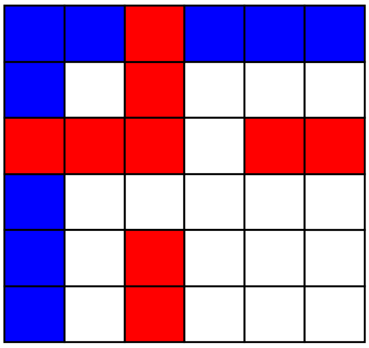


$S_2(5,5) = 15$

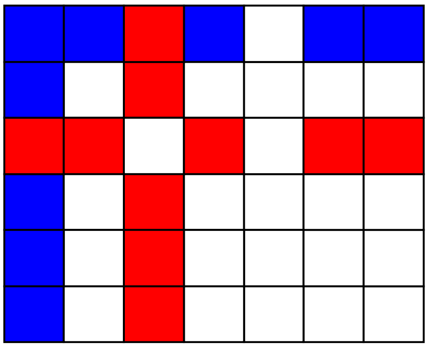


$S_2(5,6) = 17$

Figure 12



$S_2(6,6) = 18$



$S_2(6,7) = 19$

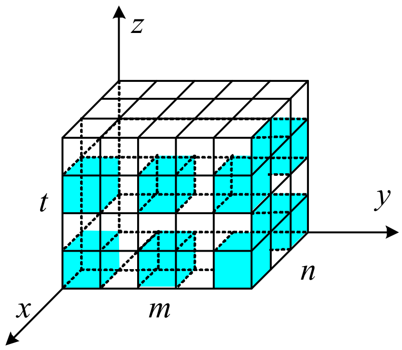
PART FIVE

Let $m \times n \times t$ cuboid (rectangular parallelepiped) is divided into $m \times n \times t$ little cubes. Some of them are coloured in blue. A three-dimensional coordinate system {*Oxyz*} is associated with the cuboid.

Problem 10. What is the maximal number of small cubes of an $m \times n \times t$ cuboid that can be coloured blue, such that no two blue cubes are symmetric with respect to any horizontal or vertical plane of the cuboid.

Solution.

Case 1. Let at least one of the numbers *m, n* и *t* is even. Consequently we could divide the cuboid into $2 \times 1 \times 1$ cuboids. In each of them could be coloured maximum one small cube (otherwise there will be a symmetry). Then the number of coloured cubes is the same as the number of cuboids – they are $\frac{m \times n \times t}{2}$. An example for such colouring is "chess colouring", i.e. the cubes are coloured through one.



Case 2. Let the numbers *m, n* and *t* be odd.

If we divide the cube into $2 \times 1 \times 1$ cuboids there will be one small cube remaining. Therefore the number of coloured cubes is not greater than the number of cuboids plus 1 i.e. $\frac{m \times n \times t + 1}{2}$. An example for such colouring again is "chess colouring", and the bigger part of the cubes are coloured.

PART SIX

Some small cubes of an $m \times n \times t$ cuboid are coloured blue. We call such a colouring specular if for any interior horizontal or vertical plane of the cuboid there are two blue small cubes that are symmetric with respect to this plane. Denote by $S(m,n,t)$ the minimal number of blue cubes in a specular colouring of an $m \times n \times t$ cuboid.

Problem 11. Prove that a specular colouring is minimal when all the coloured cubes are located on three sequences – along the axis *Ox*, along the axis *Oy*, and along the axis *Oz*.
Solution. By analogy with the plane problem we could prove that the colouring will be minimal when the coloured cubes are located in three different directions.

Problem 12. Find $S(m,n,t)$ or estimate it (give lower and upper bounds).

Solution. It follows from problem 11 that it is enough to find a minimal colouring for one sequence – along the axis *Ox*, one – along the axis *Oy*, and one – along the axis *Oz*, i.e. to find the minimal colouring for sequence of cubes $1 \times 1 \times k$, where *k* = *m*, *n* or *t*.

Consider $1 \times 1 \times k$ cuboid. It's projection on the plane is $1 \times k$ rectangle. Consequently we can look at the cubes as cells in the plane case. So $S(m,n,t) = S(1,m) + S(1,n) + S(1,t) - 1$.

Problem 13. Denote by $B(m,n,t)$ the number of different minimal specular colourings of an $m \times n \times t$ cuboid. Find or estimate $B(m,n,t)$.

Solution.

a) First we will find $B(1,1,t)$. If we look at the projection of a cuboid line on the plane, consequently $B(1,1,t) = B(1,t)$.

b) By analogy with Problem 7 we obtain that

$$B(m,n,t) = S(1,1,t) \times S(1,n,1) \times S(m,1,1) \times B(1,1,t) \times B(1,n,1) \times B(m,1,1) =$$
$$= S(1,t) \times S(1,n) \times S(m,1) \times B(1,t) \times B(1,n) \times B(m,1).$$

SUMMARY

The following results are achieved:

PART ONE

1. The maximum number of cells of an $m \times n$ grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid is found.

PART TWO

The following statements are proved:

2. A specular colouring is minimal when all the coloured cells are located on one row and one column of the grid.
3. $S(1,n) \geq 4$, when $n \geq 4$.
4. $S(1,n) \geq 6$, when $n \geq 8$.
5. $S(1,n) \geq 8$, when $n \geq 12$.
6. $S(1,n) \geq 10$, when $n \geq 20$.
7. $S(1,n) \geq 12$, when $n \geq 28$.
8. $S(m,n) = S(m,1) + S(1,n) - 1$.
9. A computer programme was produced which is able to find $S(1,n)$ when $1 < n < 31$. At the end the source code of the program is applied.
10. The value of $S(1,n)$, when $n \leq 30$ is found. Examples are given.
11. A conjecture is deduced that $S(1,n) \leq 5 + \left[\frac{n}{4}\right]$, when $n \geq 12$.

PART THREE

Several new directions of the problem are proposed.

12. The values of $B(1,n)$, when $n \leq 30$ are found, where $B(m,n)$ is a number of different minimal specular colourings of $m \times n$ grid.
13. The maximum number of the pairs of white cells that are symmetric with respect to one of the lines is found.
14. It is proved that $B(m,n) = S(m,1) \times S(1,n) \times B(m,1) \times B(1,n)$.

PART FOUR

15. The problems proposed in part one and two are considered for two colours. Formulae are derived. Examples are given.

PART FIVE

16. The problem proposed in Part one is considered for cuboid.

PART SIX

17. The problems proposed in Part two are considered for cuboid.

18. The following formulae are derived:

a. $S(m,n,t) = S(1,m) + S(1,n) + S(1,t) - 1$

b. $B(1,1,t) = B(1,t)$

c. $B(m,n,t) = S(1,1,t) \times S(1,n,1) \times S(m,1,1) \times B(1,1,t) \times B(1,n,1) \times B(m,1,1) =$
$= S(1,t) \times S(1,n) \times S(m,1) \times B(1,t) \times B(1,n) \times B(m,1)$