2009

Problem 9

Good Numbers

Any rational number x may be expressed as a continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{a_{m-1} + \cfrac{1}{a_m}}}}$$

where a_0 is the integer part of x, and the numbers $a_1, a_2, \dots, a_m$ are positive integers called partial quotients of x. We will also write
$x = [a_0; a_1, a_2, \dots, a_m]$.

1. Find all numbers $n > 2$ that can be expressed as the sum of two positive integers $n = a+b$ so that $a < b$ and the continued fraction for a/b has all its partial quotients equal to 1.

   For example, $13 = 5 + 8$ and $5/8 = [0; 1, 1, 1, 1, 1]$.

2. A number $n > 2$ is called 2-good if for some positive integers $a < b$ we have $n = a+b$ and the partial quotients of a/b are equal to 1 or 2.
For example, $11 = 4+7$ and $4/7 = [0; 1, 1, 2, 1]$.

   (a) Are there infinitely many odd numbers that are not 2-good?

   (b) Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.

   (c) Describe the set of all 2-good numbers.

3. In general, a number n is called k-good if it can be expressed as the sum of two positive integers $n = a+b$ so that $a < b$ and the continued fraction for a/b has all its partial quotients not greater than k.
Does there exist a positive integer K such that all positive integers $n > 2$ are K-good?

4. Describe the set of numbers n with the property: there exist two positive integers $b > a$ such that $n = b + a$ or $n = b - a$ and the partial quotients of a/b are equal to 1 or 2.

5. Suggest and study additional directions of research.

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6/25/2009

We denote $\frac{p_i}{q_i}$ = [0; a_i, $a_2, \ldots,$ a_i], where p_i and q_i are two relatively prime natural numbers. We will prove the following statement:

Statement 1.1. The following recurrence relations hold for $k > 1$:

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}.$$

Proof: We proceed by induction on the index k. We have $p_0 = 0$, $q_0 = 1$, $p_1 = 1$, and $q_1 = a_1$. For $k = 2$ the following equalities hold:

$$\frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}}$$

$$\frac{p_2}{q_2} = \frac{a_2}{a_1 a_2 + 1}.$$

In this case we have $p_2 = a_2 = p_1 a_2 + p_0$ и $q_2 = a_1 a_2 + 1 = q_1 a_2 + q_0$ which satisfies the given relations. Suppose the relations hold for all natural numbers k < m. For k = m + 1 we have:

$$\frac{p_{m+1}}{q_{m+1}} = [0; a_1, a_2, \ldots, a_m, a_{m+1}] = [0; a_1, a_2, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}].$$

Using the supposition we get:

$$p_{m+1} = p_{m-1}(a_m + \frac{1}{a_{m+1}}) + p_{m-2}$$

$$= \frac{1}{a_{m+1}}((p_{m-1}a_m + p_{m-2})a_{m+1} + p_{m-1})$$

$$= \frac{1}{a_{m+1}}(p_m a_{m+1} + p_{m-1}).$$

In the same way, we have:

$$q_{m+1} = q_{m-1}(a_m + \frac{1}{a_{m+1}}) + q_{m-2}$$

$$= \frac{1}{a_{m+1}}((q_{m-1}a_m + q_{m-2})a_{m+1} + q_{m-1})$$

$$= \frac{1}{a_{m+1}}(q_m a_{m+1} + q_{m-1}).$$

As p_{m+i} and q_{m+i} are relatively prime we conclude that

$$p_{m+1} = a_{m+1} p_m + p_{m-1}$$
$$q_{m+1} = a_{m+1} q_m + q_{m-1}.$$

which completes the proof.

Using the result obtained in Statement 1.1. we construct an effective algorithm for calculating the value of p_k and q_k. In the table below we enter the values of the variables a_i. The column $\begin{pmatrix} p_m \\ q_m \end{pmatrix}$ for m>2 is obtained in this way: we multiply the column $\begin{pmatrix} p_{m-1} \\ q_{m-1} \end{pmatrix}$ by a_m and then add the column $\begin{pmatrix} p_{m-2} \\ q_{m-2} \end{pmatrix}$ to the result.

Ao	a_1	a_2	a_3	…	a_{k-1}	a_k
p_0	p_1	p_2	p_3	…	p_{k-1}	p_k
q_o	q_1	q_2	q_3	…	q_{k-1}	q_k

Table 1: Calculating values of p_i and q_i

Statement 1.2. *If l is a K-good number, then every multiple of l is K-good.*

<u>*Proof.*</u> Since l is a *K-good* number, then there exist such a and b that $l=a+b$ and $\frac{a}{b}=[0, a_1, a_2, \ldots, a_k]$, where $a_i = \{1, 2, \ldots, K\}$ for $i \in$ N. Let m be an arbitrary multiple of l, such that $m=tl$. Then $m=t(a+b)=ta+tb$.

In addition, $\frac{ta}{tb} = \frac{a}{b} = [0, a_1, a_2, \ldots, a_k]$. Therefore, we conclude that m is a *K-good* number.

1. Find all numbers n > 2 that can be expressed as the sum of two positive integers n = a+b so that a < b and the continued fraction for a/b has all its partial quotients equal to 1.

We use the algorithm described above Table 1. In this case we have $a_i = 1$ for $i \in \{1, 2, \ldots\}$.

a_i	0	1	1	1	1	1	1	1	1	...
p_i	0	1	1	2	3	5	8	13	21	...
q_i	1	1	2	3	5	8	13	21	34	...

We observe that when we add the numerators p_i and denominators q_i the sums p_i+q_i for $i \in$ N equal 1, 3, 5, 8,13, 21,…, whence the numbers sought are the consecutive members of the Fibonacci sequence.

2. A number $n > 2$ is called 2-good if for some positive integers $a < b$ we have $n = a+b$ and the partial quotients of a/b are equal to 1 or 2.

(a) Are there infinitely many odd numbers that are not 2-good?

(b) Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.

(c) Describe the set of all 2-good numbers.

We use the Euclid's Algorithm to make continuous fractions. Given the numbers $a < b$ we illustrate the way we get its corresponding continuous fraction $[0; a_1, a_2, \dots, a_k]$.

$$b = aa_1 + r_1$$
$$a = r_1a_2 + r_2$$
$$r_1 = r_2a_3 + r_3$$
$$\vdots$$
$$r_{k-3} = r_{k-2}a_{k-1} + r_{k-1}$$
$$r_{k-2} = r_{k-1}a_k$$

The fraction $\frac{a}{b}$ is equal to $[0; a_i, a_2, \dots, a_k]$.

(a) There exist primes that cannot be expressed as 2-good numbers. For example: 23, 53, 59, 83, 103, 107, 113, ... We observe that all of these primes are congruent to (-1) mod 3. There are infinitely many prime numbers congruent to (-1) mod 3. Indeed, suppose that there are finitely many primes $p_1, p_2, p_3, \dots, p_s \equiv (-1) \pmod 3$. If s is even, then $p_1p_2p_3 \dots p_s + 1 \equiv (-1) \pmod 3$ and thus has a prime divisor p congruent to (-1) mod 3. From the supposition, we obtain that p must be one of the primes $p_1, p_2, p_3, \dots, p_s$. It follows that p divides 1 which is a contradiction to the supposition. If s is odd, then $p_1p_2p_3 \dots p_s + 3 \equiv (-1) \pmod 3$ and thus has a prime divisor q congruent to (-1) mod 3. From the supposition, we obtain that q must be one of the primes $p_1, p_2, p_3, \dots, p_s$. It follows that q divides 3 which is again a contradiction to the supposition.

(b) If the only not *2-good* numbers are the primes, described in 1., then every even number can be written as the sum of two odd *2-good* numbers because there cannot exist 3 consecutive odd numbers that are prime.

(c) Our aim is to describe the set of all *2-good* numbers - the numbers such that $a_i = \{1, 2\}$ for $i \in$ N. In this case the following inequalities $a < b < 3a$ hold. Therefore a $> \frac{n}{4}$, b $> \frac{n}{2}$. There are at most 2^k different 2-good numbers that can be obtained from the continuous fraction $[0; a_1, a_2, \dots, a_k]$, where $a_i = \{1, 2\}$.

We saw in point 1. that the numbers obtained when all partial quotients of $\frac{a}{b}$ are equal to 1 are the members of the Fibonacci sequence $\{f_k\}_{k \in \mathbf{N}}$ where $f_k = f_{k-1} + f_{k-2}$.

Problem 9

The numbers obtained when all partial quotients of $\frac{a}{b}$ are equal to 2 are as follows:

a_i	0	2	2	2	2	2	2	2	2	...
p_i	0	1	2	5	12	29	70	169	408	...
q_i	1	2	5	12	29	70	169	408	885	...
p_i+q_i	1	3	7	17	41	99	239	576	1391	...

These numbers are members of the sequence $\{s_k\}_{k\in\mathbf{N}}$, where $s_k=2s_{k-1}+s_{k-2}$ and $s_0=1$, $s_1=3$.

An arbitrary combination of partial quotients equal to 1 or 2 cannot give us a formula for a recurrence sequence.

3. In general, a number n is called k-good if it can be expressed as the sum of two positive integers n = a+b so that a < b and the continued fraction for a/b has all its partial quotients not greater than k.
Does there exist a positive integer K such that all positive integers n > 2 are K-good?

Hypothesis: All positive integers n>2 are 3-good.

Denote by S_K the set of *K-good* numbers. By Statement 1.2. it follows that if all prime numbers are *K-good* , then $S_K \equiv \mathbf{N}$. We have that 23, 53, 59, 83, 103, 107, 113 are *3-good*.