2009

Problem 8

Positivity of Symmetric Polynomials

A polynomial $P(x, y)$ with real coefficients is symmetric if the equality $P(x, y) = P(y, x)$ holds for all $x, y \in R$.

1. Let $P(x, y) = x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a > -1$.
2. Let $P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4.

a) Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a < -4$, $b > (a^2+8)/4$ or $a > -4$, $b > -2a - 2$.

b) Prove that $P(x, y) > 0$ for all $x, y \neq 0$ if and only if $|a| > 4$, $b > (a^2+8)/4$ or $|a|$ 6 4, $b > 2|a| - 2$.

3. Let $P(x, y)$ be one of the following symmetric polynomials:

$$x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5,$$
$$x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6,$$
$$x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7.$$

Find necessary and sufficient conditions on the coefficients such that $P(x, y) > 0$ for all (a) $x, y > 0$, (b) $x, y \neq 0$.

4. Find sufficient conditions for a homogeneous symmetric polynomial $P(x, y)$ of degree $n > 7$ to take positive values for all (a) $x, y > 0$, (b) $x, y \neq 0$.
5. Using methods developed in the previous questions, give necessary and sufficient conditions for a non-homogeneous symmetric polynomial of two real variables to be positive.

Sofia High School of Mathematics

6/21/2009

Positivity of Symmetric Polynomials

Problem 1:

Let $P(x, y) = x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a > -1$.

Solution:

We will study the polynomial:

$$(1) \qquad \boldsymbol{P(x,y) = x^3 + ax^2y + axy^2 + y^3}$$

Let $P(x,y) > 0$ for all $x, y > 0$. We will prove that $a > -1$.

From $x, y > 0$ we can conclude that $x, y \neq 0$. So we can divide both sides of equation (1) by y^3. Hence:

$$\boldsymbol{\frac{P(x,y)}{y^3} = \left(\frac{x}{y}\right)^3 + a\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1}$$

Because of $P(x,y) > 0$ and $y > 0$, $\frac{P(x,y)}{y^3} > 0$. Therefore:

$$(2) \qquad \boldsymbol{\left(\frac{x}{y}\right)^3 + a\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1} > 0, \textit{for all } x, y > 0$$

Let us replace $\frac{x}{y}$ by z. Because of $x, y > 0$, $z > 0$:

$$\boldsymbol{z^3 + az^2 + az + 1} > 0, \boldsymbol{for\ all\ z} > 0.$$

Let us group the monomials and then factorize the inequality above:

$$\boldsymbol{(z+1)(z^2 + (a-1)z + 1)} > 0, \boldsymbol{for\ all\ z} > 0 \textbf{ and } \boldsymbol{z + 1} > 0.$$

Therefore:

$$\boldsymbol{z^2 + (a-1)z + 1} > 0, \boldsymbol{for\ all\ z} > 0.$$

Now let us express a by z:

$$\boldsymbol{a} > 1 - \left(\boldsymbol{z + \frac{1}{z}}\right)$$

Using the fact that $z + \frac{1}{z} \geq 2$, for all $z > 0$, we can conclude that inequality (2) is equivalent to:

$$a > -1$$

Finally we can conclude that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a > -1$.

Problem 2:

Let $P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4.

(a) Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $\boldsymbol{a < -4, b > \frac{a^2+8}{4}}$ or $\boldsymbol{a \geq -4, b > -2a - 2}$.

(b) Prove that $P(x, y) > 0$ for all $\boldsymbol{x, y \neq 0}$ if and only if $|\boldsymbol{a}| > 4, \boldsymbol{b} > \frac{a^2+8}{4}$ or $|\boldsymbol{a}| \leq \boldsymbol{4}, \boldsymbol{b} > 2|\boldsymbol{a}| - \boldsymbol{2}$.

Solution:

Let us study the polynomial:

$$(3) \qquad \boldsymbol{P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4}$$

We can divide both sides of equation (3) by x^2y^2. We get:

$$\frac{\mathbf{P(x, y)}}{\mathbf{x^2y^2}} = \left(\frac{\mathbf{x}}{\mathbf{y}}\right)^2 + \mathbf{a}\frac{\mathbf{x}}{\mathbf{y}} + \mathbf{b} + \mathbf{a}\frac{\mathbf{y}}{\mathbf{x}} + \left(\frac{\mathbf{y}}{\mathbf{x}}\right)^2$$

From $P(x, y) > 0$ it is obvious that $\frac{P(x,y)}{x^2y^2} > 0$. Therefore:

$$\left(\frac{\mathbf{x}}{\mathbf{y}}\right)^2 + \mathbf{a}\frac{\mathbf{x}}{\mathbf{y}} + \mathbf{b} + \mathbf{a}\frac{\mathbf{y}}{\mathbf{x}} + \left(\frac{\mathbf{y}}{\mathbf{x}}\right)^2 > 0$$

a) Let us study the first case where $x, y > 0$.

Substitute $\frac{x}{y} = z$. Because of $x, y > 0, z > 0$:

$$\boldsymbol{z^2 + az + b + a\frac{1}{z} + \frac{1}{z^2}} > 0, for\ all\ z > 0.$$

Let us group the monomials and then factorize the inequality above:

$$\boldsymbol{\left(z + \frac{1}{z}\right)^2 + a\left(z + \frac{1}{z}\right) + b - 2} > 0, for\ all\ z > 0$$

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Substitute $\left(\mathbf{z}+\frac{\mathbf{1}}{\boldsymbol{z}}\right)=t$. We get:

$$\mathbf{t^2+at+b-2}>0$$

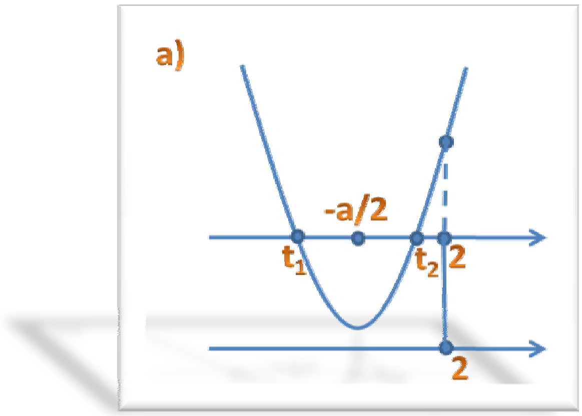
First we have to estimate the possible values of t. From $\left(\mathbf{z}+\frac{\mathbf{1}}{\boldsymbol{z}}\right)=t$ and the fact that $\boldsymbol{t}+\frac{1}{\boldsymbol{t}}\geq 2$ we can conclude that:

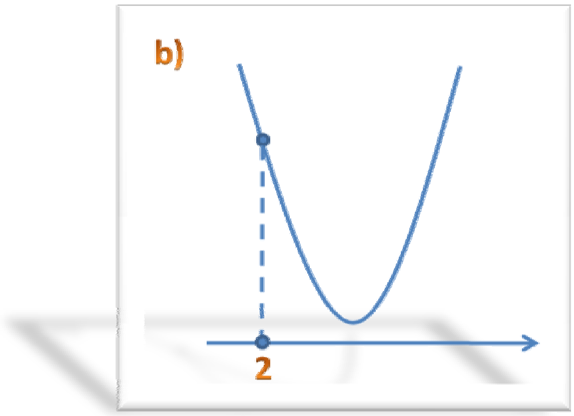
$$\boldsymbol{t}\in[\mathbf{2};+\infty)$$

Hence:

$$f(t)=t^2+at+b-2>0, for\ all\ t\in[2;+\infty)$$

This is possible only in one of these situations:





a) Therefore:

$$\begin{cases}-\frac{a}{2}\leq 2\\ f(2)>0\end{cases} \iff \begin{cases}a\geq -4\\ b>-2a-2\end{cases}$$

The last condition is the first statement that we need to prove.

b) Therefore:

$$\begin{cases}-\frac{a}{2}\leq 2\\ D=a^2-4b+8<0\end{cases} \iff \begin{cases}a\geq -4\\ b>\frac{a^2+8}{4}\end{cases}$$

The last condition is the second statement that we need to prove.

Therefore P(x, y) > 0 for all x, y > 0 if and only if $a<-4, b>\frac{a^2+8}{4}$ or $a\geq -4, b>-2a-2$.

b) Let us study the second case where $x, y\neq 0$. If x and y have the same signs we have a similar situation as in the first case because :

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P(x, y) > 0 for all x, y > 0 if and only if $\boldsymbol{a} < -4, \boldsymbol{b} > \frac{\boldsymbol{a^2+8}}{\boldsymbol{4}}$ *or* $\boldsymbol{a} \geq \boldsymbol{-4}, \boldsymbol{b} > -2a-2.$

Let us study the case where x and y are of different signs.

Let us substitute $z = -\frac{x}{y} > 0$:

$$\boldsymbol{z^2 - az + b - a\frac{1}{z} + \frac{1}{z^2}} > 0, for\ all\ z > 0.$$

Let us group the monomials and then factorize the inequality above:

$$\left(\boldsymbol{z + \frac{1}{z}}\right)^{\boldsymbol{2}} - \boldsymbol{a}\left(\boldsymbol{z + \frac{1}{z}}\right) + \boldsymbol{b - 2} > 0, for\ all\ z > 0$$

Substitute $\left(\mathbf{z} + \frac{\mathbf{1}}{z}\right) = t$. We obtain:

$$\mathbf{t^2 - at + b - 2} > 0$$

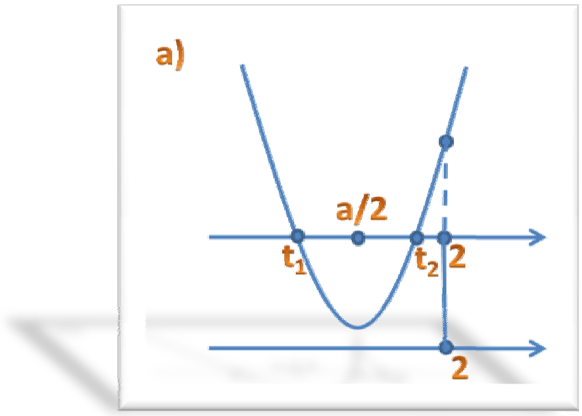
First we have to estimate the possible values of t. From $\left(\mathbf{z} + \frac{\mathbf{1}}{z}\right) = t$ and the fact that $\boldsymbol{t} + \frac{1}{\boldsymbol{t}} \geq 2$ we can conclude that:

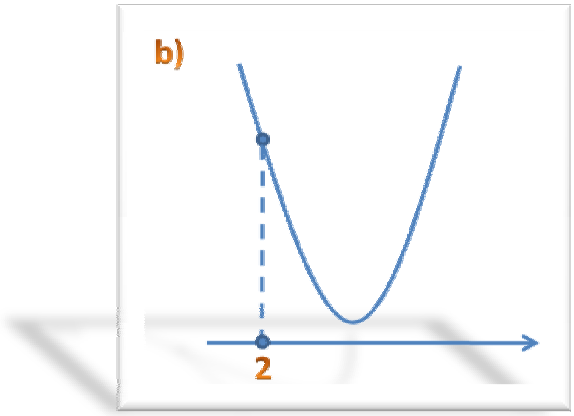
$$\boldsymbol{t} \in [\mathbf{2}; +\infty)$$

Therefore:

$$f(t) = t^2 - at + b - 2 > 0, for\ all\ t \in [2; +\infty)$$

This is possible only in one of these situations:





a) Therefore:

$$\left|\begin{array}{l} -\frac{a}{2} \le 2 \\ f(2) > 0 \end{array}\right. \quad \Longleftrightarrow \quad \left|\begin{array}{l} a \le 4 \\ b > 2a - 2 \end{array}\right.$$

b) Therefore:

$$\left|\begin{array}{l} \frac{a}{2} > 2 \\ D = a^2 - 4b + 8 < 0 \end{array}\right. \quad \Longleftrightarrow \quad \left|\begin{array}{l} a > 4 \\ b > \frac{a^2+8}{4} \end{array}\right.$$

Therefore:

$P(x, y) > 0$ for all x and y of same signs if and only if $a < -4, b > \frac{a^2+8}{4}$ or $a \ge -4, b > -2a - 2$;

$P(x, y) > 0$ for all x and y of different signs if and only if $a > 4, b > \frac{a^2+8}{4}$ or $a \le 4, b > 2a - 2$.

If we combine both cases we can conclude that:

$P(x, y) > 0$ for all $\boldsymbol{x, y \neq 0}$ if and only if $|a| > 4, b > \frac{a^2+8}{4}$ or $|a| \le 4, b > 2|a| - 2$.

Problem 3:

Let $P(x, y)$ be one of the following symmetric polynomials:

$$x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5,$$

$$x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6,$$

$$x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7.$$

Find necessary and sufficient conditions on the coefficients such that $P(x, y) > 0$ for all (a) $x, y > 0$, (b) $\boldsymbol{x, y \neq 0}$.

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Solution:

Let us study the polynomial:

$$(4) \qquad \mathbf{P(x,y) = x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5}$$

We will try to find necessary and sufficient conditions for $P(x,y) > 0$.

a) Let $x, y > 0$. So we can divide both sides of (4) by y^5. We get:

$$\mathbf{\frac{P(x,y)}{y^5} = \left(\frac{x}{y}\right)^5 + a\left(\frac{x}{y}\right)^4 + b\left(\frac{x}{y}\right)^3 + b\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1}$$

We know that $P(x,y) > 0$ and $y > 0$, so it is obvious that $\frac{P(x,y)}{y^5} > 0$. Therefore:

$$\mathbf{\left(\frac{x}{y}\right)^5 + a\left(\frac{x}{y}\right)^4 + b\left(\frac{x}{y}\right)^3 + b\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1} > 0, \textit{for all } x, y > 0.$$

Substitute $\frac{x}{y} = z$. Because of $x, y > 0$, $z > 0$. We obtain:

$$\mathbf{z^5 + az^4 + bz^3 + bz^2 + az + 1} > 0, \textit{for all } z > 0.$$

Let us group the monomials and then factorize the inequality above:

$$\boldsymbol{(z+1)(z^4 + (a-1)z^3 + (b} + 1 - a\boldsymbol{)z^2 + (a-1)z + 1)} > 0, \textit{for all } z > 0\,.$$

Because of $z + 1 > 0$, the following has to be true:

$$K(z) = z^4 + (a-1)z^3 + (b + 1 - \mathrm{a})z^2 + (a-1)z + 1 > 0, \textit{for all } z > 0$$

The polynomial K(z) is symmetric (from problem two). Therefore, it will be positive if and only if:

$$\left|\begin{array}{l} a - 1 \geq -4 \\ \mathbf{b + 1 - a} > -2(\mathbf{a - 1}) - \mathbf{2} \end{array}\right. \iff \left|\begin{array}{l} a - 1 < -4 \\ b + 1 - a > \frac{(a-1)^2 + 8}{4} \end{array}\right.$$

$$\left|\begin{array}{l} \mathbf{a \geq -3} \\ b > -a - 1 \end{array}\right. \quad \text{or} \quad \left|\begin{array}{l} \mathbf{a < -3} \\ b > \frac{a^2 + 2a + 5}{4} \end{array}\right.$$

Therefore, $P(x,y) > \mathbf{0}$ for all $x, y > 0$ if and only if $a < -3, b > \frac{a^2+2a+5}{4}$, or $a \geq -3, b > \mathbf{-a-1}$

b) We will show that there are no necessary and sufficient conditions $P(x,y) > 0$ for all $x, y \neq 0$.

Let us substitute $y = 1$ in the equation (4). We obtain:

$$\mathbf{P(x,1) = x^5 + ax^4 + bx^3 + bx^2 + ax} + 1$$

Hence:

$$\lim_{x \to +\infty} P(x,1) = +\infty$$

$$\lim_{x \to -\infty} P(x,1) = -\infty$$

This means that there are no necessary and sufficient conditions $P(x,y) > 0$ for all $x, y \neq 0$.

Let us study the polynomial:

$$(5) \qquad \mathbf{P(x,y) = x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6}$$

We will try to find necessary and sufficient conditions $P(x,y) > 0$.

a) Let $x, y > 0$. We can divide both sides of equation (5) by x^3y^3. Hence:

$$\mathbf{\frac{P(x,y)}{x^3y^3} = \left(\frac{x}{y}\right)^3 + a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c + b\frac{y}{x} + a\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3}$$

It is obvious that $\frac{P(x,y)}{x^3y^3} > 0$. Therefore:

$$\left(\frac{x}{y}\right)^3 + a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c + b\frac{y}{x} + a\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 > 0, \textit{for all } x, y > 0$$

Substitute $\frac{x}{y} = z$. Because of $x, y > 0, z > 0$. Therefore:

$$z^3 + az^2 + bz + c + b\frac{1}{z} + a\frac{1}{z^2} + \frac{1}{z^3} > 0, \textit{for all} > 0.$$

Let us group the monomials and then factorize the inequality above:

$$\left(z + \frac{1}{z}\right)^3 + a\left(z + \frac{1}{z}\right)^2 + (b-3)\left(z + \frac{1}{z}\right) - 2a + c > 0, \textbf{\textit{for all }} z > 0.$$

Substitute $\left(z + \frac{1}{z}\right) = t$. Therefore:

$$\mathbf{t^3 + at^2 + (b-3)t - 2a + c} > 0$$

Problem 8

First we have to estimate the possible values of t. From $\left(\mathbf{z}+\frac{\mathbf{1}}{\boldsymbol{z}}\right)=t$ and the fact that $\boldsymbol{t}+\frac{1}{\boldsymbol{t}}\geq 2$ we can conclude that:

$$\boldsymbol{t}\in[\mathbf{2};+\infty)$$

Hence:

$$\boldsymbol{f(t)}=\boldsymbol{t^3}+\boldsymbol{at^2}+(\boldsymbol{b}-\mathbf{3})\boldsymbol{t}-\mathbf{2}\boldsymbol{a}+\boldsymbol{c}>0, for\ all\ t\in[2;+\infty)$$

Let us study the function f(t).

- **Finding of the discriminant**

This is the discriminant of f(t):

$$108+9a^2+8a^4-108b-42a^2b+36b^2+a^2b^2-4b^3+54ac-4a^3c+18abc-27c^2$$

- **Case one**

Let us study the case when the polynomial f(t) has one or two roots.

The polynomial has one or two roots when the discriminant is nonpositive. Therefore:

$$b<\frac{1}{3}(9+a^2)$$

and

$$c\leq \frac{1}{27}(27a-2a^3+9ab)-\frac{2}{27}\sqrt{729+243a^2+27a^4+a^6-729b-162a^2b-9a^4b+243b^2+27a^2b^2-27b^3}$$

or

$$c\geq\frac{1}{27}(27a-2a^3+9ab)+\frac{2}{27}\sqrt{729+243a^2+27a^4+a^6-729b-162a^2b-9a^4b+243b^2+27a^2b^2-27b^3}$$

or

$$b\geq\frac{1}{3}(9+a^2)$$

Then f(t) will be positive for all t∈[2;+¥) if f(2)>0. Therefore:

$$b>\frac{1}{2}(-2-2a-c)$$

Finally, in this case f(t) will be positive for all t∈[2;+¥) if:

$a < -6$

and

$b < \frac{1}{4}(-4a + a^2)$ and $c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$
$+ \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$b = \frac{1}{4}(-4a + a^2)$ and $c > \frac{1}{27}(27a - 2a^3 + 9ab) +$
$+ \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$b > \frac{1}{4}(-4a + a^2)$ and $c > -2 - 2a - 2b$

or

$a == -6$

and

$b < 15$ and $c \geq \frac{1}{27}(270 - 54b) + \frac{2}{27}\sqrt{91125 - 18225b + 1215b^2 - 27b^3}$

or

$b \geq 15$ and $c > 10 - 2b$

or

$a > -6$

and

$b < -9 - 4a$ and $c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$

$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$b = -9 - 4a$ and $c > -2 - 2a - 2b$

or

$-9 - 4a < b \leq \frac{1}{4}(-4a + a^2)$ and $c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$

$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$\frac{1}{4}(-4a + a^2) < b < \frac{1}{3}(9 + a^2)$ and $-2 - 2a - 2b < c \leq \frac{1}{27}(27a - 2a^3 + 9ab) -$

$-\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$

$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$b \geq \frac{1}{3}(9 + a^2)$ and $c > -2 - 2a - 2b$

- Case two

Let us study the case when the polynomial f(t) has three roots.

The polynomial will have three roots when the discriminant is positive. Therefore:

$b < \frac{1}{3}(9 + a^2)$

and

$$\frac{1}{27}(27a - 2a^3 + 9ab) - \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3} < c < \frac{1}{27}(27a - 2a^3 + 9ab) + \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

Then f(t) will be positive for all $t \in [2; +\infty)$ if f(2)>0. We get:

$$b > \frac{1}{2}(-2 - 2a - c)$$

And if $t_1' \leq t_2' < 2$:

$$a > -6 \&\& - 9 - 4a < b \leq \frac{1}{3}(9 + a^2)$$

Finally, in this case f(t) will be positive for all t∈[2;+¥) if:

$a > -6$

and

$$-9 - 4a < b \leq \frac{1}{4}(-4a + a^2) \text{ and } -2 - 2a - 2b < c < \frac{1}{27}(27a - 2a^3 + 9ab) + \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

$$\frac{1}{4}(-4a + a^2) < b < \frac{1}{3}(9 + a^2) \text{ and } \frac{1}{27}(27a - 2a^3 + 9ab) - \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3} < c < \frac{1}{27}(27a - 2a^3 + 9ab) + \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

Necessary and sufficient conditions

Finally, combining both cases, we conclude that:

$a < -6$

and

$$b < \frac{1}{4}(-4a + a^2)\ and\ c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$$

$$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

or

$$b == \frac{1}{4}(-4a + a^2)\ and\ c > \frac{1}{27}(27a - 2a^3 + 9ab) +$$

$$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

or

$$b > \frac{1}{4}(-4a + a^2)\ andc > -2 - 2a - 2b$$

or

$a == -6$

and

$$((b < 15\&\&c \geq \frac{1}{27}(270 - 54b) + \frac{2}{27}\sqrt{91125 - 18225b + 1215b^2 - 27b^3})||(b \geq 15\&\&c > 10 - 2b)))$$

or

$a > -6$

and

$$b < -9 - 4a\ and\ c \geq \frac{1}{27}(27a - 2a^3 + 9ab) +$$

$$+\frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$$

or

$b = -9 - 4a \; and \; c$
$> \frac{1}{27}(27a - 2a^3 + 9ab)$
$+ \; + \frac{2}{27}\sqrt{729 + 243a^2 + 27a^4 + a^6 - 729b - 162a^2b - 9a^4b + 243b^2 + 27a^2b^2 - 27b^3}$

or

$b > -9 - 4a \; and \; c > -2 - 2a - 2b$))

b) Let $x, y \neq 0$. If x and y are of same signs we have a similar situation as in point a). Let x and y be of different signs.

Without loss of generality let y<0. Substitute in equation (5) $z = -y$:

$$\mathbf{P(x,y) = L(x,z) = x^6 - ax^5z + bx^4z^2 - cx^3z^3 + bx^2z^4 - axz^5 + z^6}$$

Therefore we need to find necessary and sufficient conditions for L(x, z) to be positive for all $x, y > 0$. We can also conclude that L(x,z) is a symmetric polynomial of degree six and therefore we can find these necessary and sufficient conditions using the results of point a).

We study the polynomial:

$$(6) \qquad \mathbf{P(x,y) = x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7}$$

We will try to find necessary and sufficient conditions for $P(x,y) > 0$.

a) Let $x, y > 0$. Therefore we can divide both sides of equation (5) by y^7. Therefore:

$$\mathbf{\frac{P(x,y)}{y^7} = \left(\frac{x}{y}\right)^7 + a\left(\frac{x}{y}\right)^6 + b\left(\frac{x}{y}\right)^5 + c\left(\frac{x}{y}\right)^4 + c\left(\frac{x}{y}\right)^3 + b\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1}$$

It is obvious that $\frac{P(x,y)}{y^7} > 0$. Therefore

$$\mathbf{\left(\frac{x}{y}\right)^7 + a\left(\frac{x}{y}\right)^6 + b\left(\frac{x}{y}\right)^5 + c\left(\frac{x}{y}\right)^4 + c\left(\frac{x}{y}\right)^3 + b\left(\frac{x}{y}\right)^2 + a\frac{x}{y} + 1} > 0$$

Substitute $\frac{x}{y} = z$. Because of $x, y > 0, z > 0$. We get:

$$\mathbf{z^7 + az^6 + bz^5 + cz^4 + cz^3 + bz^2 + az + 1} > 0$$

Let us group the monomials and then factorize the inequality above:

$$\boldsymbol{(z+1)\left(z^6 + (a-1)z^5 + (b+a-1)z^4 + (a+c-b-1)z^4 + (b+a-1)z^2 + (a-1)z + 1\right)} > 0, \boldsymbol{for\ all\ z} > 0 \,.$$

Because of $z+1>0$, the following must be true:

$$\boldsymbol{K(z) = z^6 + (a-1)z^5 + (b+a-1)z^4 + (a+c-b-1)z^4 + (b+a-1)z^2 + (a-1)z + 1} > 0, \boldsymbol{for\ all\ z} > 0$$

The polynomial K(z) is symmetric polynomial of degree six. Therefore we can find necessary and sufficient condition $P(x,y)>0$ for all $x,y>0$, using the results for a polynomial of degree six.

b) Let $x,y \neq 0$. We will show that there are no necessary and sufficient conditions for $P(x,y)>0$ for all $x,y \neq 0$.

Substitute $\mathrm{y}=1$ in the equation (6). We get:

$$\mathbf{P(x,1) = x^7 + ax^6 + bx^5 + cx^4 + cx^3 + bx^2 + ax + 1}$$

Therefore:

$$\lim_{x\to+\infty} P(x,1) = +\infty$$

$$\lim_{x\to-\infty} P(x,1) = -\infty$$

This means that there are no necessary and sufficient conditions for $P(x,y)>0$ for all $x,y \neq 0$.

Problem 4

Find sufficient conditions for a homogeneous symmetric polynomial P(x, y) of degree $n > 7$ to take positive values for all (a) $x, y > 0$, (b) $x, y \neq 0$.

Solution

Statement for the polynomials of odd degree

1) *If n is odd, then the necessary and sufficient conditions for the polynomial* $P(x, y)$ *of degree* $n = 2k + 1$,*with coefficients* $1, a_1, a_2, a_3, \dots, a_{k-1}, a_k, a_{k-1}, \dots, a_3, a_2, a_1$ *to be positive, when* $x, y > 0$, *are the same as the necessary and sufficient conditions for the polynomial* $P(x, y)$ *of degree* $n = 2k$, *with coefficients:*

1

$-1 + a_1$

$1 - a_1 + a_2$

$-1 + a_1 - a_2 + a_3$

...

...

...

$\mp\mathbf{1} \pm \mathbf{a_1} \mp \mathbf{a_2} \pm \mathbf{a_3} \mp \cdots + \mathbf{a_{k-1}}$

$\pm\mathbf{1} \mp \mathbf{a_1} \pm \mathbf{a_2} \mp \mathbf{a_3} \pm \cdots - \mathbf{a_{k-1}} + \mathbf{a_k}$

$\mp\mathbf{1} \pm \mathbf{a_1} \mp \mathbf{a_2} \pm \mathbf{a_3} \mp \cdots + \mathbf{a_{k-1}}$

...

...

...

$-\mathbf{1} + \mathbf{a_1} - \mathbf{a_2} + \mathbf{a_3}$

$\mathbf{1} - \mathbf{a_1} + \mathbf{a_2}$

$-1 + a_1$

1,

To be positive, for $x, y > 0$.

2) *There are no necessary and sufficient condition for* $P(x, y)$ *of degree* $n = 2k + 1$ *to be positive for* $x, y \neq 0$.

This statement can be easily received using the same substitutions as we used for a polynomials of degree five and seven.

Considering this statement we can conclude that it is enough to study only the polynomial of even degree.

Let us study the polynomial:

(7) $$P(x,y) = x^{2k} + a_1x^{2k-1}y + a_2x^{2k-2}y^2 + a_3x^{2k-3}y^3 + \cdots + a_kx^ky^k + \cdots + a_{2k-3}x^3y^{2k-3} + a_{2k-2}x^2y^{2k-2} + a_{2k-1}xy^{2k-1} + y^{2k}$$

a) Let $x, y > 0$. We can divide both sides of the equation (7) by x^ky^k. We get:

$$\frac{P(x,y)}{x^ky^k} = \left(\frac{x}{y}\right)^k + a_1\left(\frac{x}{y}\right)^{k-1} + a_2\left(\frac{x}{y}\right)^{k-2} + a_3\left(\frac{x}{y}\right)^{k-3} + \cdots + a_k + \cdots + a_3\left(\frac{y}{x}\right)^{k-3} + a_2\left(\frac{y}{x}\right)^{k-2} + a_1\left(\frac{y}{x}\right)^{k-1} + \left(\frac{y}{x}\right)^{2k}$$

It is obvious that $\frac{P(x,y)}{x^ky^k} > 0$. Therefore:

$$\left(\frac{x}{y}\right)^k + a_1\left(\frac{x}{y}\right)^{k-1} + a_2\left(\frac{x}{y}\right)^{k-2} + a_3\left(\frac{x}{y}\right)^{k-3} + \cdots + a_k + \cdots + a_3\left(\frac{y}{x}\right)^{k-3} + a_2\left(\frac{y}{x}\right)^{k-2} + a_1\left(\frac{y}{x}\right)^{k-1} + \left(\frac{y}{x}\right)^{2k} > 0, for\ all\ \boldsymbol{x}, \boldsymbol{y} > 0$$

Substitute $\frac{x}{y} = z$. Because of $x, y > 0, z > 0$. We get:

$$z^k + a_1z^{k-1} + a_2z^{k-2} + a_3z^{k-3} + \cdots + a_k + \cdots + a_3\left(\frac{1}{z}\right)^{k-3} + a_2\left(\frac{1}{z}\right)^{k-2} + a_1\left(\frac{1}{z}\right)^{k-1} + \left(\frac{1}{z}\right)^k > 0, \boldsymbol{for\ all\ z > 0}.$$

Let us group the monomials and then factorize the inequality above:

$$\left(z^k + \left(\frac{1}{z}\right)^k\right) + a_1\left(z^{k-1} + \left(\frac{1}{z}\right)^{k-1}\right) + a_2\left(z^{k-2} + \left(\frac{1}{z}\right)^{k-2}\right) + a_3\left(z^{k-3} + \left(\frac{1}{z}\right)^{k-3}\right) + \cdots + a_k > 0, \boldsymbol{for\ all\ z > 0}.$$

Let us express every addend of the sort $a_r\left(z^{k-r} + \left(\frac{1}{z}\right)^{k-r}\right)$ in the following way:

$$a_r\left(\mathrm{z}^{\mathrm{k-r}}+\left(\frac{1}{z}\right)^{k-r}\right)$$

$$= a_r\left(z+\frac{1}{z}\right)^{k-r}$$

$$-\left(\binom{k-r}{1}\left(\mathrm{z}^{(\mathrm{k-r})-2.1}+\left(\frac{1}{z}\right)^{(\mathrm{k-r})-1.2}\right)+\binom{k-r}{2}\left(\mathrm{z}^{(\mathrm{k-r})-2.2}+\left(\frac{1}{z}\right)^{(\mathrm{k-r})-2.2}\right)\right.$$

$$+\binom{k-r}{3}\left(\mathrm{z}^{(\mathrm{k-r})-2.3}+\left(\frac{1}{z}\right)^{(\mathrm{k-r})-2.3}\right)+\cdots$$

$$\left.+\binom{k-r}{\left[\frac{\mathrm{k-r}}{2}\right]}\left(\mathrm{z}^{(\mathrm{k-r})-2.\left[\frac{(\mathrm{k-r})}{2}\right]}+\left(\frac{1}{z}\right)^{(\mathrm{k-r})-2.\left[\frac{(\mathrm{k-r})}{2}\right]}\right)\right)$$

herefore:

$$\left(\mathrm{z}^{\mathrm{k}}+\left(\frac{1}{z}\right)^{k}\right)+a_1\left(z+\frac{1}{z}\right)^{k-1}$$

$$-\left(\binom{k-1}{1}\left(\mathrm{z}^{\mathrm{\kappa-3}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-3}}\right)+\binom{k-1}{2}\left(\mathrm{z}^{\mathrm{\kappa-5}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-5}}\right)+\binom{k-1}{3}\left(\mathrm{z}^{\mathrm{\kappa-7}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-7}}\right)+\cdots\right.$$

$$\left.+\binom{k-1}{\left[\frac{k-1}{2}\right]}\left(\mathrm{z}^{(\mathrm{k-1})-2.\left[\frac{(\mathrm{k-1})}{2}\right]}+\left(\frac{1}{z}\right)^{(\mathrm{k-1})-2.\left[\frac{(\mathrm{k-1})}{2}\right]}\right)\right)+a_2\left(z+\frac{1}{z}\right)^{k-2}$$

$$-\left(\binom{k-2}{1}\left(\mathrm{z}^{\mathrm{\kappa-4}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-4}}\right)+\binom{k-2}{2}\left(\mathrm{z}^{\mathrm{\kappa-6}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-6}}\right)+\binom{k-2}{3}\left(\mathrm{z}^{\mathrm{\kappa-8}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-8}}\right)+\cdots\right.$$

$$\left.+\binom{k-2}{\left[\frac{k-2}{2}\right]}\left(\mathrm{z}^{(\mathrm{k-2})-2.\left[\frac{(\mathrm{k-2})}{2}\right]}+\left(\frac{1}{z}\right)^{(\mathrm{k-2})-2.\left[\frac{(\mathrm{k-2})}{2}\right]}\right)\right)+a_3\left(z+\frac{1}{z}\right)^{k-3}$$

$$-\left(\binom{k-3}{1}\left(\mathrm{z}^{\mathrm{\kappa-5}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-5}}\right)+\binom{k-3}{2}\left(\mathrm{z}^{\mathrm{\kappa-7}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-7}}\right)+\binom{k-3}{3}\left(\mathrm{z}^{\mathrm{\kappa-9}}+\left(\frac{1}{z}\right)^{\mathrm{\kappa-9}}\right)+\cdots\right.$$

$$\left.+\binom{k-3}{\left[\frac{k-3}{2}\right]}\left(\mathrm{z}^{(\mathrm{k-3})-2.\left[\frac{(\mathrm{k-3})}{2}\right]}+\left(\frac{1}{z}\right)^{(\mathrm{k-3})-2.\left[\frac{(\mathrm{k-3})}{2}\right]}\right)\right)+\cdots+a_k$$

Again we express the addends of the type $\left(\mathbf{z^{k-r}} + \left(\frac{1}{z}\right)^{k-r}\right)$ in the way that we just have shown. Then we apply this operation until we obtain a polynomial which is a sum of monomial of the type $\left(z+\frac{1}{z}\right)^{p}$:

$$\left(z+\frac{1}{z}\right)^{\mathbf{k}} + b_1\left(z+\frac{1}{z}\right)^{\mathbf{k-1}} + b_2\left(z+\frac{1}{z}\right)^{\mathbf{k-2}} + \mathbf{b_3}\left(z+\frac{1}{z}\right)^{k-3} + \cdots + \mathbf{b_k} > \mathbf{0}, \boldsymbol{for\ all\ z} > \mathbf{0}$$

Where $b_1, b_2, b_3, \ldots, b_k$ are functions of the coefficients $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \ldots, \mathrm{a}_k$.

Substitute $\left(\mathbf{z}+\frac{1}{z}\right) = t$. Therefore:

$$t^{\mathbf{k}} + b_1\mathbf{t}^{\mathbf{k-1}} + b_2\mathbf{t}^{\mathbf{k-2}} + \mathbf{b_3}\mathbf{t}^{k-3} + \cdots + \mathbf{b_k} > \mathbf{0}, \boldsymbol{for\ all\ z} > \mathbf{0}$$

First we have to estimate the possible values of t. From $\left(\mathbf{z}+\frac{1}{z}\right) = t$ and the fact that $t+\frac{1}{t} \geq 2$ we can conclude that:

$$t \in [\mathbf{2}; +\infty)$$

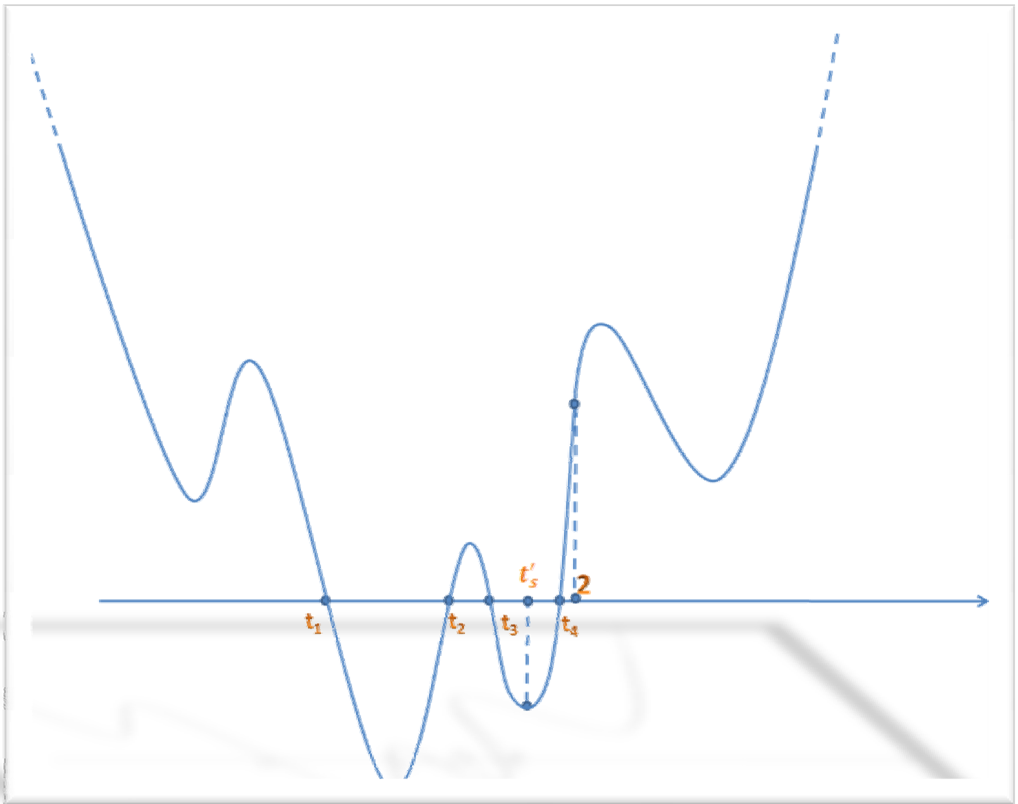
Therefore:

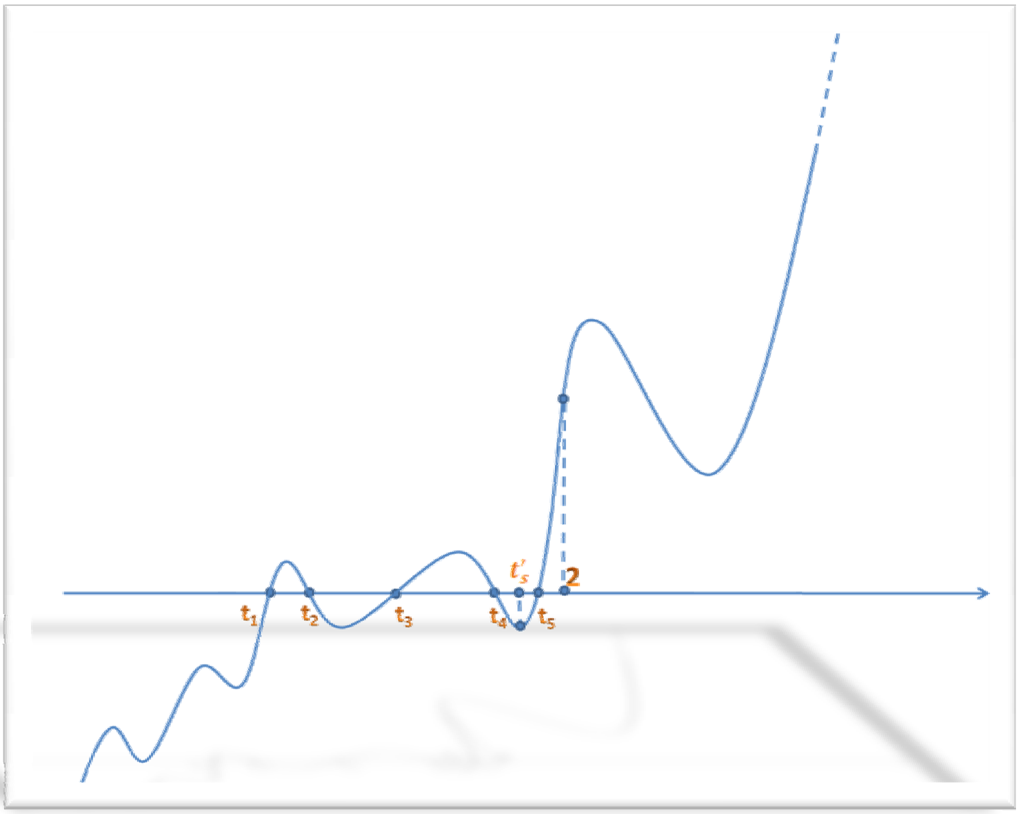
$$f(t) = t^{\mathbf{k}} + b_1\mathbf{t}^{\mathbf{k-1}} + b_2\mathbf{t}^{\mathbf{k-2}} + \mathbf{b_3}\mathbf{t}^{k-3} + \cdots + \mathbf{b_k} > \mathbf{0}, \boldsymbol{for\ all\ t} \in [2; +\infty)$$

Let us study the function $f(t)$.

$$f'(t) = kt^{\mathbf{k-1}} + (\mathbf{k}-\mathbf{1})b_1\mathbf{t}^{\mathbf{k-2}} + (k-\mathbf{2})b_2\mathbf{t}^{\mathbf{k-3}} + (\mathbf{k}-\mathbf{3})\mathbf{b_3}\mathbf{t}^{k-4} + \cdots + \mathbf{b_{k-1}}$$

Let the roots of the derivative be $t'_1, t'_2, t'_3, \ldots, t'_{2k}$ and let, without lost of generality, $t'_1 \leq t'_2 \leq t'_3 \leq \ldots \leq t'_{k-1}$. Let s be the smallest number for which it is true that for all $l > s$, $f(t'_l) \geq 0$:


t'5
2
t1
t2
t3
t4


t'5
2
t1
t2
t3
t4
t5

It is obvious that the possible values of s are 1, 2, 3, 4, ..., k-1. Therefore we have to solve k-1 systems of this type:

$$\begin{cases} \boldsymbol{t'_s} < 2 \\ \boldsymbol{f(2)} > 0 \end{cases}$$

The solutions of these systems give the necessary and sufficient condition for $\frac{P(x,y)}{y^7} > 0$ for all $x, y > 0$.

b) Let $x, y \neq 0$. We will show that there are no necessary and sufficient conditions for $P(x,y) > 0$ for all $x, y \neq 0$.

Substitute $y = 1$ in the equation (7). We obtain:

$$\mathbf{P(x,1) = x^{2k} + a_1x^{2k-1} + a_2x^{2k-2} + a_3x^{2k-3} + \cdots + a_kx^k + \cdots + a_{2k-3}x^3 + a_{2k-2}x^2 + a_{2k-1}x + 1}$$

Therefore:

$$\lim_{x \to +\infty} P(x,1) = +\infty$$

$$\lim_{x \to -\infty} P(x,1) = -\infty$$

This means that there are no necessary and sufficient conditions for $P(x,y) > 0$ for all $x, y \neq 0$.

References:

Software: Wolfram Mathematica 7