2009

Problem 6

Pattern graphs

Let n be a positive integer. A pattern of length n is a two-line table

$a_1\ a_2 \dots a_n$

$b_1\ b_2 \dots b_n$

, where $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are some rearrangements of the numbers 1, 2, . . . , n. Define two operations A and B on patterns as follows

A : replace each number a of the first line with the number that is in the a'th place (from left to right) of the second line,

B : replace each number b of the second line with the number that is in the b'th place (from left to right) of the first line.

We can construct an oriented labelled graph Gn whose vertices are all the patterns of length n, and such that for any two vertices v and w there is an A-arrow (resp. a B-arrow) from v to w if the pattern w is obtained from the pattern v by applying the operation A (resp. the operation B)

1. Can the pattern

$2\ 1\ 3\ 4 \dots n-1\ n$

$2\ 3\ 4\ 5 \dots n\ 1$

be obtained from the pattern

$2\ 3\ 1\ 4 \dots n-1\ n$

$2\ 3\ 4\ 5 \dots n\ 1$

using the operations A and B?

2. How many connected components of the graph Gn have exactly (a) 2 patterns? (b) 3 patterns?
3. Denote by gn the number of connected components of the graph Gn. Find gn (give a formula) or estimate it (give lower and upper bounds).
4. Study geometric properties of the connected components of Gn. Can they be drawn nicely in the plane: without intersections of the edges, symmetrically, etc.?
5. As a generalisation one could consider three-line patterns with $3 \cdot 2 = 6$ operations on them. Investigate this generalisation.

Sofia High School of Mathematics

6/22/2009

Pattern Graphs

Used symbols and terminology

1) Let n be a positive integer. A pattern of length n is a two-line table $\binom{a_1\, a_2 \ldots a_n}{b_1\, b_2 \ldots b_n}$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are some rearrangements of the numbers 1, 2, . . . , n.
2) An A-ensnared (resp. B-ensnared) pattern is a pattern that is obtained from itself by applying the operation A (resp. the operation B). Snare is the operation of ensnaring of a pattern.
3) Cycle is a configuration of patterns $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$, where the pattern x_i is obtained from the pattern x_{i-1} for every $\boldsymbol{i} \in \{\mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n}\}$ and the pattern x_1 is obtained from the pattern x_n by the operation A or the operation B. A-cycle (resp. B-cycle) is a cycle in which every pattern Is obtained from the one before it by the operation A (resp. the operation B).
4) Ordered queue a is a queue which fulfill the following condition:
   - *When the number k is on the p* th *position in the queue a, than the number p is on the k* th *position in the queue a .The numbers k and p can be equal.*

Introduction to the theory of permutation group

In mathematics, a permutation group is a group *G* whose elements are permutations of a given set *M*, and whose group operation is the composition of permutations in *G* (which are thought of as bijective functions from the set *M* to itself); the relationship is often written as (*G*,*M*). Note that the group of *all* permutations of a set is the symmetric group; the term *permutation group* is usually restricted to mean a subgroup of the symmetric group. The symmetric group of *n* elements is denoted by S_n; if *M* is any finite or infinite set, then the group of all permutations of *M* is often written as Sym(*M*).

Closure properties

As a subgroup of a symmetric group, all that is necessary for a permutation group to satisfy the group axioms is that it contain the identity permutation, the inverse permutation of each permutation it contains, and be closed under composition of its permutations. A general property of finite groups implies that a finite subset of a symmetric group is again a group if and only if it is closed under the group operation.

Examples

Permutations are often written in *cyclic form*, e.g. during cycle index computations, so that given the set *M* = {1,2,3,4}, a permutation *g* of *M* with *g*(1) = 2, *g*(2) = 4, *g*(4) = 1 and *g*(3) = 3 will be written as (1,2,4)(3), or more commonly, (1,2,4) since 3 is left unchanged; if the objects are denoted by a single letter or digit, commas are also dispensed with, and we have a notation such as (1 2 4).

Consider the following set *G* of permutations of the set *M* = {1,2,3,4}:

- e = (1)(2)(3)(4)
  - This is the identity, the trivial permutation which fixes each element.
- a = (1 2)(3)(4) = (1 2)
  - This permutation interchanges 1 and 2, and fixes 3 and 4.
- b = (1)(2)(3 4) = (3 4)
  - Like the previous one, but exchanging 3 and 4, and fixing the others.
- ab = (1 2)(3 4)
  - This permutation, which is the composition of the previous two, exchanges simultaneously 1 with 2, and 3 with 4.

G forms a group, since $aa = bb = e$, $ba = ab$, and $baba = e$. So (G,M) forms a permutation group.

More generally, *every* group G is isomorphic to a permutation group by virtue of its regular action on G as a set; this is the content of Cayley's theorem.

Isomorphisms

If G and H are two permutation groups on the same set X, then we say that G and H are *isomorphic as permutation groups* if there exists a bijective map $f : X \to X$ such that $r \longmapsto f^{-1} \circ r \circ f$ defines a bijective map between G and H; in other words, if for each element g in G, there is a unique h_g in H such that for all x in X, $(g \circ f)(x) = (f \circ h_g)(x)$. This is equivalent to G and H being conjugate as subgroups of S_X. In this case, G and H are also isomorphic as groups.

Notice that different permutation groups may well be isomorphic as abstract groups, but not as permutation groups. For instance, the permutation group on {1,2,3,4} described above is isomorphic as a group (but not as a permutation group) to {(1)(2)(3)(4), (12)(34), (13)(24), (14)(23)}. Both are isomorphic as groups to the Klein group V_4.

Transpositions, simple transpositions, inversions and sorting

A 2-cycle is known as a transposition. A *simple transposition* in S_n is a 2-cycle of the form $(i \;\; i + 1)$.

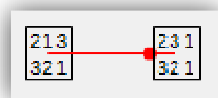
For a permutation p in S_n, a pair $(i\,,j) \in I_n$ is a *permutation inversion*, if when $i<j$, we have $p(i) > p(j)$.

It can be shown that every permutation can be written as a product of simple transpositions; furthermore, the number of simple transpositions one can write a permutation p in S_n can be the number of inversions of p and if the number of inversions in p is odd or even the number of transpositions in p will also be odd or even corresponding to the oddness of p, and that it is possible to find such a product—in fact, this is what insertion sort does implicitly (instead of giving the simple transpositions as output, it applies them to the input list).

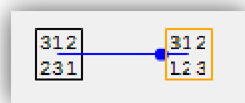
Elementary properties of the pattern graphs

Let $x \equiv \binom{a_1}{b_1}$ and $y = \binom{a_2}{b_2}$ are two different patterns.

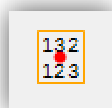
1) By definition, if the pattern x is obtained from the pattern y by applying the operation **A**, than$\boldsymbol{b_1} \equiv \boldsymbol{b_2}$.
   An example:



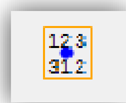
2) By definition, if the pattern x is obtained from the pattern y by applying the operation **A**, than $\boldsymbol{a} \equiv \boldsymbol{a_2}$.
   An example:



3) If the pattern x is A-ensnared,than $b_1 \equiv 1, 2, 3, 4, ..., n$.
   An example:



4) If the pattern x is B-ensnared,than $a_1 \equiv 1, 2, 3, 4, ..., n$.
   An example:



5) If the pattern x is obtained from the pattern y by applying the operation **A** (resp. the operation B), it is impossible the pattern y to be obtained from the pattern x by the operation B (resp. the operation A).
   Prove: Let assume that the opposite statement is true. Than using the properties 1) and 2) we can conclude that x $\equiv$ y.
6) The functions defined by the operations A and B are reversible: if A(x) $\equiv$ A(y) or B(x) $\equiv$B(y) , than x $\equiv$ y.

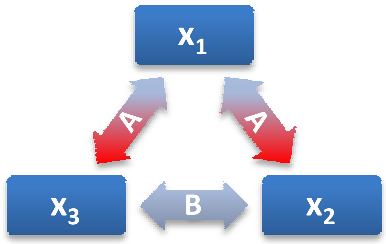
Problem 2:

How many connected components of the graph G_n have exactly (a) 2 patterns? (b) 3 patterns?

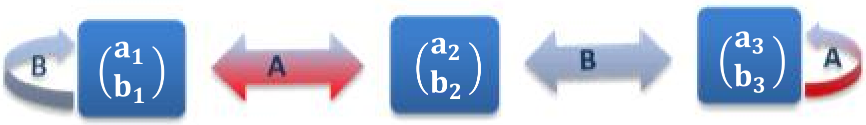
Solution:

a. We will prove that there are no components of the graph G_n containing exactly two patterns. Let assume that the opposite statement is true. It is obvious that if there is a component that has exactly two patterns, both patterns have to be ensnared and each one has to be obtained by the other. Let these two patterns are $x \equiv \binom{a_1}{b_1}$ and $y = \binom{a_2}{b_2}$. Let without lost of generalization the pattern x is A-ensnared. Using property 3) we can conclude that $b_1 \equiv 1, 2, 3, 4, ..., n$. We can also conclude that both patterns are obtained from one another by the operation B. Therefore from pattern 2), $a_1 \equiv a_2$. It is also obvious that the second pattern is A-ensnared. Because of that, again from property 3), we can conclude that $b_2 \equiv 1, 2, 3, 4, ..., n$. Finally, x ≡ y, which is an obvious contradiction. *So we prove that the number of the components of the graph Gn that contain exactly two patterns is zero.*

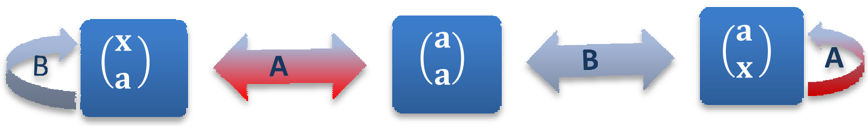
b. In this point will find the number of the components of the graph G_n that contains exactly three patterns. Firstly, let study the configuration when the patterns x_1, x_2 and x_3 form a cycle:



Without lost of generalization let the pattern x_2 to be obtained from the pattern x_1 by the operation A and vice versa. Therefore the pattern x_3 is obtained from x_2 by the operation B and vice versa. Therefore the pattern x_1 is obtained from the pattern x_3 by the operation A and vice versa, which is an obvious contradiction, because the operation A is already applied on the pattern x_1, and therefore the pattern x_2 is obtained (We use the properties 5) and 6)).
Than we can conclude that the only possible configuration for a component exactly three patterns $x_1 \equiv \binom{a_1}{b_1}$, $x_2 \equiv \binom{a_2}{b_2}$, $x_3 \equiv \binom{a_3}{b_3}$ is the following:



Let without lost of generalization the pattern x_1 is B-ensnared. Therefore $a_1 \equiv 1, 2, 3, 4, ..., n$ and also x_2 is obtained from x_1 by the operation A and vice versa. Therefore $b_1 \equiv b_2$ and also x_3 is obtained from x_2 by operation B and vice versa. Therefore $a_2 \equiv a_3$ and also $b_3 \equiv 1, 2, 3, 4, ..., n$. We can also conclude that this configuration is possible if $a_2 \equiv b_2$. So $a_3 \equiv a_2 \equiv b_2 \equiv b_1$. Therefore the component is of the following sort:



Where $x \equiv 1, 2, 3, 4, ..., n$ and a is an ordered queue. It is important to emphasize that the queue x is also an ordered queue. Therefore the number of different components that contains exactly three patterns is one less than the number of different ordered queues because the queues x and a have to be different.

We will study two cases considering the parity of n, to find the number of ordered queues.

- Firstly, let n is even. We can divide the queue a into r pares of numbers of the sort $(p_1, k_1), (p_2, k_2), (p_3, k_3), ..., (p_r, k_r)$, for which the statement defining an ordered queue holds, and $n/2 - r$ pares of the sort $(k_{r+1}, k_{r+2}), (k_{r+3}, k_{r+4}), ..., (k_{n/2-1}, k_{n/2})$, where the numbers $k_{r+1}, k_{r+2}, k_{r+3}, ..., k_{n/2}$ are on positions as follows: k_{r+1}-st, k_{r+2}-nd, k_{r+3}-rd, ..., $k_{n/2}$-th. Therefore, there are $n/2$ pares. There are $(n(n-1))$ possibilities for the first pair, $((n-2)(n-3))$ for the second one, and so on, there are (2.1) possibilities for $(n/2)$-th pair. We also have to divide the multiplication by $\left(\frac{n}{2}\right)!$ because the pairs are unordered. Therefore, the number of the ordered queues a is:

$$\frac{(n(n-1)).((n-2)(n-3)).....(3.4).(2.1)}{\left(\frac{n}{2}\right)!} = \frac{n!}{\left(\frac{n}{2}\right)!}$$

- In analogical way let divide the queue a into r groups of the sort $(p_1, k_1), (p_2, k_2), (p_3, k_3), ..., (p_r, k_r)$, for which the statement defining an ordered queue holds, and $(n-1)/2 - r$ groups of the sort $(k_{r+1}, k_{r+2}), (k_{r+3}, k_{r+4}), ..., (k_{(n-1)/2-1}, k_{(n-1)/2})$, where the numbers $k_{r+1}, k_{r+2}, k_{r+3}, ..., k_{(n-1)/2}$ are on positions as follows: k_{r+1}-st, k_{r+2}-nd, k_{r+3}-rd, ..., $k_{(n-1)/2}$-th and one group that contains exactly one number $k_{(n+1)/2}$, which is on $k_{(n+1)/2}$-th position. Therefore there are $(n+1)/2$ groups. Следователно Therefore, the number of the ordered queues a is:

$$\frac{n.((n-1)(n-2)).((n-3)(n-4)).....(3.4).(2.1)}{\left(\frac{n+1}{2}\right)!} = \frac{n!}{\left(\frac{n+1}{2}\right)!}$$

Finally, we can conclude that the number of the components of the graph G_n is:

Problem 6

$\frac{(n(n-1)).((n-2)(n-3)).....(3.4).(2.1)}{\left(\frac{n}{2}\right)!} = \frac{n!}{\left(\frac{n}{2}\right)!} - 1$, *for even n and* $\frac{n.((n-1)(n-2)).((n-3)(n-4)).....(3.4).(2.1)}{\left(\frac{n+1}{2}\right)!} = \frac{n!}{\left(\frac{n+1}{2}\right)!} - 1$, *for odd n.*

Hypothesis

1) If the patterns $x \equiv \binom{a}{c}$ and $y \equiv \binom{b}{c}$ are part of one component then they are also part of one **A**-cycle.
2) If the patterns $x \equiv \binom{c}{a}$ and $y \equiv \binom{c}{b}$ are part of one component then they are also part of one B-cycle.

Statements

1) Every patterns $x \equiv \binom{a}{c}$ and $y \equiv \binom{b}{c}$ are part of two **A**-cycles with equal length.
2) Every patterns $x \equiv \binom{c}{a}$ and $y \equiv \binom{c}{b}$ are part of two B-cycles with equal length.

Proof

We will prove only the first statement. The other statement can be proved in an analogical way.

Let study the properties of the patterns using the theory of permutation groups. Every pattern is an ordered pair of permutations which can be expressed as multiplication of cycles. Let study the pattern $x \equiv \binom{a}{c}$. Let z $\equiv \binom{a'}{c}$ is obtained from the pattern x by operation A. We can conclude that by definition of the operation A

$$a' = ca.$$

Let us study the A-cycle of the pattern x. From the conclusion above, we can conclude that every pattern that is part of the A-cycle of the pattern x is of the following sort:

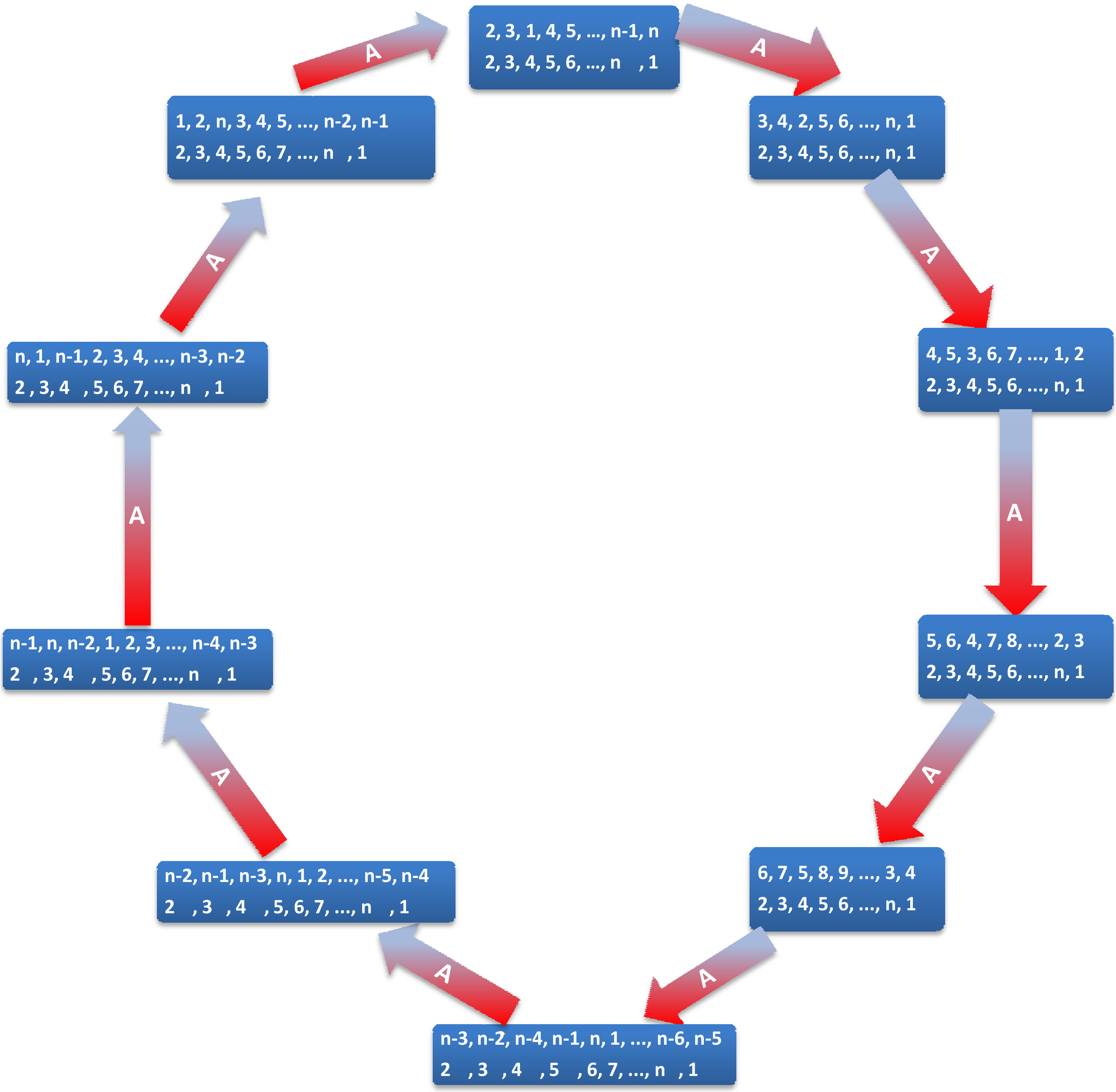
$$\binom{c^i a}{c}$$

Let the order of c to be k. Therefore, the A-cycle of the pattern x contains k patterns. Then we can conclude that the length of an A-cycle doesn't depend on the first row of the pattern. Therefore the pattern $x \equiv \binom{a}{c}$ and $y \equiv \binom{b}{c}$ are part of two **A**-cycles with length k (equal length), what we have to prove.

Problem 1

Can the pattern $\begin{matrix} 2\;1\;3\;4\ldots n-1\;n \\ 2\;3\;4\;5\ldots n\;1 \end{matrix}$ be obtained from the pattern $\begin{matrix} 2\;3\;1\;4\ldots n\;-\;1\;n \\ 2\;3\;4\;5\ldots n\;1 \end{matrix}$ using the operations A and B?

Let a ≡ 2, 3, 1, 4, ..., n-1, n; b ≡ 2, 1, 3, 4, ..., n-1, n; c ≡ 2,3,4, ..., n, 1. We will prove that the pattern $x \equiv \binom{a}{c}$ cannot be obtained from the pattern $y \equiv \binom{b}{c}$, using the operations A and B.



Let assume that the opposite statement is true. The pattern y can be obtained from the pattern x, if and only if both patterns are part of one component of the graph. Using hypothesis 1) we can conclude that the patterns x and y must be part of one A-cycle. Let study the **A**-cycle of the pattern x (see the figure on the previous page).
We can see that the pattern y is not part of the A-cycle of the pattern x, which is an obvious contradiction. *Therefore, we prove that it is impossible to obtain the pattern y from the pattern x.*

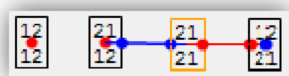
Problem 3

Denote by g_n the number of connected components of the graph G_n. Find g_n (give a formula) or estimate it (give lower and upper bounds).

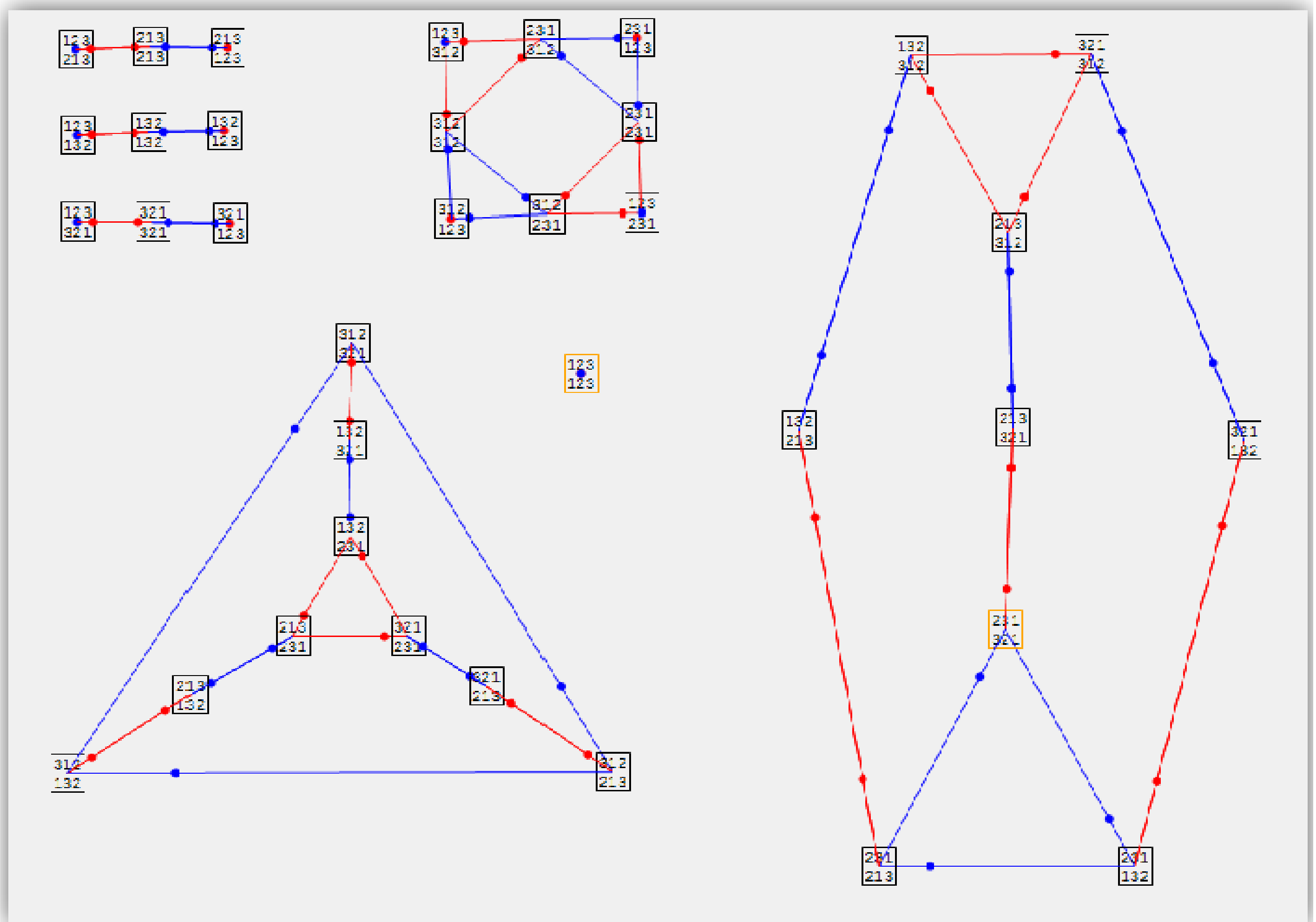
Hypothesis for the solution

$$\boldsymbol{g(n) = (n-1)^{n-1} + (n-1)^{n-2} + (n-1)^{n-3} + \dots + (n-1)^2 + (n-1)^1 + (n-1)^0}$$

Let verify the formula for $n = 2$. Let substitute n = 3. Therefore $g(2) = 2$, which is obviously true:



Let verify the formula for n = 3 . Let substitute n = 3 . Therefore g(3) = 7 , which is true from the text of the problem:



Estimation of g(n)

Let study the patterns of the sort x = a|(1, 2, 3, 4, ..., n-1, n). They are n!. Each of the patterns is an A-cycle with length one. From the hypothesis 1) we can conclude that each of these patterns is in a different component of the graph G_n. Therefore:

$$g(n) \geq n!$$

The number of the components has to be no more than the number of different patterns. Therefore: $g(n) \leq (n!)^2$

Finally, $(\boldsymbol{n}!)^2 \geq \boldsymbol{g}(\boldsymbol{n}) \geq \boldsymbol{n}!$.

References

1) John D. Dixon and Brian Mortimer. *Permutation Groups*. Number 163 in Graduate Texts in Mathematics. Springer-Verlag, 1996.
2) Peter J. Cameron. *Permutation Groups*. LMS Student Text 45. Cambridge University Press, Cambridge, 1999.
3) Siderov Plamen. *Notes of Algebra, groups, rings and polynomials.* VEDI, Sofia, 1995 /in Bulgarian/
4) Grossman I., Magnus V. *Groups and Their Graphs.* MIR, Moscow, 1971 /in Russian/
5) Alexandrov P. S. *Introduction to the Group Theory.* KVANTUM, Moscow, 2008 /in Russian/
6) Software: Manilov Stanislav. *Graph Generator*