2009

Problem 4

Minimality of Inscribed Polygons

We say that a polygon P is inscribed in a polygon Q if the vertices of P lie on the edges of Q, no two on the same edge. (A vertex of an inscribed polygon P is not allowed to coincide with a vertex of the polygon Q.)

1. A triangle T is inscribed in a triangle ABC, so that ABC is divided into four triangles T_1, T_2, T_3 and T (see the picture).
   (a) Is it always true that area(T) $\geq$ min{area(T_1), area(T_2), area(T_3)}?
   (b) Can T have a bisector (median, perimeter, angle, inscribed or circumscribed circle, etc.) smaller than all bisectors (medians, perimeters, angles, inscribed or circumscribed circles, etc.) triangles T_1, T_2 and T_3?

2. A convex polygon $P = P_1P_2...P_m$ is inscribed in a convex polygon $Q = Q_1Q_2...Q_n$, where $3 \leq m \leq n$, so that Q is divided into $m + 1$ parts. Can the polygon P possess a minimality property compared to the parts (for instance, can P have the smallest area, perimeter, angle, diagonal, etc.)?

3. Formulate and investigate 3-dimentional analogs of the problem.

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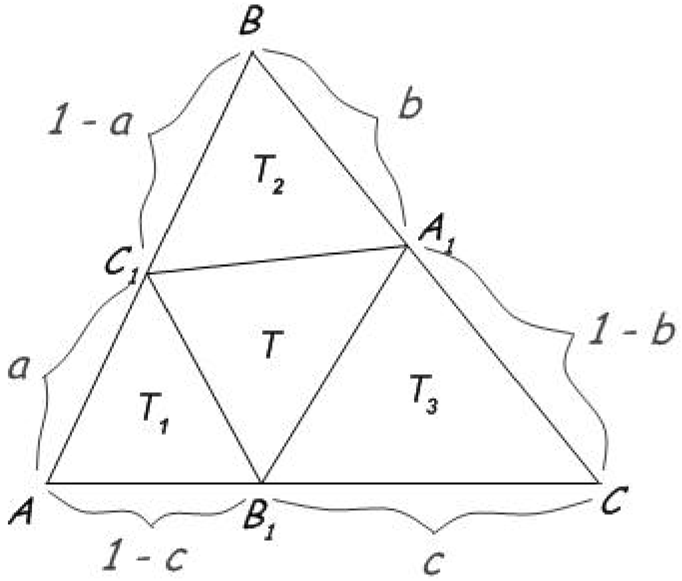
1. A triangle T is inscribed in a triangle ABC, so that ABC is divided into four triangles T_1, T_2, T_3 and T (see the picture).

(a) Is it always true that area(T) ≥ min{area(T_1), area(T_2), area(T_3)}?

(b) Can T have a bisector (median, perimeter, angle, inscribed or circumscribed circle, etc.) smaller than all bisectors (medians, perimeters, angles, inscribed or circumscribed circles, etc.) triangles T_1, T_2 and T_3?

Let the given inscribed triangle T be $A_1B_1C_1$, where points A_1, B_1 and C_1 are as shown in the following figure:



(a) Let S = area(ABC), and a, b and c be real positive numbers less than 1 such that

$$a = \frac{AB_1}{AC}, b = \frac{CA_1}{CB}, c = \frac{BC_1}{BA}$$

$\Rightarrow$ area(T_1) = $a(1 - c)S$, area(T_2) = $b(1 - a)S$, area(T_3) = $c(1 - b)S$

Without lost of generality, let $a \le b \le c \Rightarrow 1 - a \ge 1 - b \ge 1 - c$

$\Rightarrow$ min{ area(T_1), area(T_2), area(T_3)} = area(T_1)

area(T_1) = $(a - ac)$ area(ABC)

area(T) = S - area(T_1) - area(T_2) - area(T_3) =

$= S - a(1 - c)S - b(1 - a)S - c(1 - b)S =$

$= S\,(1 - a - b - c + ab + bc + ca)$

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We want to check whether the statement 'area(T) $\geq$ area(T_1)' holds, so we will investigate the following function:

$$f(a, b, c) = area(T) - area(T_1) = S(1 - a - b - c + ab + bc + ca) - (a - ac)S =$$

$$= 1 - 2a - b - c + ab + bc + 2ac$$

The necessary condition (*see the explanation at the end of (a)*) for the function f(a, b, c) to have an extremum is the following:

$$\begin{vmatrix} f'_a(a,b,c) = 0 \\ f'_b(a,b,c) = 0 \\ f'_c(a,b,c) = 0 \end{vmatrix} \Rightarrow \begin{vmatrix} -2 + b + 2c = 0 \\ -1 + a + c = 0 \\ -1 + b + 2a = 0 \end{vmatrix} \Rightarrow \begin{vmatrix} a = \frac{1}{4} \\ b = \frac{1}{2} \\ c = \frac{3}{4} \end{vmatrix}$$

The sufficient condition for f(a, b, c) to be an extremum is:

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0,$$

where $a_{11} = f''_{aa}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $a_{12} = f''_{ab}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $a_{13} = f''_{ac}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $a_{22} = f''_{bb}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, $a_{33} = f''_{cc}(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$ and so on.

$a_{11} = a_{22} = a_{33} = 0$, $a_{12} = a_{21} = 1$, $a_{13} = a_{31} = 2$, $a_{23} = a_{32} = 1$

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4 > 0 \Rightarrow \text{this part of the sufficient condition is true.}$$

However, $a_{11} = 0$ and $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

$\Rightarrow$ It is not clear whether $f(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$ is an extremum or not, because $a_{11} = 0$. If a_{11} were less than 0, then we would be sure that $f(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$ is not an extremum.

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Now let's investigate the cases in which a, b, and c tend towards 0 or 1.

If a, b and c tend towards 1, then the area of T tends towards the area of triangle ABC $\Rightarrow$ $f(a, b, c) = area(T) - area\,(T_1)$ tends towards the area of triangle ABC.

If a, b and c tend towards 0, the case is the same as the previous one, because 1-a, 1-b and 1-c tend towards 1.

If *a* and *b* tend towards 0 and c tends towards 1.

In this case the smallest areas would be the ones of T and T_1. The areas of triangles T and T_1 both tend towards 0, but the significant in this case is how "fast" each of the areas tends towards 0. T and T_1 share side B_1C_1. As a and b tend towards 0 and c tends towards 1, the height of T tends towards the height of triangle ABC perpendicular to the side AB, and the height of T_1 tends towards 0. Therefore, the area of triangle T_1 tends "faster" towards 0 than the area of triangle T $\Rightarrow$ $f(a, b, c) > 0$

If *c* and *b* tend towards 1 and *a* tends towards 0.

This case is analogous to the previous one, and the result is the same.

As a conclusion, when a, b and c tend towards 0 or 1 $f(a, b, c) > 0$.

$f(a, b, c)$ can be 0. This can be achieved when $a = b = c = \frac{1}{2}$.

In addition, the only way to get a something close to an extremum is the following case: $f(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$. But even if it was a minimum, the value of $f(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$ is $\frac{1}{8} > 0$.

On the basis of our arguments so far, we are inclined to think that $f(a, b, c) \geq 0$ for all possible values of a, b and c.

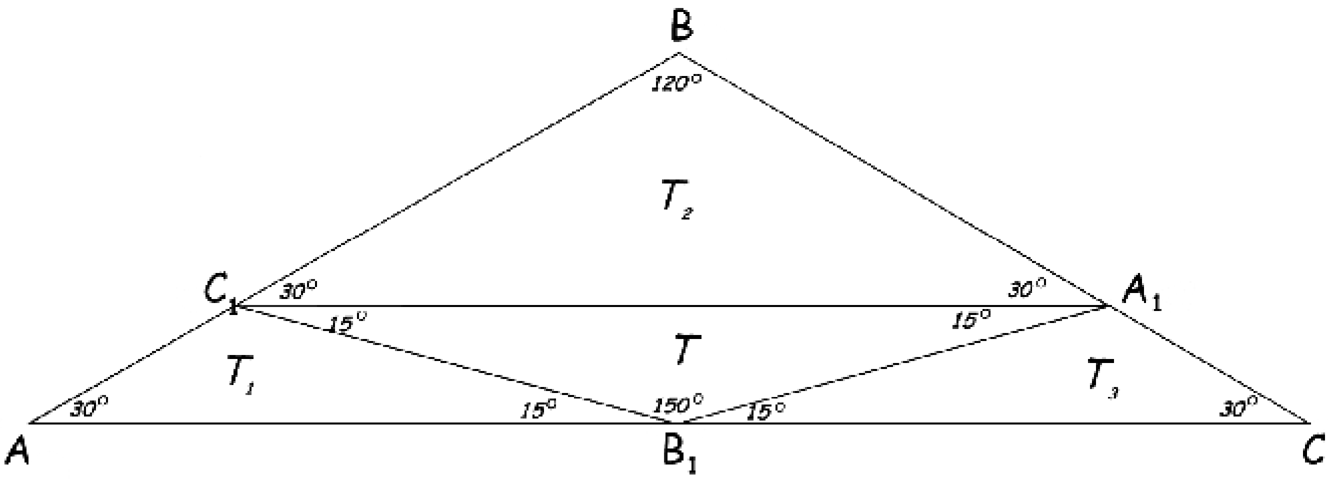
Explanation: The necessary and sufficient conditions that are used in the problem are taken from *Г. М. Фихтенгольц*, volume 1 (from Russian).

Problem 4

(b)

Can T have a bisector (median) smaller than all bisectors (medians) of the trianglesT_1, T_2 and T_3?

The case about the bisector and the one about the median are put together since the solution for both of them is the same. Our assumption is that T can have the smallest bisector (median), so we will need an example in order to prove the statement. Let's have a look at the following triangle:



Let l, l_1, l_2 and l_3 are the smallest bisectors of triangles T, T_1, T_2 and T_3, respectively. Each one of the bisectors is drawn towards the biggest side of the respective triangle (this is a well-known theorem). For example, l is drawn from point B_1 towards the side A_1C_1.

Bisector l is perpendicular to A_1C_1, because triangle $A_1B_1C_1$ is isosceles ($A_1B_1 = B_1C_1$). Therefore, l is the smallest segment with one of the vertices in point B_1 and the other vertex lying on the line defined by the side A_1C_1.

$AC \parallel A_1C_1$, because triangle ABC is also isosceles ($AB = BC$). Consequently, the bisector l is the smallest distance between the lines defined by the segments AC and A_1C_1.

The bisectors l_1 and l_3 are with one of the vertices at point C_1 and point A_1, respectively, and the other vertex on the side AC. l_1 and l_3 are not perpendicular to the side AC, because the triangles T_1 and T_3 are not isosceles. Therefore, l_1 and l_3 are smaller than l.

All that is left is to compare l with l_2. The bisector l_2 has one of its vertices in point B and the other one on the side A_1C_1. Moreover, l_2 is perpendicular to the side A_1C_1, because triangle T_2 is isosceles. For l and l_2 we have:

$$l = \frac{1}{2}\operatorname{tg}15^{\circ} A_1C_1; \qquad l_2 = \frac{1}{2}\operatorname{tg}30^{\circ} A_1C_1$$

$$\Rightarrow l < l_2$$

As a conclusion, triangle T can have the smallest bisector.

The case regarding the medians is analogous.

Let m, m_1, m_2 and m_3 are the smallest medians of triangles T, T_1, T_2 and T_3, respectively. Each one of the medians is drawn towards the biggest side of the respective triangle. For example, m is drawn from point B_1 towards the side A_1C_1.

The median m is perpendicular to side A_1C_1, because triangle $A_1B_1C_1$ is isosceles ($A_1B_1 = B_1C_1$). Therefore, m is the smallest segment with one of its vertices at point B_1 and the other vertex lying on the line defined by the side A_1C_1.

$AC \parallel A_1C_1$, because triangle ABC is also isosceles ($AB = BC$). Consequently, the bisector m is the smallest distance between the lines defined by the segments AC and A_1C_1.

The medians m_1 and m_3 are with one of the vertices at point C_1 and point A_1, respectively, and the other vertex on the side AC. m_1 and m_3 are not perpendicular to the side AC, because the triangles T_1 and T_3 are not isosceles. Therefore, m_1 and m_3 are smaller than m.

All that is left is to compare m with m_2. The median m_2 has one of the vertices at point B and the other one on the side A_1C_1. Moreover, m_2 is perpendicular to side A_1C_1, because triangle T_2 is isosceles. For m and m_2 we have:

$$m = \frac{1}{2}\operatorname{tg}15^{\circ} A_1C_1 \qquad m_2 = \frac{1}{2}\operatorname{tg}30^{\circ} A_1C_1$$

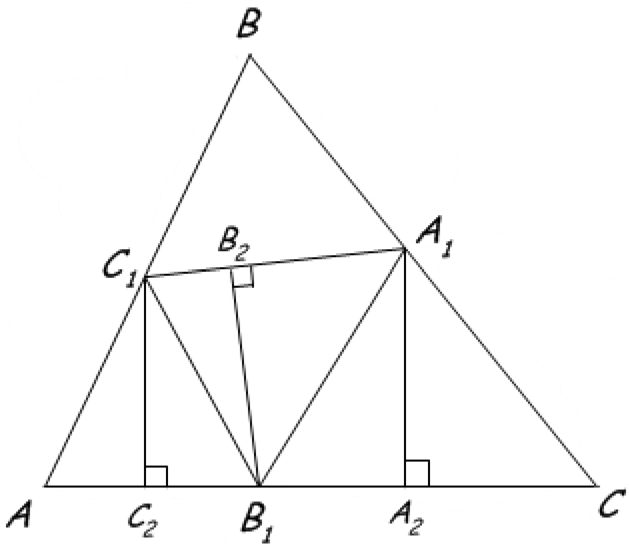
$$\Rightarrow m < m_2$$

As a conclusion, triangle T can have the smallest median.

Can T have a height smaller than all heights of the triangles T_1, T_2 and T_3?

The case about the height is not among one of the mentioned in Problem 4, but we are going to look at it since it is an interesting case.

Let A_1A_2, B_1B_2 and C_1C_2 be heights in, respectively, triangles AB_1C_1, $B_1C_1A_1$ and CA_1B as shown in the figure below and B_1B_2 is the smallest height in triangle $A_1B_1C_1$.



Let's assume that B_1B_2 is the smallest of all heights. $\Rightarrow$ $B_1B_2 < A_1A_2$ and $B_1B_2 < C_1C_2$

Let's look at the triangles $B_1B_2A_1$ and $B_1A_1A_2$. They are both right triangles with equal hypotenuses. $B_1B_2 < C_1C_2 \Rightarrow \angle B_2A_1B_1 < \angle A_1B_1A_2$

By analogy $B_1B_2 < A_1A_2 \Rightarrow \angle B_2C_1B_1 < \angle C_1B_1C_2$

$180^0 = \angle C_1B_1C_2 + \angle C_1B_1A_1 + \angle A_1B_1A_2 > \angle B_1C_1B_2 + \angle C_1B_1A_1 + \angle B_1A_1B_2 = 180^0$

$\Rightarrow$ B_1B_2 cannot be smaller than A_1A_2 and C_1C_2 at the same time.

Therefore, T cannot have the smallest height.

This raises the question: Can T have a height bigger than all heights of the triangles T_1, T_2 and T_3?

In order to solve this problem we will use the same symbols, only this time B_1B_2 is by assumption the biggest of all heights.

$\Rightarrow$ $B_1B_2 > A_1A_2$ and $B_1B_2 > C_1C_2$

We again look at the triangles $B_1B_2A_1$ and $B_1A_1A_2$. They are both right triangles with equal hypotenuses. $B_1B_2 > C_1C_2 \Rightarrow \angle B_2A_1B_1 > \angle A_1B_1A_2$

Analogously $B_1B_2 > A_1A_2 \Rightarrow \angle B_2C_1B_1 > \angle C_1B_1C_2$

$180^0 = \angle C_1B_1C_2 + \angle C_1B_1A_1 + \angle A_1B_1A_2 < \angle B_1C_1B_2 + \angle C_1B_1A_1 + \angle B_1A_1B_2 = 180^0$

$\Rightarrow$ B_1B_2 cannot be bigger than A_1A_2 and C_1C_2 at the same time.

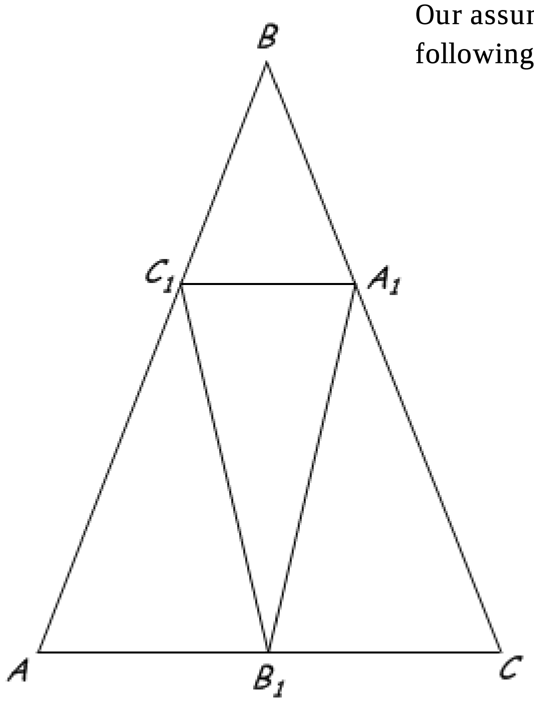
Therefore, T cannot have the biggest height.

Conclusion: T cannot have the biggest or the smallest height.

Can T have a side smaller than all sides of the triangles T_1, T_2 and T_3?

This is another case not mentioned in the problem.

The case is about the smallest side of the triangles T_1, T_2 and T_3. Though each side of triangle T is a side of one of the other triangles as well, we will consider each side as a side of both triangles. In other words, if the smallest side of the triangles T_1, T_2 and T_3 is also a side of T, then T has the smallest side and this case is solved.



Our assumption this time is that T can have the smallest side, and the following example proves us right:

In this figure the following holds:

- $\angle ABC < 60^0$
- triangle ABC is isosceles ($AB = BC$)
- $A_1C_1 \parallel AC$
- $BC_1 < \frac{1}{2} BA$

$\angle ABC$ is the smallest in the triangle A_1BC_1; whence, A_1C_1 is smaller than BC_1 and A_1B.

$A_1C_1 < BC_1 = A_1B < A_1C = AC_1$

triangle ABC ~ triangle A_1BC_1

(because $A_1C_1 \parallel AC$) In addition, $BC_1 < \frac{1}{2} BA$

$\Rightarrow A_1C_1 < \frac{1}{2} AC = AB_1 = B_1C$

All that is left is to prove that A_1C_1 is smaller than B_1C_1 and A_1B_1 for A_1C_1 to be the smallest of all sides. However, we do not need A_1C_1 to be the smallest. Even if one of the sides B_1C_1 and A_1B_1 is smaller than B_1C_1 and A_1B_1, we have already proved that the smallest side is part of the triangle T, which is enough for our assumption to be proved.

As a conclusion, triangle T can have the smallest side.

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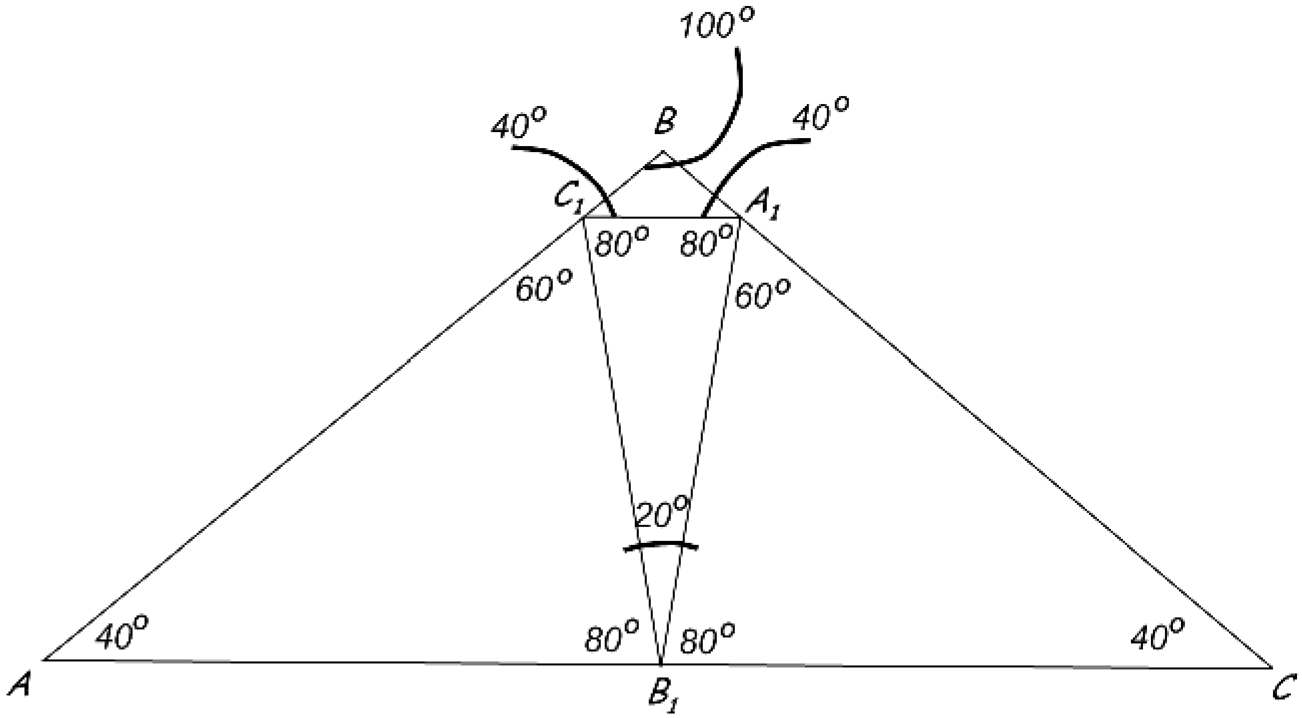
Can T have an angle smaller than all angles of the triangles T_1, T_2 and T_3?

Let's look at the following figure:

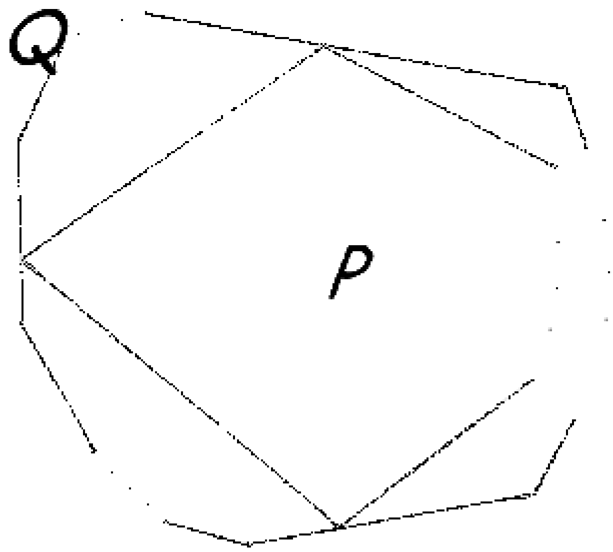


Such a triangle exists and $\angle A_1B_1C_1$ – an angle from triangle T, is obviously the smallest of all.

As a conclusion, triangle T can have the smallest angle.

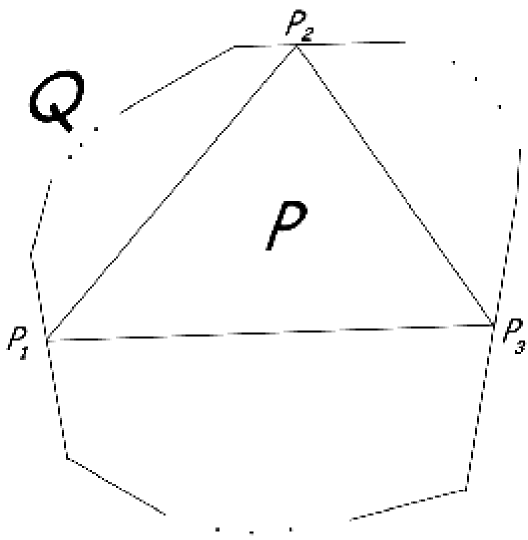
2. A convex polygon $P = P_1P_2...P_m$ is inscribed in a convex polygon $Q = Q_1Q_2...Q_n$, where $3 \leq m \leq n$, so that Q is divided into m + 1 parts. Can the polygon P possess a minimality property compared to the parts (for instance, can P have the smallest area, perimeter, angle, diagonal, etc.)?

The figure of the convex polygon $P = P_1P_2...P_m$ inscribed in the convex polygon $Q = Q_1Q_2... Q_n$ is of the sort:



<u>Can *P* have area smaller than the areas of all other polygons into which *Q* is divided by *P*?</u>

To solve this problem first let's investigate the case <u>$m = 3$</u>.



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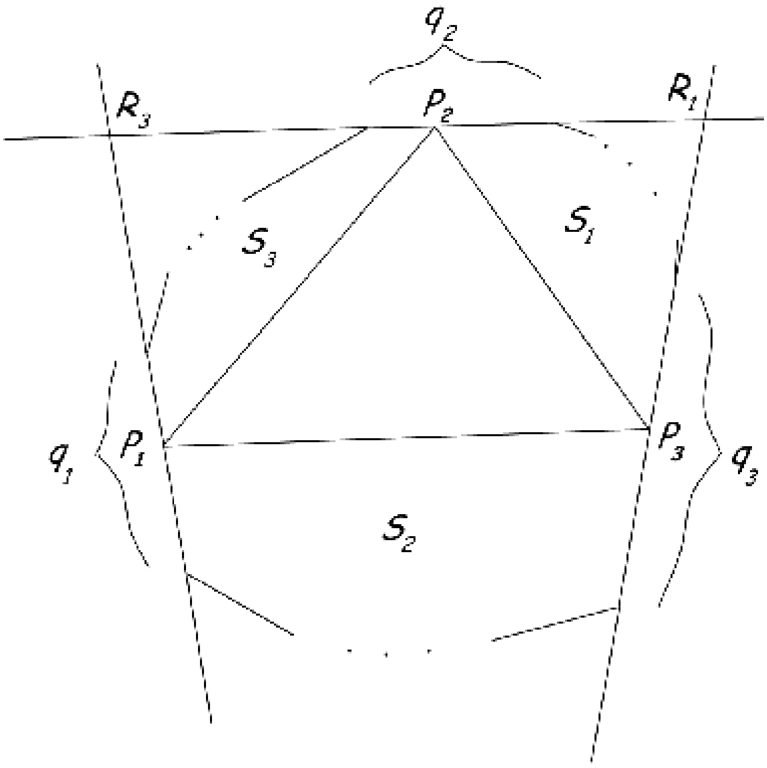
Let the following be true:

- q_1, q_2 and q_3 are the sides on which the points P_1, P_2 and P_3 lie, respectively;
- $q_1 \cap q_2 = R_3$, $q_3 \cap q_2 = R_1$, $q_1 \cap q_3 = R_2$;
- S_1, S_2 and S_3 are the tree parts other than P into which Q is divided (look at the figure bellow)



(In the figure above point R_2 is not shown, but the idea can still be conveyed clearly.)

Let's look at triangle $R_1R_2R_3$. Triangle $P_1P_2P_3$ is inscribed in $R_1R_2R_3$. In other words, we have the same case as in problem 4-1-(a). Therefore, we can use the result from 4-1-(a).

$\Rightarrow$ area($P_1P_2P_3$) $\geq$ min{area($P_1P_2R_3$), area($P_1R_2P_3$), area($R_1P_2P_3$)}

We also have:

area($R_1P_2P_3$) $\geq$ area(S_1)

area($P_1R_2P_3$) $\geq$ area(S_2)

area($P_1P_2R_3$) $\geq$ area(S_3)

$\Rightarrow$ area($P_1P_2P_3$) $\geq$ min{area(S_1), area(S_2), area(S_3)}

Thus, the problem is solved in the case $m = 3$.

Problem 4

<u>For $m > 3$</u>

In this case it is enough to choose any triangle $P_iP_jP_k$ (i, j and $k \in \mathbb{N}$ and i, j and $k \leq m$). For $P_iP_jP_k$ it is true that $P_iP_jP_k$ cannot have minimal area compared to the parts into which Q is divided.

$\text{area}(P) \geq \text{area}(P_iP_jP_k)$

$\Rightarrow P$ cannot have minimal area compared to the parts into which Q is divided.