2009

Problem 3

Monotonic squares

We say that a positive integer $a = a_k a_{k-1} \ldots a_1 a_0$, where $0 \le a_i \le 9$ are the digits of a in base 10, is an increasing square if $a = b^2$ for some integer b and $a_k \ge a_{k-1} \ge \ldots \ge a_1 \ge a_0$. For instance, $13456 = 116^2$.
If we have the reverse inequality $a_k \le a_{k-1} \le \ldots \le a_1 \le a_0$, then the square a is called *decreasing* . For instance, $8874441 = 2979^2$.
Let $a = b^2$ and $c = d^2$ be two squares with the following representation in base 10:

$a = a_k a_{k-1} \ldots a_1 a_0$, $b = b_k b_{k-1} \ldots b_1 b_0$ and $c = c_k c_{k-1} \ldots c_1 c_0$, $d = d_k d_{k-1} \ldots d_1 d_0$.

We say that a pair of squares a and c is ordered, and write $a \prec c$, if the sequence $a_0, a_1, \ldots, a_{k-1}, a_k$ is a subsequence of $c_0, c_1, \ldots, c_{k-1}, c_k$ and the sequence $b_0, b_1, \ldots, b_{k-1}, b_k$ is a subsequence of $d_0, d_1, \ldots, d_{k-1}, d_k$. For instance, $1156 = 34^2 \prec 111556 = 334^2$.
A set of squares F is called a *family* if any pair of squares from F is ordered.

1. Find infinite families of increasing squares. For instance, $1156 = 34^2 \prec 111556 = 334^2 \prec 11115556 = 3334^2 \prec \ldots$

2. Is there any infinite family of decreasing squares?

3. How many elements are in maximal increasing family? For example, can it have exactly (a) one increasing square? (b) two increasing squares? (A *maximal increasing* family F is a family of increasing squares, such that any increasing square a with the property "either $a \prec c$ or $c \prec a$ for all $c \in F$" is already in F.)

4. How many elements can a maximal family of decreasing squares have?

5. Investigate the problem in other bases.

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1. Find infinite families of increasing squares. For instance, $1156 = 34^2 \prec 111556 = 334^2 \prec 11115556 = 3334^2 \prec \ldots$

After investigating some one-digit and two-digit numbers, we find the following:

I. $3^2 = 9 \prec 37^2 = 1369 \prec 337^2 = 113569 \prec 3337^2 = 11135569 \prec \ldots$

II. $4^2 = 16 \prec 34^2 = 1156 \prec 334^2 = 111556 \prec 3334^2 = 11115556 \prec \ldots$

III. $5^2 = 25 \prec 35^2 = 1225 \prec 335^2 = 112225 \prec 3335^2 = 11122225 \prec \ldots$

IV. $7^2 = 49 \prec 67^2 = 4489 \prec 667^2 = 444889 \prec 6667^2 = 44448889 \prec \ldots$

Now we are going to make sure that the above are infinite families of increasing squares.

Let n be a natural number or 0.

$$\underbrace{1\ldots1}_{n+1}3\underbrace{5\ldots5}_{n}69 = \underbrace{1\ldots1}_{n+1}.10^{n+3} + 3.10^{n+2} + \underbrace{5\ldots5}_{n}.10^2 + 66.10 + 3 =$$

$$= \frac{10^{n+1}-1}{9}10^{n+3} + 3.10^{n+2} + 5.\frac{10^n-1}{9}10^2 + 6\frac{100-1}{9} + 3 =$$

$$= \frac{1}{9}(10^{2n+4} - 10^{n+3} + 27.10^{n+2} + 5.10^{n+2} - 500 + 600 - 6 + 27) =$$

$$= \frac{1}{9}(10^{2n+4} - 22.10^{n+2} + 121) = \left(\frac{10^{n+2}+11}{3}\right)^2 = \left(3\underbrace{\ldots}_{n+1}37\right)^2$$

$\Rightarrow$The first sequence is an infinite family of increasing squares.

$$\underbrace{1\ldots1}_{n+1}\underbrace{5\ldots5}_{n}6 = \underbrace{1\ldots1}_{n+1}.10^{n+1} + \underbrace{5\ldots5}_{n}.10 + 6 = \frac{10^{n+1}-1}{9}10^{n+1} + 5.\frac{10^n-1}{9}10 + 6 =$$

$$= \frac{1}{9}(10^{2n+2} - 10^{n+1} + 5.10^{n+1} - 50 + 64) = \frac{1}{9}(10^{2n+2} + 4.10^{n+1} + 4) =$$

$$= \frac{1}{9}(10^{2n+2} + 2.2.10^{n} + 2^2) = \left(\frac{10^{n+1}+2}{3}\right)^2 = \left(3\underbrace{\ldots}_{n}34\right)^2$$

$\Rightarrow$The second sequence is an infinite family of increasing squares.

$$\underbrace{1\ldots1}_{n}\underbrace{2\ldots2}_{n+1}5 = \underbrace{1\ldots1}_{n}.10^{n+2} + \underbrace{2\ldots2}_{n+1}.10 + 5 = \frac{10^{n}-1}{9}10^{n+2} + 2.\frac{10^{n+1}-1}{9}10 + 5 =$$

$$= \frac{1}{9}(10^{2n+2} - 10^{n+2} + 2.10^{n+2} - 20 + 45) = \frac{1}{9}(10^{2n+2} + 10^{n+2} + 25) =$$

$$= \frac{1}{9}(10^{2n+2} + 2.5.10^{n+1} + 5^2) = \left(\frac{10^{n+1}+5}{3}\right)^2 = \left(3\underbrace{\ldots}_{n}35\right)^2$$

$\Rightarrow$The third sequence is an infinite family of increasing squares.

$$\underbrace{4\ldots4}_{n+1}\underbrace{8\ldots8}_{n}9 = \underbrace{4\ldots4}_{n+1}.10^{n+1} + \underbrace{8\ldots8}_{n}.10 + 9 = 4.\frac{10^{n+1}-1}{9}10^{n+1} + 8.\frac{10^n-1}{9}10 + 9 =$$

$$= \frac{1}{9}(4.10^{2n+2} - 4.10^{n+1} + 8.10^{n+1} - 80 + 81) = \frac{1}{9}(4.10^{2n+2} + 4.10^{n+1} + 1) =$$

$$= \frac{1}{9}(2^2.10^{2n+2} + 2.2.10^{n} + 1^2) = \left(\frac{2.10^{n+1}+1}{3}\right)^2 = \left(6\underbrace{\ldots}_{n}67\right)^2$$

$\Rightarrow$The forth sequence is an infinite family of increasing squares.

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2. Is there any infinite family of decreasing squares?

Let's look at the following families of decreasing squares:
$1^2 = 1$ p $10^2 = 100$ p $100^2 = 10000$ p $1000^2 = 1000000$ p ...
$1^2 = 1$ p $21^2 = 441$ p $210^2 = 44100$ p $2100^2 = 4410000$ p ...
As a whole, if we have $a = b^2$ such that a is a decreasing square, we can add two zeros at the end of a and one zero at the end of b. Thus, we have a new decreasing square which is from the same family as a is. If we continue adding zeros the same way, we will have an infinite number of decreasing squares, and all of those squares will form a family.
As a conclusion, there exists an infinite family of decreasing squares.

3. How many elements are in maximal increasing family? For example, can it have exactly (a) one increasing square? (b) two increasing squares? (A maximal increasing family F is a family of increasing squares, such that any increasing square a with the property "either $a \prec c$ or $c \prec a$ for all $c \in F$" is already in F.)

To solve this point we are going to give an algorithm which will be used in order to find all possible families of increasing squares.

Let $X^2 = \underbrace{1\ldots1}_{a1}\underbrace{2\ldots2}_{a2}\underbrace{3\ldots3}_{a3}\underbrace{4\ldots4}_{a4}\ldots\underbrace{9\ldots9}_{a9}$ is an increasing square, where a_i ($i = 1, 2, \ldots, 9$) is a natural number or zero.
Note: zero is not included in X^2, because, obviously, a natural number cannot begin with zero.
Because of this reason and the fact that X^2 is an <u>*increasing*</u> square, nor X^2, or X can have a zero.

$$X^2 = \underbrace{1\ldots1}_{a1}\underbrace{2\ldots2}_{a2}\underbrace{3\ldots3}_{a3}\underbrace{4\ldots4}_{a4}\ldots\underbrace{9\ldots9}_{a9} = \sum_{i=1}^{9} i\frac{10^{a_i}-1}{9}10^{a_{i+1}+a_{i+2}+\ldots+a_9} =$$

$$= \frac{1}{9}(10^{a_1+\ldots+a_9} - 10^{a_2+\ldots+a_9} + 2.\,10^{a_2+\ldots+a_9} - 2.\,10^{a_3+\ldots+a_9} + \ldots + 9.\,10^{a_9} - 9) =$$

$$= \frac{1}{9}(10^{a_1+\ldots+a_9} + 10^{a_2+\ldots+a_9} + 10^{a_3+\ldots+a_9} + 10^{a_9} - 9)$$

Let: $A = 10^{a_1+\ldots+a_9} + 10^{a_2+\ldots+a_9} + 10^{a_3+\ldots+a_9} + 10^{a_9} - 9$
$B = 10^{a_1+\ldots+a_9} + 10^{a_2+\ldots+a_9} + 10^{a_3+\ldots+a_9} + 10^{a_9}$
X^2 and 9 are squares $\Rightarrow$ A is also a square $\Rightarrow$ A can end in 0, 9, 6, 5, 4 or 1 (because each square can end only on 0, 9, 6, 5, 4 or 1). Now we are going to investigate each of those cases.

<u>CASE 1:</u> Let A end in 0 $\Rightarrow$ B ends in 9 $\Rightarrow$ 9 of the members of B are equal to 1. b has 9 members $\Rightarrow$ all of them are equal to 1 $\Rightarrow a_i = 0$ ($i = 1, 2, \ldots, 9$)
This leads us to contradiction. As a conclusion, A cannot end in 9.

<u>CASE 2:</u> Let A end in 9 $\Rightarrow$ B ends in 8 $\Rightarrow$ 8 of the members of B are equal to 1.
$\Rightarrow a_i = 0$ ($i = 2, \ldots, 9$)
$\Rightarrow X^2 = \underbrace{1\ldots1}_{a1} \Rightarrow$ X end in 9 or 1.
<u>If X ends in 1</u>

If X = 1 we have a solution.

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If $X > 1 \Rightarrow X$ ends on at least 11 $\Rightarrow X^2$ ends on the last two digits of 11^2 which is 21 $\Rightarrow$ contradiction with $a_2 = 0$.

<u>If X ends in 9</u>

Let's make the following table:

X ends in	X . X ends in
9	81
19	61
29	41
39	21
49	01
59	81
69	61
79	41
89	21
99	01

$\Rightarrow$ contradiction.

<u>CASE 1:</u> Let A end in 6 $\Rightarrow$ B ends in 5 $\Rightarrow$ 5 of the members of B are equal to 1.

$\Rightarrow a_i = 0$ ($i = 5, \ldots, 9$)

$\Rightarrow X^2 = 1\underbrace{\ldots}_{a1}1\;2\underbrace{\ldots}_{a2}2\;3\underbrace{\ldots}_{a3}3\;4\underbrace{\ldots}_{a4}4$

X^2 is a square and X can be divided by 2 (because X^2 ends on an even number) $\Rightarrow X^2$ can be divided by 4. A number can be divided by 4 if and only if the last two numbers of the given number form a number divisible by 4 (This fact is well-known; therefore we wont write its proof) $\Rightarrow X^2$ can end only on 24 or 44.

X^2 ends in 4 $\Rightarrow$ X ends in 2 or 8.

<u>If X ends in 2</u>

$\Rightarrow X = 1\ldots12\ldots2$

If X ends in 22 $\Rightarrow X^2$ ends in 84 $\Rightarrow$ contradiction

$\Rightarrow$ X ends in 12 $\Rightarrow X^2$ ends in 44.

If $X = 12 \Rightarrow X^2 = 144 \Rightarrow$ we have a solution

If $X > 12 \Rightarrow$ X ends in 112 $\Rightarrow X^2$ ends in 544 $\Rightarrow$ contradiction

<u>If X ends in 8</u>

X^2 ends on 24 or 44

If X^2 ends in 24 $\Rightarrow$ X ends in 18 or 68

If $X = 18 \Rightarrow X^2 = 324 \Rightarrow$ contradiction

If $X > 18$ and X ends in 18 $\Rightarrow$ X ends in 118 $\Rightarrow X^2$ ends in 924

If X ends in 68
In order for X^2 to end in 24 or 44 $\Rightarrow$ X ends in 668 or 168

Thus, by trying all possible cases, we can find all possible increasing squares. Then we can investigate the result and check whether some of the squares form finite families.

4. How many elements can a maximal family of decreasing squares have?

Let's look back at point 2. From our explanations is clear that any maximum family of decreasing squares is infinite (because of the constant adding of zeros as shown in point 2.)
To find all possible decreasing squares which do not end in 0 we can use the method from point 3. Thus, we may be able to find the number of all maximum families of decreasing squares.

5. Investigate the problem in other bases.

Let's have the following example: $Y = 1\underbrace{0...0}_{2k}{}_{(p)}$, where Y is a number beginning with 1, with even number of zeroes after the digit 1 (k is a natural number or 0) and in base p.
$Y = 1\underbrace{0...0}_{2k}{}_{(p)} = p^{2k}$ which is in base p $\Rightarrow$ Y is a decreasing square. Since the fact that Y is a square no matter the value of k $\Rightarrow$ for any $k \in \mathbb{N}$ we have a decreasing square in base p and all of these squares belong to one and the same family. Thus, for any base p we always have infinite family of decreasing squares.

In point 4 we proved that in base 10 all maximum families of decreasing squares are infinite. The same goes for any base p. This can be easily verified by using the same method from point4.

As a conclusion, for any base p there exists a family of decreasing squares and all maximum families of decreasing squares are infinite.