2009

Problem 2

A Functional Equation

Let k be a constant real number.

1. Find some (all) functions $f: \mathbb{R} \to \mathbb{R}$ with the property that $f(f(x) + kx) = xf(x)$ for all real numbers x.

2. Find all solutions $f: \mathbb{R} \to \mathbb{R}$ of the functional equation

$$(1)\quad f(f(x) + f(y) + kxy) = xf(y) + yf(x),\ x, y \in \mathbb{R}.$$

Consider the case when f is (a) a polynomial, (b) a continuous function, (c) an arbitrary function.

3. Let $n > 2$ be a positive integer.
Find some (all) functions $f: \mathbb{R} \to \mathbb{R}$ such that
$f(f(x_1) + f(x_2) + \ldots + f(x_n) + kx_1x_2\ldots x_n) = x_1f(x_2) + x_2f(x_3) + \ldots + x_nf(x_1)$
for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

4. Suggest and investigate other generalizations of the functional equation (1).

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1. Find some (all) functions $f: R \rightarrow R$ with the property that $f(f(x) + kx) = xf(x)$ for all real numbers x.

The functional equation $f(f(x)+kx)=xf(x)$ has the following solutions:

$$f=\begin{cases} f(x)=0, x\neq 0 \\ f(0)=a, a\in R. \end{cases}$$

When k=0 we have additional solutions $f_1(x)=x^{\frac{1-\sqrt{5}}{2}}$, and $f_2(x)=x^{\frac{1+\sqrt{5}}{2}}$.

2. Find all solutions $f: R \rightarrow R$ of the functional equation(1) $f(f(x) + f(y) + kxy) = xf(y) + yf(x)$, $x, y \in R$. Consider the case when f is (a) a polynomial, (b) a continuous function, (c) an arbitrary function.

We consider the case when $f(x)$ is an arbitrary function. We will consider the following cases for *k:*

I. *k=0*

In this case we have that $f(f(x)+f(y))=xf(y)+yf(x)$.

If we take x=0, it follows that $f(f(0)+f(y))=yf(0)$. Then, if we take y=0 we obtain that $f(2f(0))=0$.

First, we suppose that $f(0)\neq 0$. We take $x=y=2f(0)$ and substitute in

$f(f(x)+f(y))=xf(y)+yf(x)$.

From the fact that $f(2f(0))=0$, it follows that $f(0)=0$ which contradicts the supposition.

Therefore we have that $f(0)=0$. If we substitute x with 0 in $f(f(x)+f(y))=xf(y)+yf(x)$, we get that $f(f(y))=0$. Then, if we take $x=f(x)$ and substitute it in $f(f(x)+f(y))=xf(y)+yf(x)$, it follows that $0=f(f(y))=f(f(f(x))+f(y))=f(x)f(y)$. When we take $x=y$, we get that $f^2(x)=0$. Hence, we get that $f\equiv 0$.

II. $k\neq 0$

Suppose there exists a function $f(x)$ satisfying the initial equation. We substitute $g(x)=kf(\frac{x}{k})$ for every

$x\in\mathbb{R}$. It is clear that $g(x)$ is a function, $g:\mathbb{R}\rightarrow\mathbb{R}$. It follows that $f(\frac{x}{k})=\frac{g(x)}{k}$.

The original functional equation is $f(f(x)+f(y)+kxy)=xf(y)+yf(x)$. We substitute in it x with $\frac{x}{k}$ and y with $\frac{y}{k}$. The result is:

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$$f(f(\frac{x}{k})+f(\frac{y}{k})+\frac{xy}{k})=\frac{x}{k}f(\frac{y}{k})+\frac{y}{k}f(\frac{x}{k}).$$

Using the fact that $f\left(\frac{x}{k}\right)=\frac{g(x)}{k}$ we obtain

$$g(g(x)+g(y)+xy)=\frac{xg(y)+yg(x)}{k}. \text{ (1)}$$

If we take x=0, we get:

$$g(g(0)+g(y))=\frac{yg(0)}{k}.$$

First, if $g(0)=0$, then $g(g(y))=0$ for every $y\in\mathbb{R}$. We substitute x with $\frac{-g(y)}{y}$ in (1):

$$g(g(x))=\frac{\frac{-g(y)}{y}g(y)}{k}+\frac{yg\left(\frac{-g(y)}{y}\right)}{k}=g(g(x))=0.$$

Therefore, $g\left(\frac{-g(y)}{y}\right)=\left(\frac{g(y)}{y}\right)^2$. If we take $\frac{-g(y)}{y}=a$, then $g(a)=a^2$. Furthermore, we substitute x=a, y=-a in (1) and we get $g(g(-a)+a^2-a^2)=ag(-a)-a^3$. From $g(g(y))=0$ for every $y\in$ **R**, it follows that $g(-a)=a^2$ (supposing that there exists $a\neq 0$). We substitute $x=y=a$ and $x=y=-a$ in (1). We get $g(3a^2)=\frac{-2a^3}{k}$ and $g(3a^2)=\frac{2a^3}{k}$. Therefore, a=0. Hence, $\frac{g(y)}{y}=0$ for every $y\in R/\{0\}$. Thus, $g\equiv 0$ and $f\equiv 0$.

Second, if $g(x)\neq 0$, from $g(g(0)+g(y))=\frac{yg(0)}{k}$ it follows that $g(y)$ is injective and surjective, meaning it is bijective. We have that $g(2g(0))$ from above. We substitute $x=y=2g(0)$ in (1) and get $g(4g^2(0))=0$. Therefore, $2g(0)=4g^2(0)$. Since $g(0)\neq 0$, we get $g(0)=\frac{1}{2}$. If we take x=0 in (1), then $g(g(y)+\frac{1}{2})=\frac{y}{2k}$. Then if y=0, from the previous result follows $g(1)=0$. We take x=1 in (1) and get $g(g(y)+y)=\frac{g(y)}{k}$. Therefore, we have $g(\frac{1}{2})=\frac{1}{2k}$.

We substitute x=1, $y=\frac{1}{2}$ in (1) and get $g(g(\frac{1}{2})+\frac{1}{2})=\frac{g(\frac{1}{2})}{k}=\frac{1}{2k^2}$. From $g(g(y)+\frac{1}{2})=\frac{y}{2k}$ when $y=\frac{1}{2}$, we have $g(g(y)+\frac{1}{2})=\frac{1}{4k}$. According to the results obtained up to now, the following relations hold:

(*i*) $g(g(x)+g(y)+xy)=\frac{xg(y)+yg(x)}{2}$

(*ii*) $g(g(y)+\frac{1}{2})=\frac{y}{4}$

(*iii*) $g(g(y)+y)=\frac{g(y)}{2}$

From (*ii*), $g(g(y)+\frac{1}{2})+\frac{1}{2}=\frac{y}{4}+\frac{1}{2}$. Therefore, $g(\frac{y}{4}+\frac{1}{2})=\frac{g(y)+\frac{1}{2}}{4}$.

From (*iii*), $\frac{g(y)}{2}=g(g(g(y)+y)+\frac{1}{2})=g(\frac{g(y)}{2}+\frac{1}{2})$. Hence, from this result and (*ii*)

$g(\frac{g(y)}{2}+\frac{1}{2})=\frac{g(y)+y}{4}$. From $g(\frac{y}{4}+\frac{1}{2})=\frac{g(y)+\frac{1}{2}}{4}$ when $y=2g(y)$, we get $g(\frac{g(y)}{2}+\frac{1}{2})=\frac{g(2g(y))+\frac{1}{2}}{4}$. Therefore, from the last two equalities, we have $g(2g(y))=g(y)+y-\frac{1}{2}$.

We will now prove the equality $g(\frac{y}{2})=\frac{y}{4}+g(y)$. From $g(2g(y))=g(y)+y-\frac{1}{2}$ for $y=g(y)+\frac{1}{2}$ and (*ii*), it follows that $\frac{g(y)}{2}=g(g(y)+\frac{1}{2})+g(y)+\frac{1}{2}-\frac{1}{2}$. Hence, $g(\frac{y}{2})=\frac{y}{4}+g(y)$. If we substitute $y=2y-2$ in $g(\frac{y}{4}+\frac{1}{2})=\frac{g(y)+\frac{1}{2}}{4}$, from $g(\frac{y}{2})=\frac{y}{4}+g(y)$ we obtain

$$\frac{-y+1}{2}+g(y-1)+\frac{1}{2}=g(2y-2)+\frac{1}{2}=4g(\frac{y}{2})=y+4g(y).$$

Therefore, we get $g(y-1)=\frac{3}{2}y-1+4g(y)$.

Now, form $g(\frac{y}{2})=\frac{y}{4}+g(y)$ for $y=4y+2$, we have

$$g(y+1)=\frac{g(4y+2)+\frac{1}{2}}{4}=g(2y+1)-\frac{2y+1}{4}+\frac{1}{2}=\frac{\frac{g(2y)}{4}-\frac{3}{2}(2y+1)-1-\frac{2y+1}{4}+\frac{1}{2}}{4}$$

$$=\frac{\frac{g(2y)}{4}-\frac{5}{2}y-\frac{1}{4}}{4}=\frac{\frac{g(y)}{4}-\frac{21}{8}y-\frac{1}{4}}{4}$$

and $g(y+1)=\frac{g(y)}{4}-\frac{3}{2}(y+1)-\frac{1}{4}$ from $g(y-1)=\frac{3}{2}y-1+4g(y)$.

Therefore, we obtain $g(y)=\frac{-y+1}{2}$ and thus $f(x)=\frac{-2x+1}{4}$.

Finally, we have the following result:

$$\text{for } \begin{cases} k\neq 2, f\equiv 0; \\ k=2, f_1\equiv 0, f_2=\frac{-2x+1}{4}. \end{cases}$$

3. In the given equation:

$$f(f(y_1)+f(y_2)+\ldots f(y_n)+ky_1y_2\ldots y_n)=y_1f(y_2)+y_2f(y_3)+\ldots y_nf(y_1),$$

we switch y_1 and y_2 and get

$$y_1f(y_2)+y_2f(y_3)+y_nf(y_1)=y_2f(y_1)+y_1f(y_3)+y_nf(y_2)$$

We fix y_2 to be equal to a constant $c\neq 0$. Let $y_1=x$, y_3 and y_n be constants, different from 0 and $y_3\neq y_n$. We have $xf(y_2)+y_nf(x)+y_2f(y_3)=y_2f(x)+xf(y_3)+y_nf(y_2)$. Therefore, $(y_n-y_2)f(x)=cx+b$. Hence, $f(x)$ is linear. Let $f(x)=ax+b$. From

$$f(f(y_1)+f(y_2)+\ldots f(y_n)+ky_1y_2\ldots y_n)=y_1f(y_2)+y_2f(y_3)+\ldots y_nf(y_1)$$

we obtain

$$a^2(y_1+\mathbf{K}+y_n)+nba+aky_1y_2\mathbf{K}y_n+b=f(a(y_1+y_2+\mathbf{K}+y_n)+nb+ky_1y_2\mathbf{K}y_n)$$

$$=a(y_1y_2+\mathbf{K}+y_ny_1)+(y_1+\mathbf{K}+y_n)b.$$

Since $y_1,y_2,\mathbf{K},y_n$ are variables, we use the method of comparison of the coefficients. We get

$$\begin{cases} nab+b=0 \\ a^2=b \\ a=0 \\ ak=0. \end{cases}$$

Hence, $a=b=0$. Therefore, $f\equiv 0$.